COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Lecture 8 **(3.31.08)**

Morphological Image Processing

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A number of figures used in this presentation are courtesy of
“Morphological Image Analysis” by P. Soille

Outline

- What is Mathematical Morphology?
- Background Notions
- Introduction to Set Operations on Images
- Basic operation
  - Erosion, Dilation, Opening, Closing, Hit-or-Miss
- Algorithms
- Morphological operations on gray-level images

Morphological Image Processing

- Started in 1960s by G. Matheron and J. Serra
- Analysis of form and structure of objects
- Tools/Operations for describing/characterizing image regions and image filtering
- Images are treated as sets

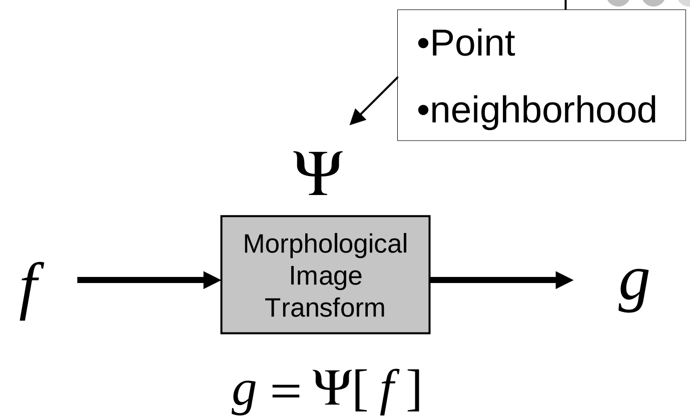
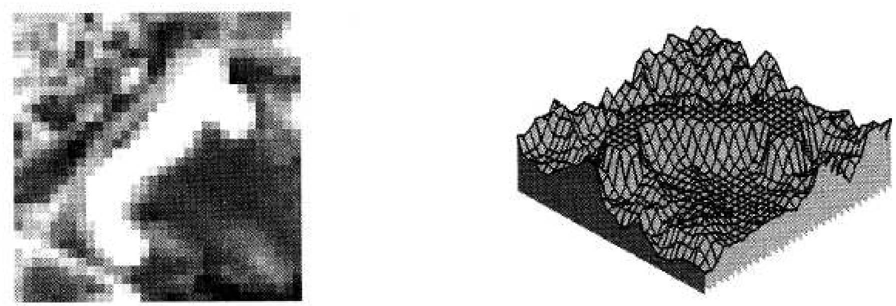


Image-to-image transform

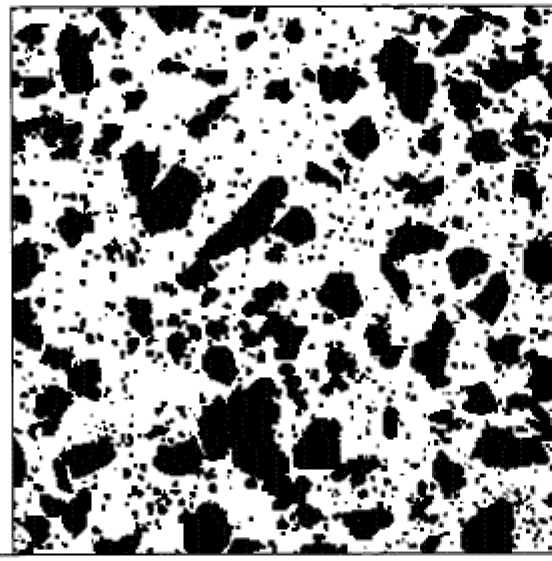


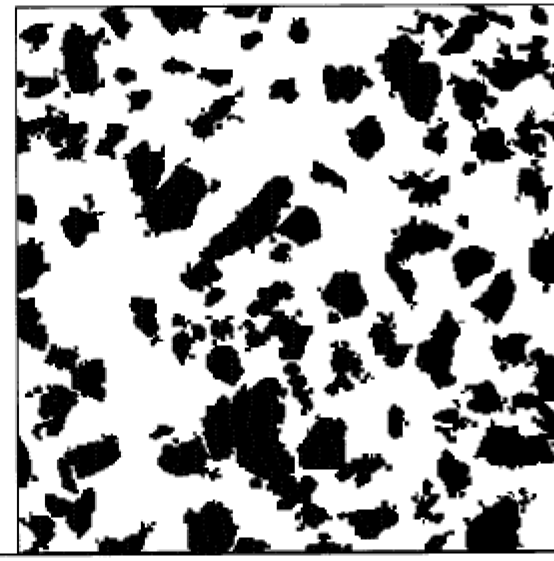
(a) Grey tone image.

(b) Set representation of *(a)*.

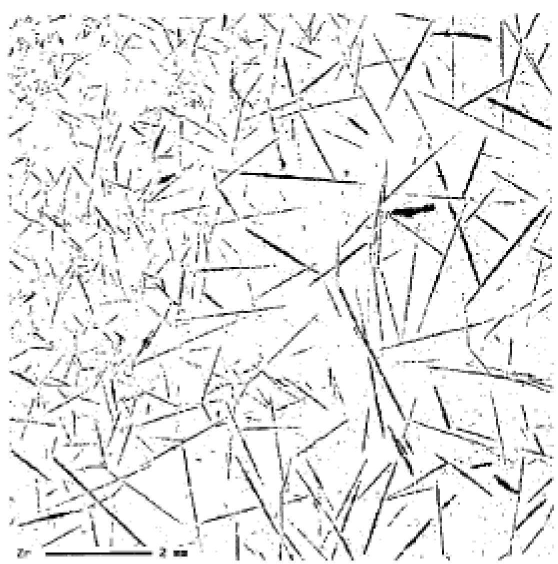
Applications - filtering

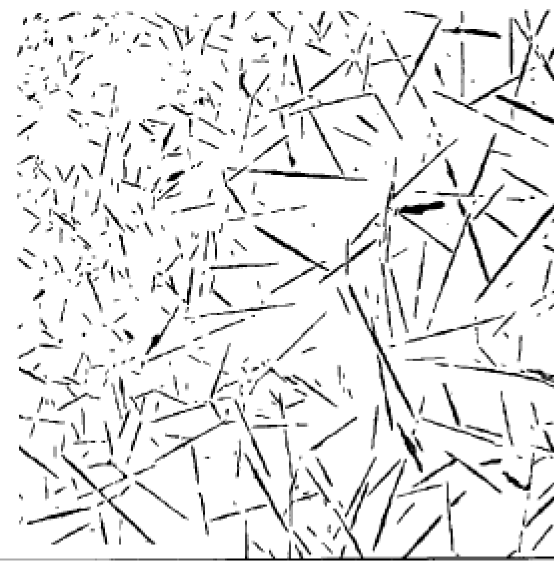
1. Removal of small blobs



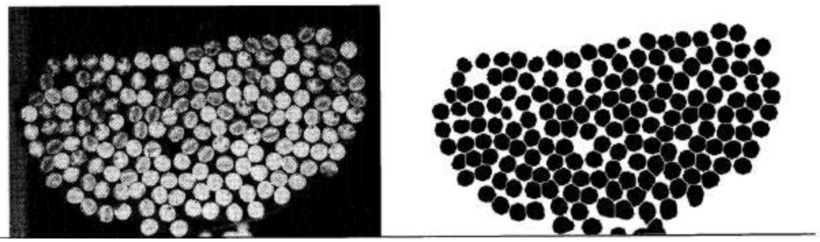


2. Extraction and grouping of linear objects

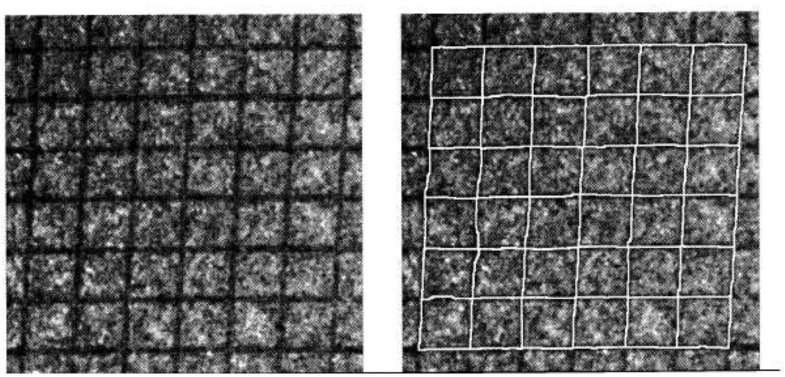




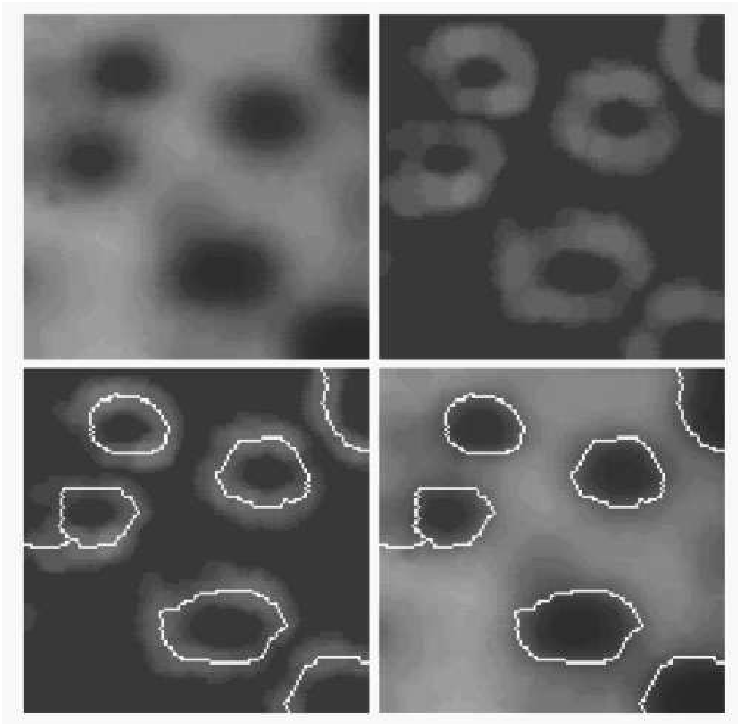
Applications - segmentation



1. Separation of connected blobs

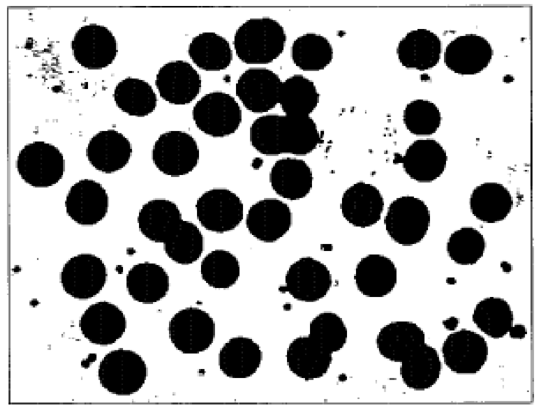


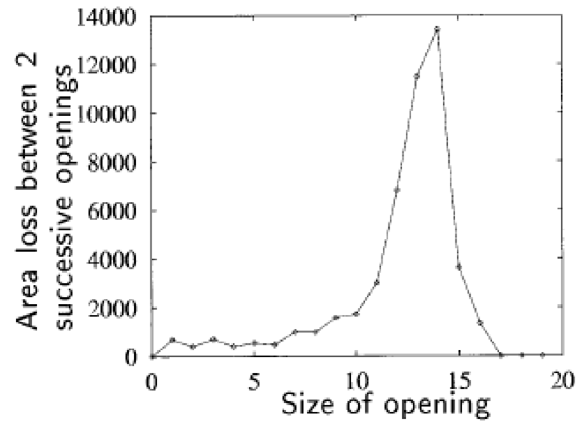
5. Extraction of grid lines



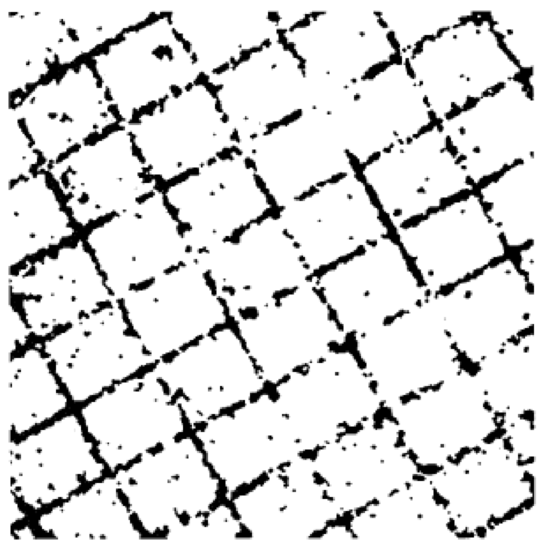
Applications - quantification

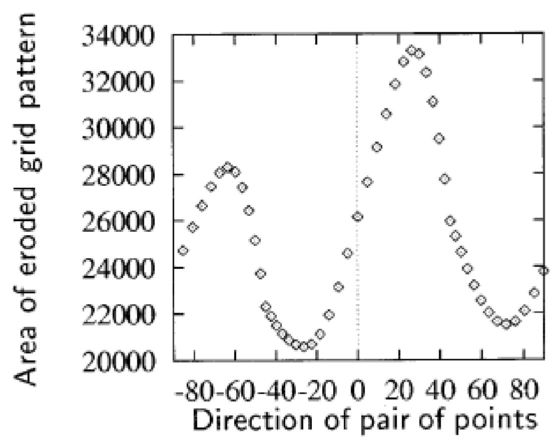
1. Pattern spectrum or granulometries





2. Analysis of directions





Background Notions: Image as a Set

Binary image

$$f : D_f \subset Z^n \to \{0,1\}$$

definition domain of f –or- image plane

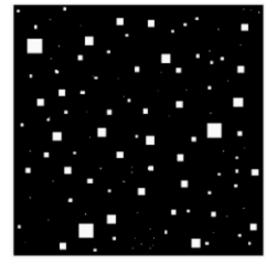
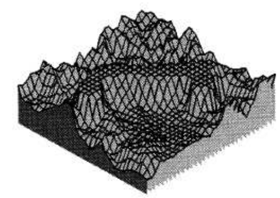


Image as a set: Set of all white pixels

Grey-level image

$$f : D_f \subset Z^n \to \{0,1,\ldots,t_{\max}\}$$

(a) Grey tone image.



(b) Set representation of (a).

Image as Digital Elevation Map (DEM)

- How could the grey-level image be treated as a set?

Image Graph

$$G(f) = \{(x,t) \in Z^n \times N_0 \mid t = f(x)\}$$

Image Sub-Graph

$$SG(f) = \{(x,t) \in Z^n \times N_0 \mid 0 \le t \le f(x)\}$$

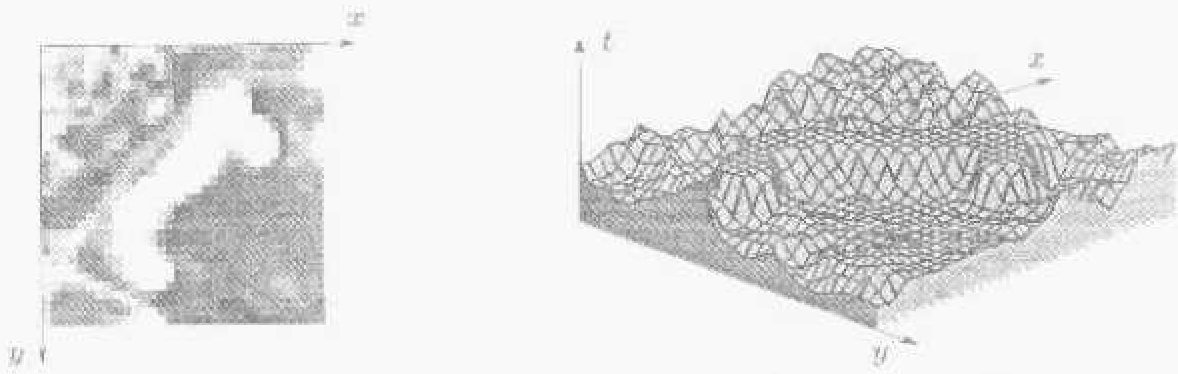
Background Notions:
Gray-level Image as a Set

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$f(x)$	0	0	1	3	2	2	4	4	0	5	5	3	0	0

(a) 1-D discrete signal f.

(b) Graph of the signal f defined in (a).

(c) Subgraph of f.

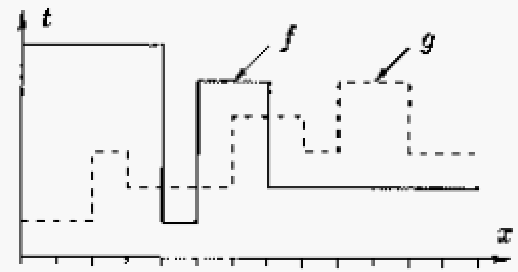


(d) Grey tone image.

(e) Subgraph of (d).

Set Operations on Images - Union & Intersection

Union	$(f \vee g)(x) = \max[f(x), g(x)]$	$SG(f \vee g) = SG(f) \bigcup SG(g)$
Intersection	$(f \wedge g)(x) = \min[f(x), g(x)]$	$SG(f \wedge g) = SG(f) \bigcap SG(g)$



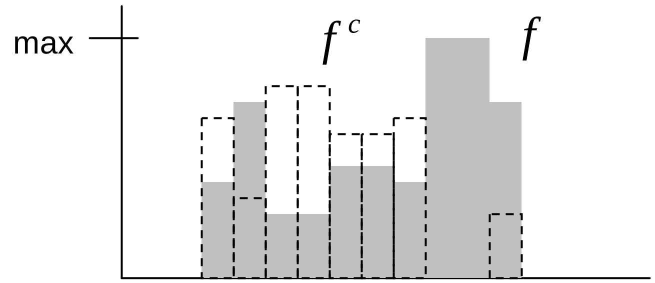
(a) 1-D signals f and g. (b) Point-wise maximum $\vee$ and point-wise minimum $\wedge$.

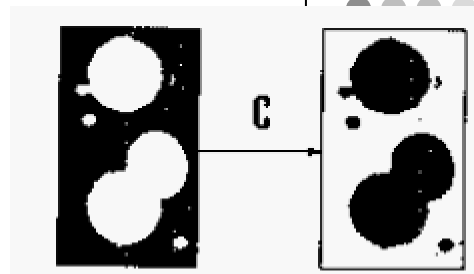
$$(\Psi_1 \vee \Psi_2)(f) = \Psi_1(f) \vee \Psi_2(f)$$
$$(\Psi_1 \wedge \Psi_2)(f) = \Psi_1(f) \wedge \Psi_2(f)$$

Set Operations on Images

Complementation

$$f^{c}(x) = t_{\max} - f(x)$$





(a) Binary image complementation.

Set Difference

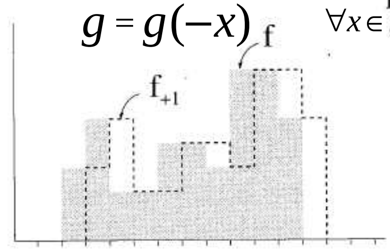
$$X \setminus Y = X \bigcap Y^{c}$$

Note: Only on binary images

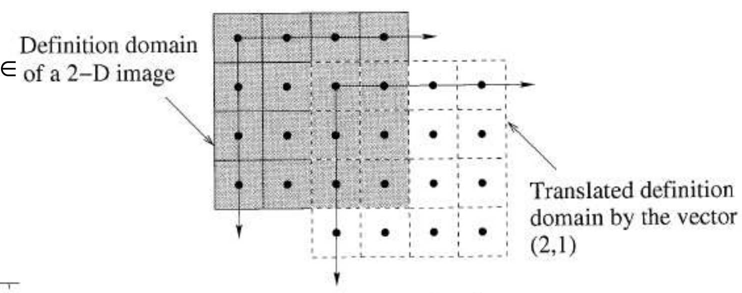
Translation

$$f_b(x) = f(x-b)$$

$$\check{g} = g(-x) \quad \forall x \in$$



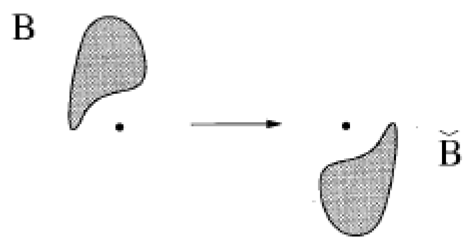
(a) $f_{+1}(x) = f(x-1)$



(b) $f_{(2,1)}(1,1) = f(3,2)$

Reflection

$$\check{B} = \{-b \mid b \in B\}$$



Morphological Image Operations

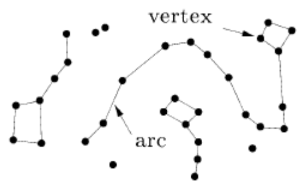
- All morphological image operations are the result of interaction between a set representing an image and a set representing a structuring element

- All interactions are based on combination of intersection, union, complementation and translation

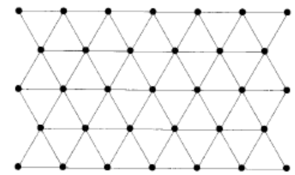
Graphs

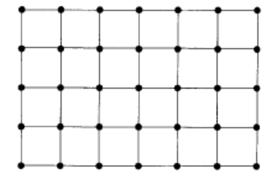
Graph is a pair of vertices and edges (V,E), where:

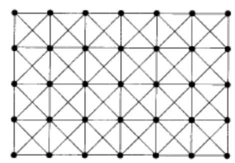
$$V = (v_1, v_2, \cdots, v_n)$$
$$E = (e_1, e_2, \cdots, e_m)$$

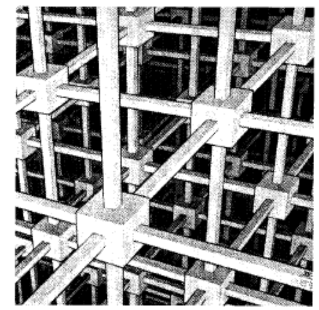


(a) General graph.


(b) Hexagonal graph.


(c) 4-connected graph.


(d) 8-connected graph.


(e) 6-connected graph in the 3-D cubic grid by M.C. Escher ©Cordon Art-Baarn-Holland.

- **Planar graph**
- **Simple graph**

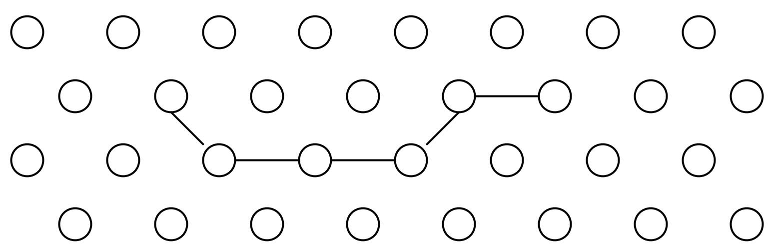
Neighborhood of vertex v:

$$N_G(v) = \{v' \in V \mid (v, v') \in E\}$$

Path P in graph G:

$$P_G = (v_0, v_1, \cdots, v_l) \quad ,(v_i, v_{i+1}) \quad neighbors$$

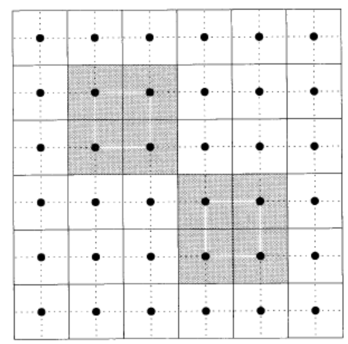
$$P_G^h$$



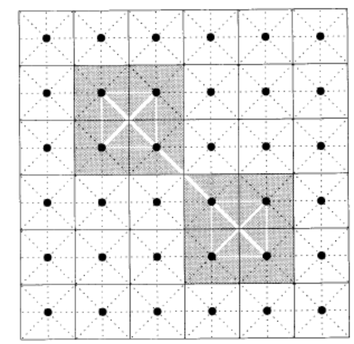
Grids & Connectivity

Connectivity:
a set is connected if each pair of its points can be joined by a path completely in the set

G^h-Connectivty:
2 pixel p and q of image f are G^h-connected iff there exists a P_G^h path with end points p and q

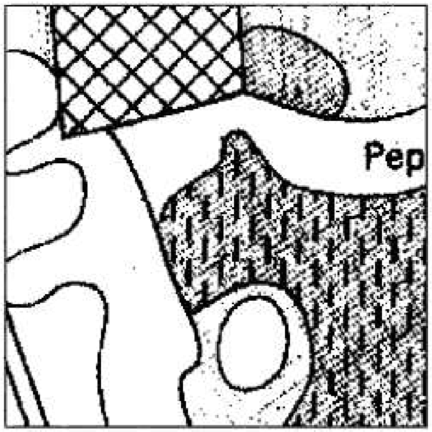


(a) A 6 × 6 discrete binary image and its representation in the 4-connected graph.

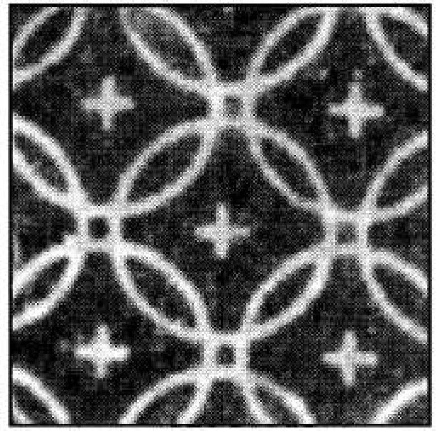


(b) Same image as in *(a)* but represented in 8-connectivity.

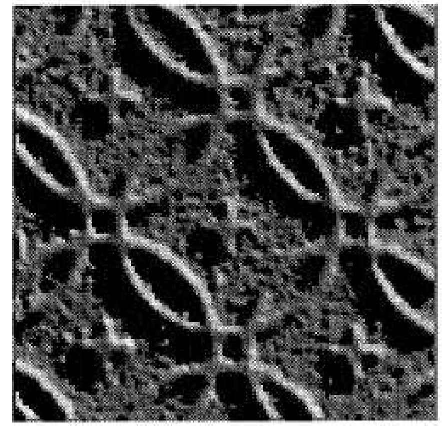
Structuring Element (SE): A Small Set for Probing Images



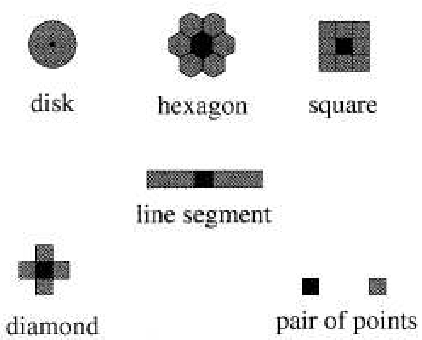
(a) A binary image.



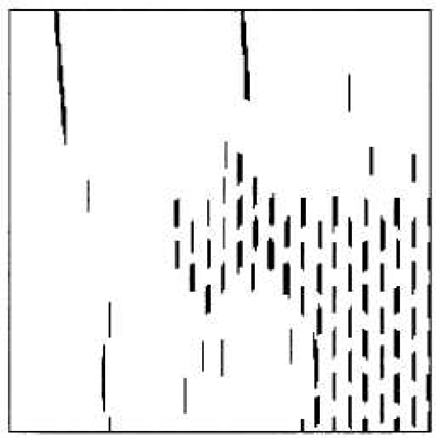
(b) A grey scale image.



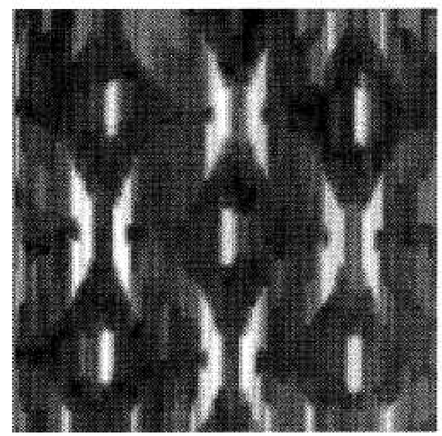
(c) Topographic representation of *(b)*.



(d) Shape of some common structuring elements.



(e) Extraction of vertical structures of *(a)* using a vertical SE.



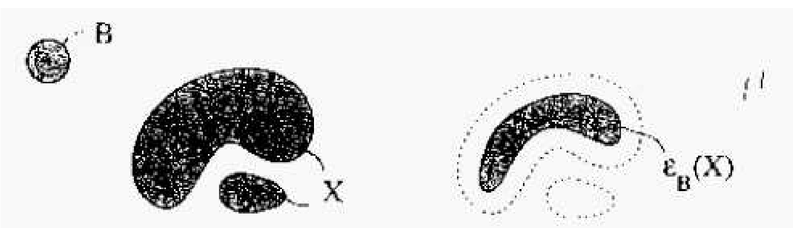
(f) Extraction of vertical structures of *(b)* using a vertical SE.

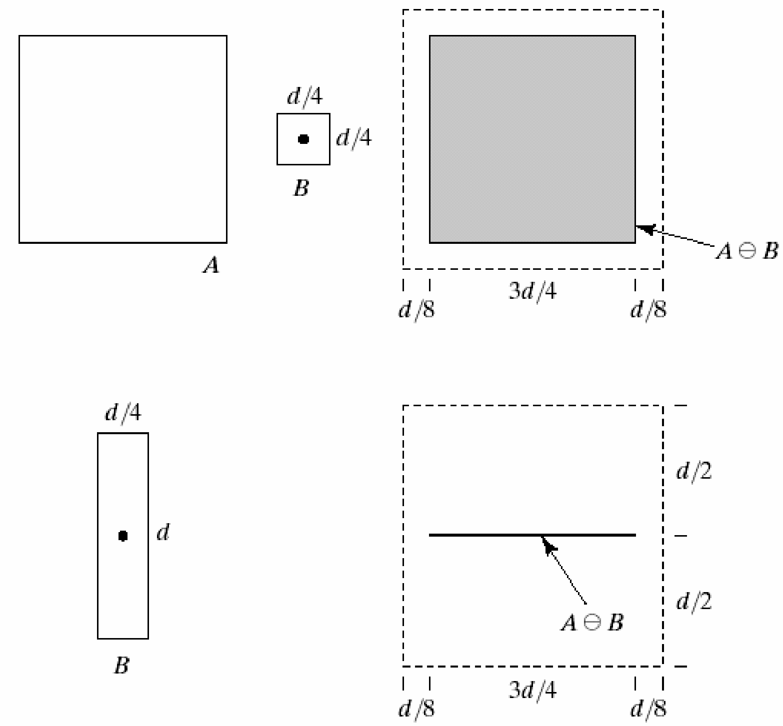
Erosion: "*Does the SE fit the set?*"

$$\varepsilon_B(X) = \{x \mid B_x \subseteq X\}$$: Eroding set X with SE B

$$\varepsilon_B(X) = \bigcap_{b \in B} X_{-b}$$

$$\varepsilon_B(X) = X \ominus \check{B}$$





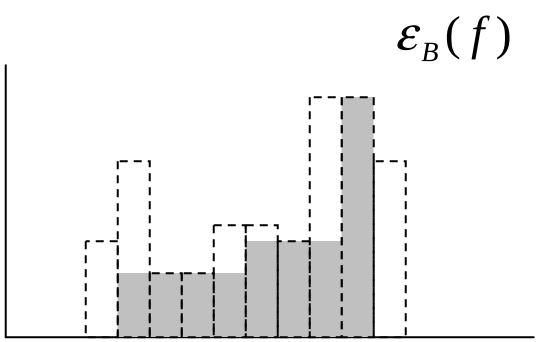
a b c
d e

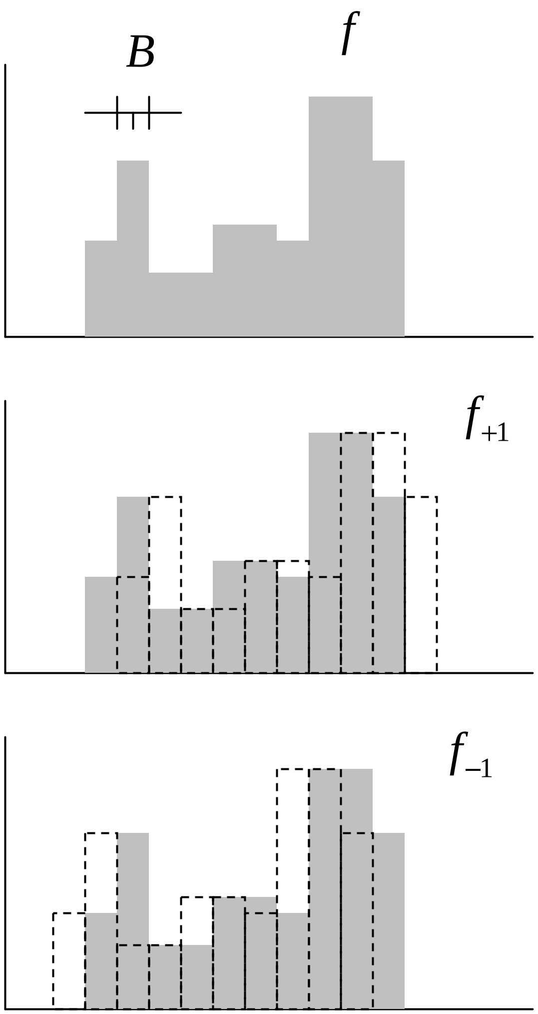
FIGURE 9.6 (a) Set A. (b) Square structuring element. (c) Erosion of A by B, shown shaded. (d) Elongated structuring element. (e) Erosion of A using this element.

Erosion: *Implementation*

$$\varepsilon_B(f) = \bigwedge_{b \in B} f_{-b}$$

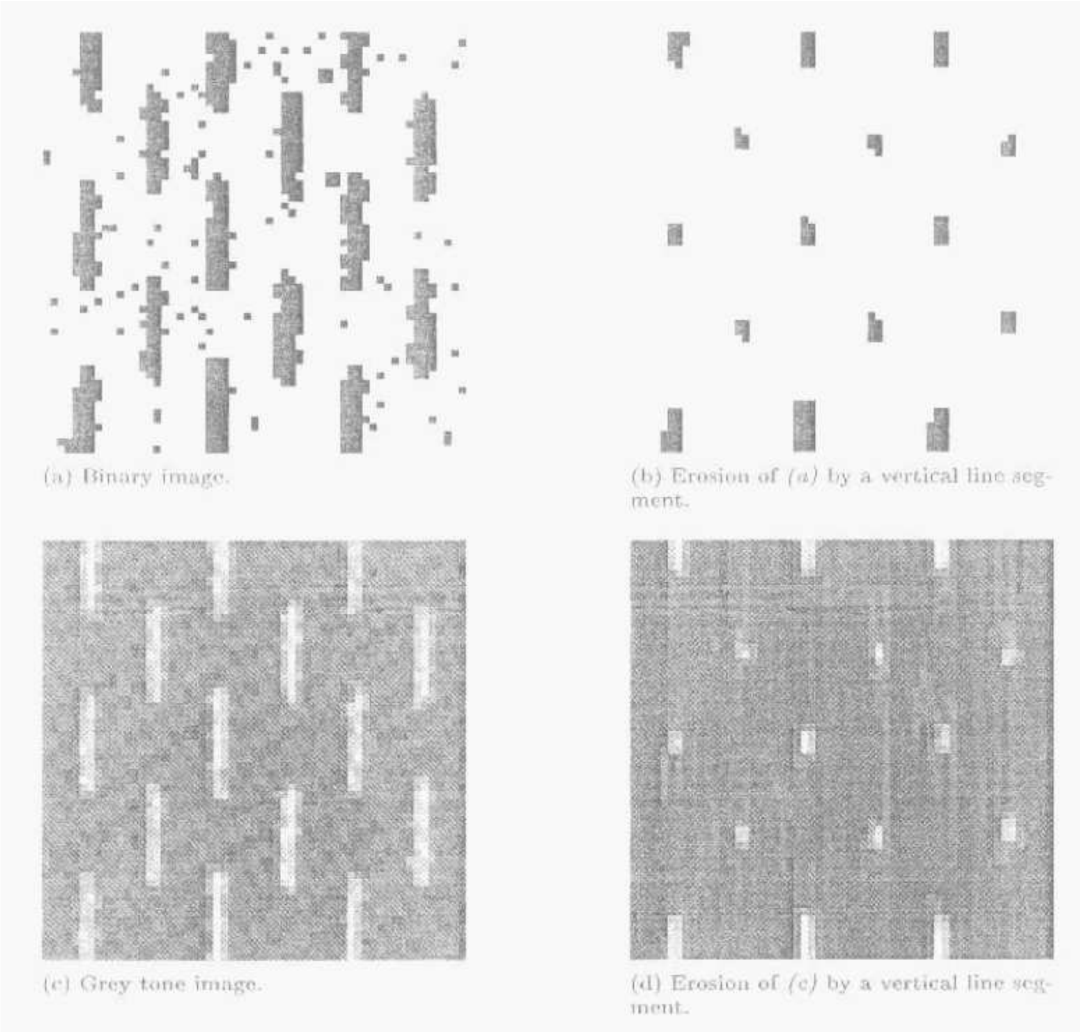
$$\Rightarrow [\varepsilon_B(f)](x) = \min_{b \in B} f(x+b)$$





Erosion: "*Does the SE fit the set?*" ***Grey-level image***

$$\varepsilon_B[SG(f)] = \{(x,t) \mid B_{(x,t)} \subseteq SG(f)\}$$

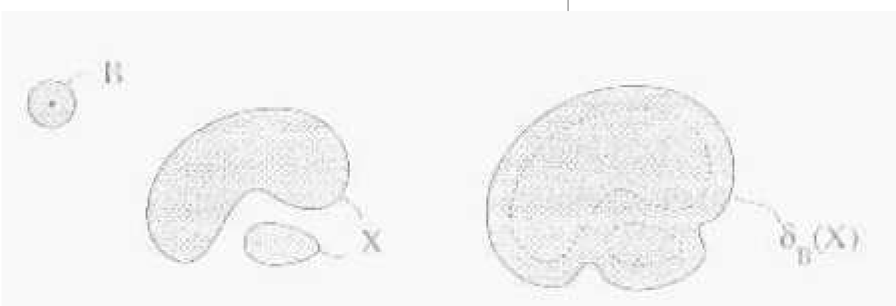


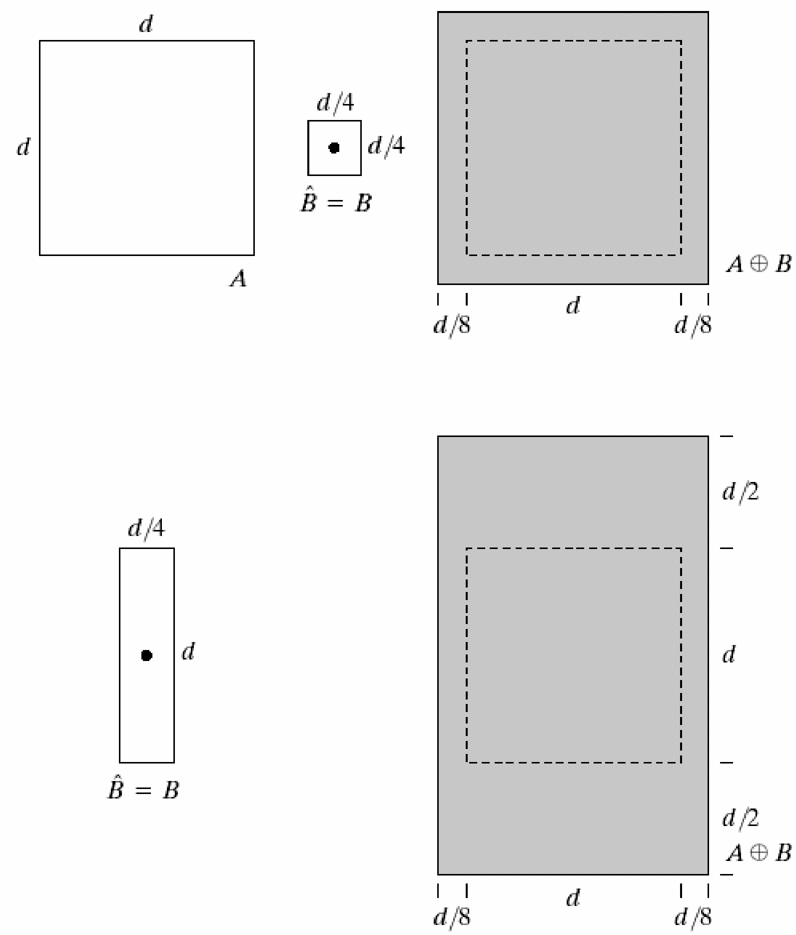
Dilation: "*Does the SE hit the set?*"

$$\delta_B(X) = \{x \mid B_x \cap X \neq \phi\}$$: Dilating set X with SE B

$$\delta_B(X) = \bigcup_{b \in B} X_{-b}$$

$$\delta_B(X) = X \oplus \check{B}$$

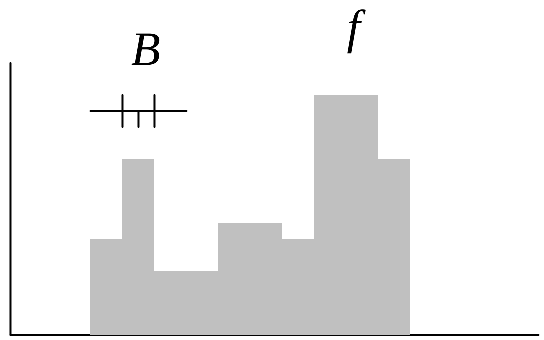


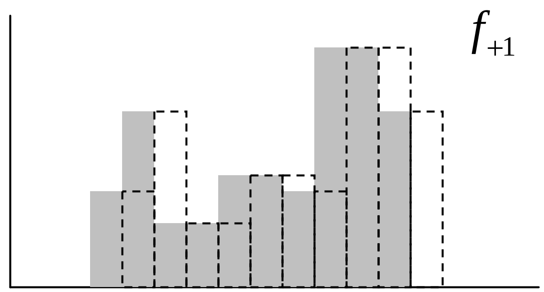


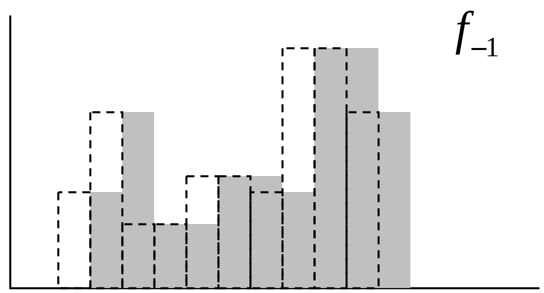
Dilation: *Implementation*

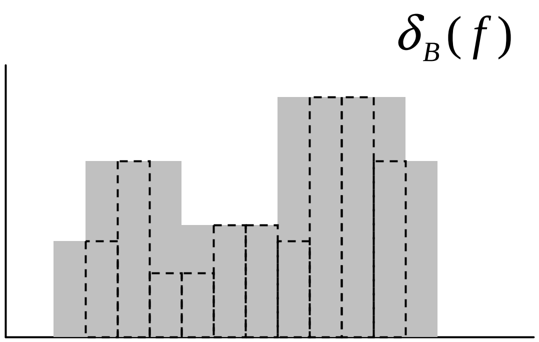
$$\delta_B(f) = \bigvee_{b \in B} f_{-b}$$

$$\Rightarrow [\delta_B(f)](x) = \max_{b \in B} f(x+b)$$



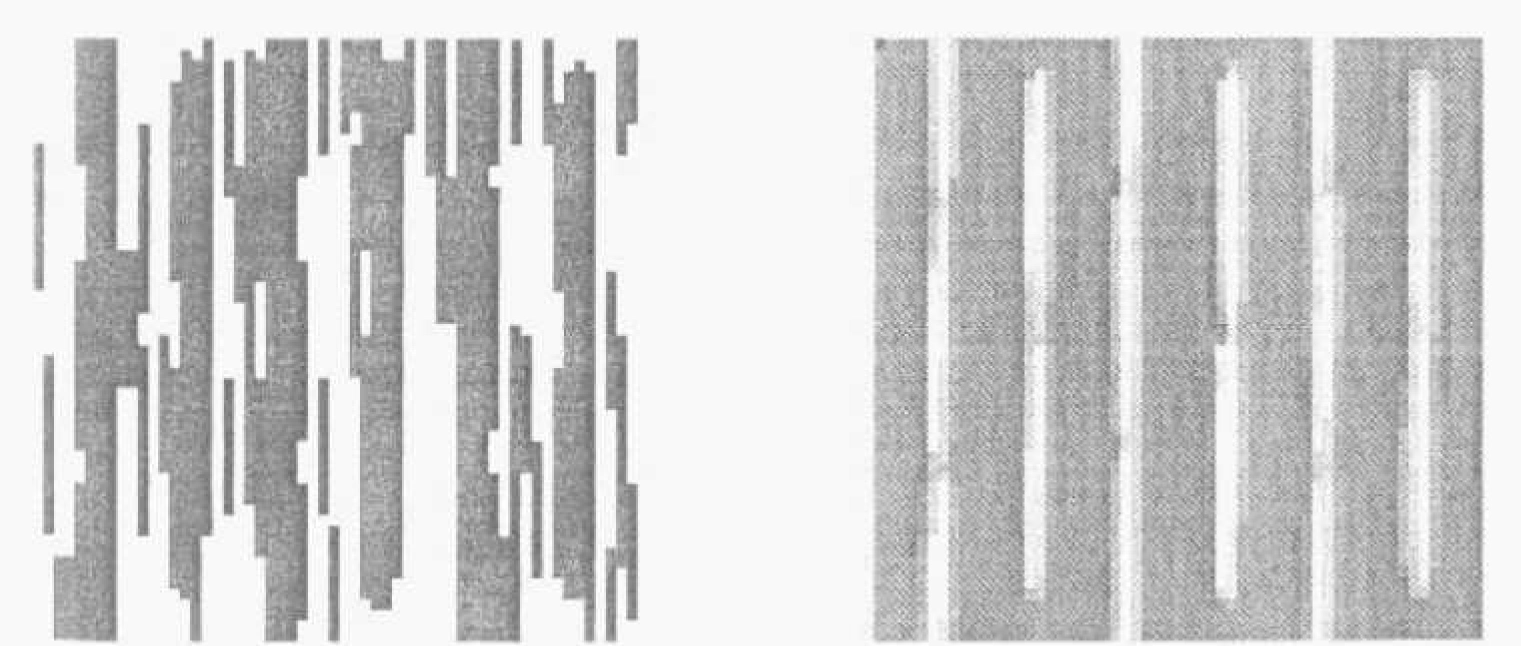






Dilation: "*Does the SE hit the set?*"
Grey-level image

$$\delta_B[SG(f)] = \{(x,t) \mid B_{(x,t)} \cap SG(f) \neq \phi\}$$

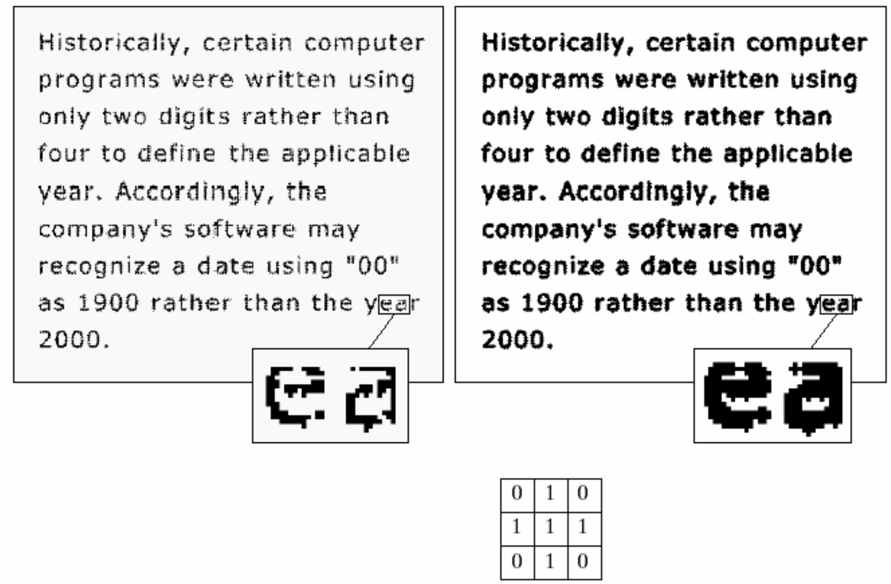


(a) Dilation of Fig. 3.5a by a vertical line segment.

(b) Dilation of Fig. 3.5c by a vertical line segment.

Erosion and Dilation: Examples

Dilation



a c
b

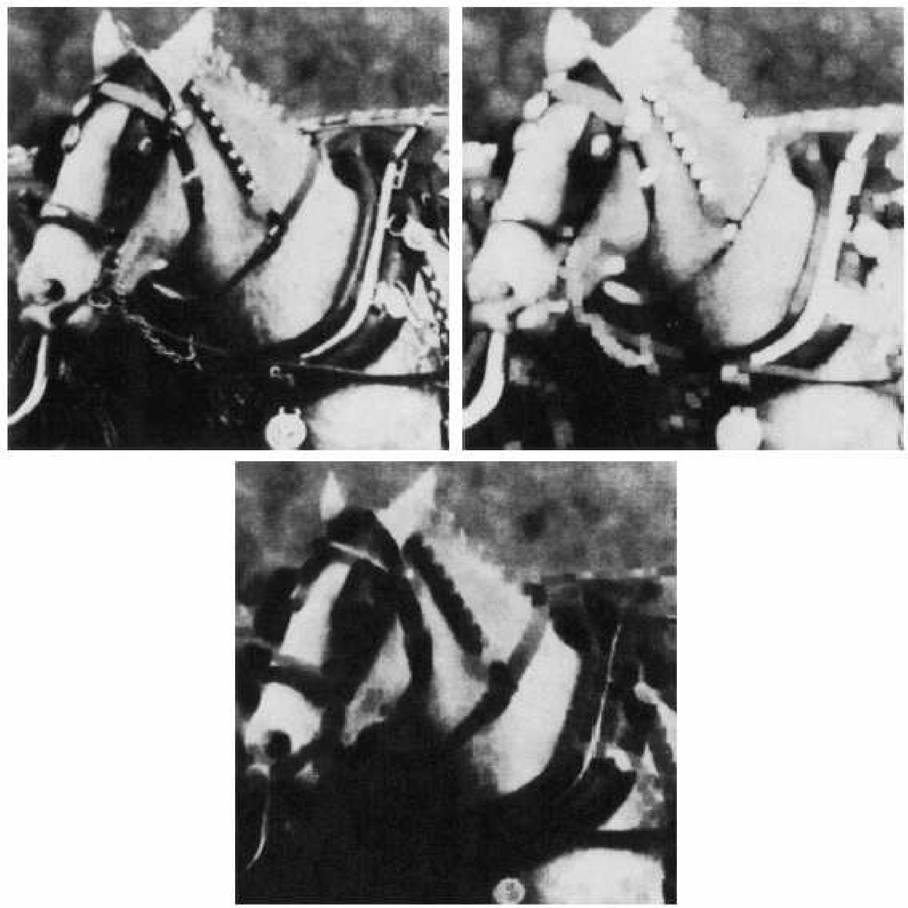
FIGURE 9.5 (a) Sample text of poor resolution with broken characters (magnified view). (b) Structuring element. (c) Dilation of (a) by (b). Broken segments were joined.

Erosion then Dilation

a b c

FIGURE 9.7 (a) Image of squares of size 1, 3, 5, 7, 9, and 15 pixels on the side. (b) Erosion of (a) with a square structuring element of 1's, 13 pixels on the side. (c) Dilation of (b) with the same structuring element.

Erosion and Dilation: Example



a b
c

FIGURE 9.29
(a) Original image. (b) Result of dilation.
(c) Result of erosion.
(Courtesy of Mr. A. Morris, Leica Cambridge, Ltd.)

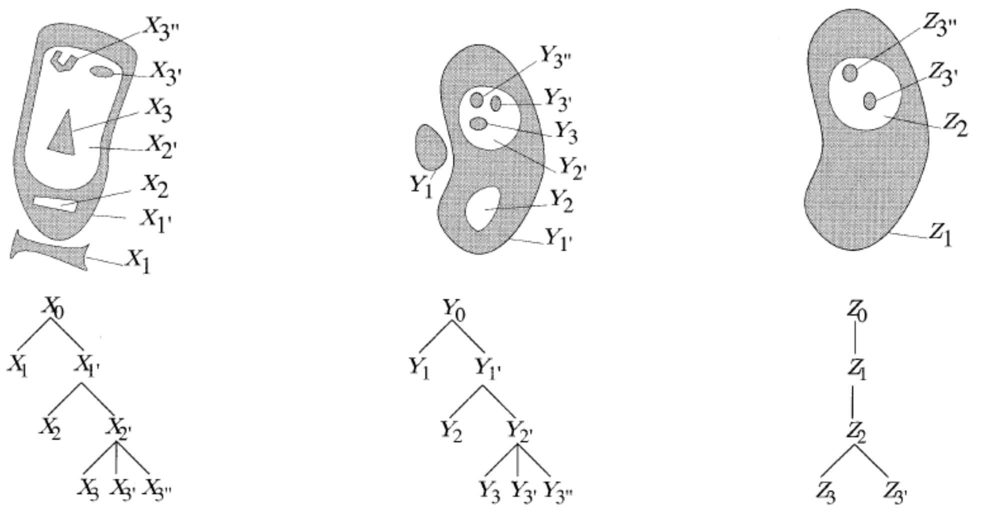
Properties of Erosion and Dilation

Duality

$$\varepsilon_B = C\delta_B C$$

$$\begin{aligned}\delta_B(f^c) &= \vee_{b\in B}[t_{\max} - f_{-b}] \\ &= t_{\max} - \wedge_{b\in B}[f_{-b}] \\ &= t_{\max} - \varepsilon_B(f) \\ &= [\varepsilon_B(f)]^c\end{aligned}$$

- Erosion and Dilation are irreversible operations
- Homotopy is not preserved under either one



(a) A set X and its homotopy tree

(b) A set Y and its homotopy tree

(c) A set Z and its homotopy tree

Properties of Erosion and Dilation

Increasingness

$$f \leq g \quad \Rightarrow \quad \begin{cases} \varepsilon(f) \leq \varepsilon(g) \\ \delta(f) \leq \delta(g) \end{cases}$$

Distributivity

$$\delta(\bigvee_i f_i) = \bigvee_i \delta(f_i)$$

$$\varepsilon(\bigwedge_i f_i) = \bigwedge_i \varepsilon(f_i)$$

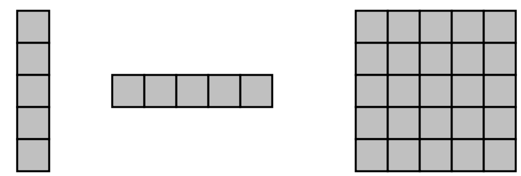
Properties of Erosion and Dilation

Composition

$$\delta_{B_2}\delta_{B_1}(f) = \delta_{(\delta_{\check{B}_2}(B_1))}(f)$$

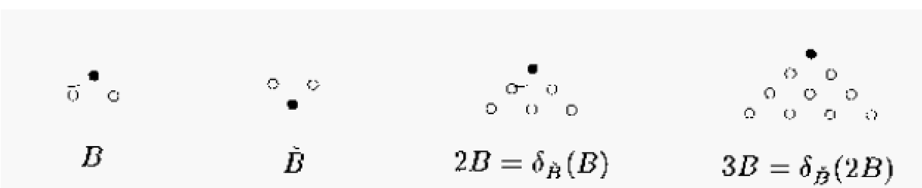
$$\varepsilon_{B_2}\varepsilon_{B_1}(f) = \varepsilon_{(\delta_{\check{B}_2}(B_1))}(f)$$

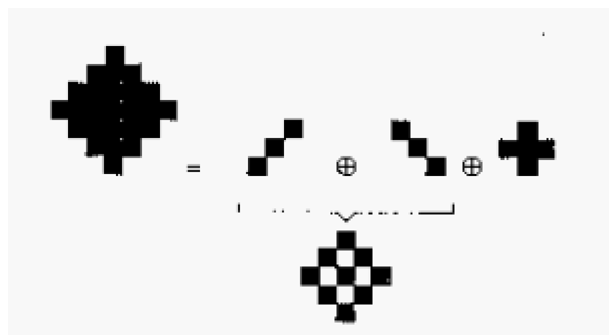
Break down operations using large SE with multiple operations with small SE.



B_1 B_2 $\delta_{B_2}(B_1)$

$$\delta_{nB} = \delta_B^{(n)}$$





Making a circular disk

Opening –
“If SE fits image then keep all SE!”

$$\gamma_B(f) = \delta_{\check{B}}[\varepsilon_B(f)]$$

$$\gamma_B(X) = \bigcup_x \{B_x \mid B_x \subseteq X\}$$

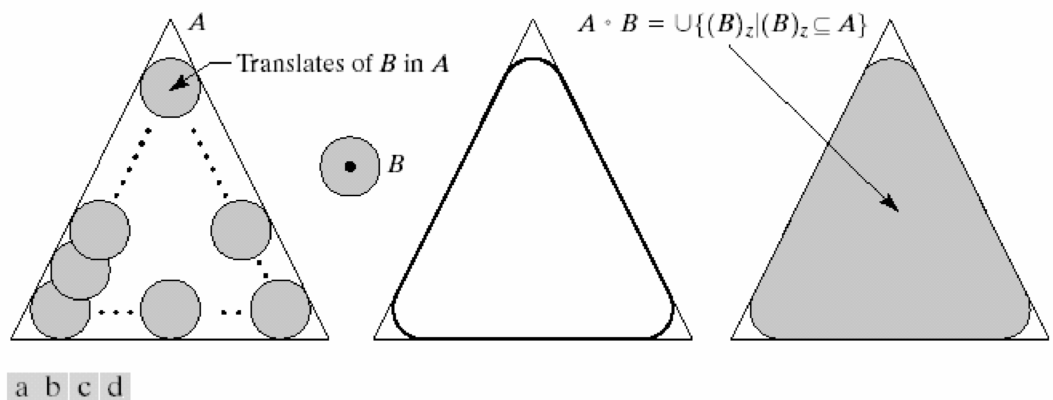
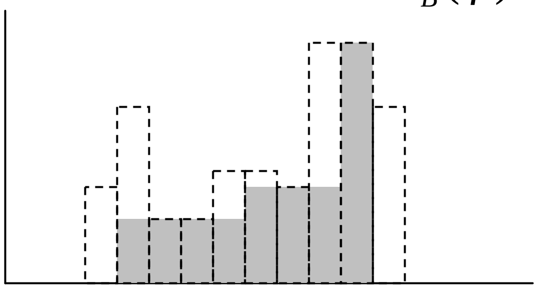


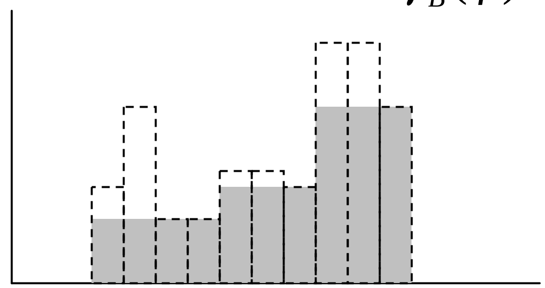
FIGURE 9.8 (a) Structuring element B “rolling” along the inner boundary of A (the dot indicates the origin of B). (c) The heavy line is the outer boundary of the opening. (d) Complete opening (shaded).

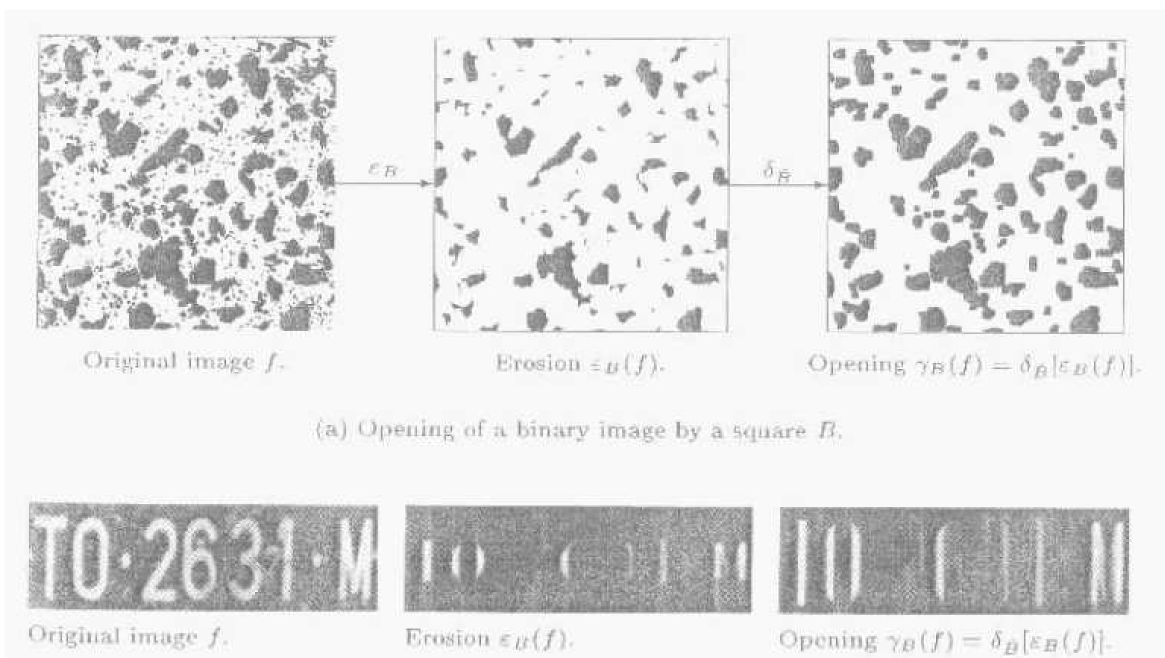
Opening –
"If SE fits the image then keep all SE!"

$\varepsilon_B(f)$



$\gamma_B(f)$

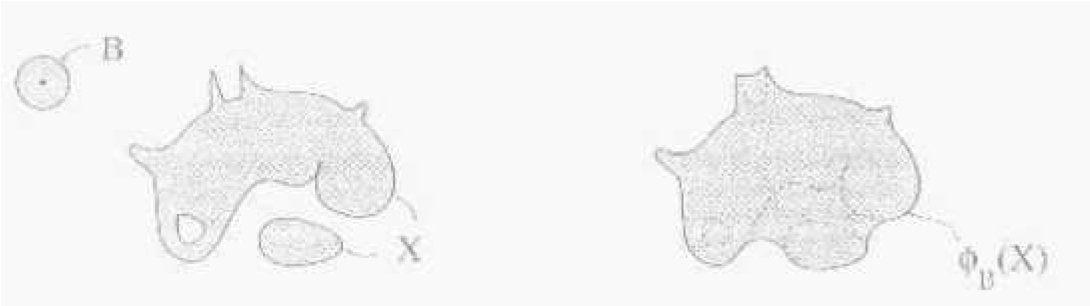




Closing – "If SE fits the background then all points in SE belong to the complement of closing!"

$$\phi_B(f) = \varepsilon_{\check{B}}[\delta_B(f)]$$

$$\phi_B(X) = \bigcap_x \{B_x^c \mid X \subseteq B_x^c\}$$



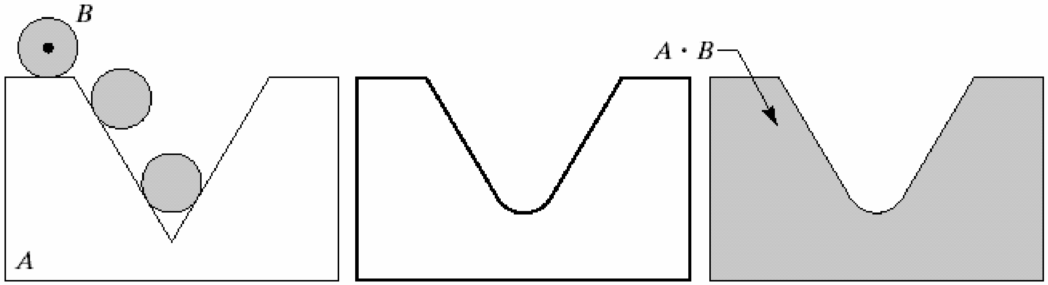
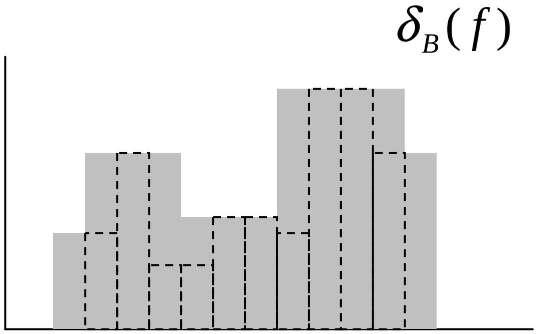
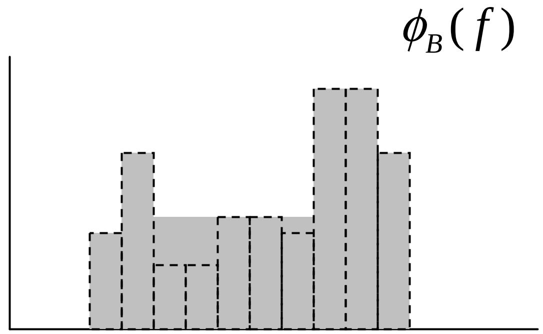
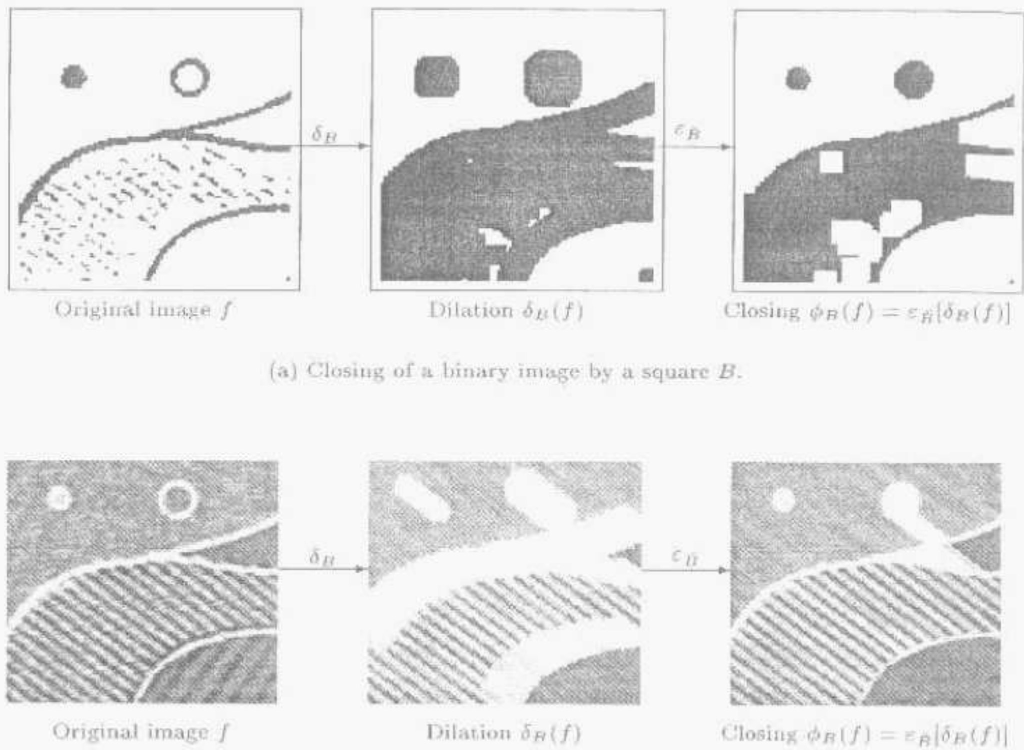


FIGURE 9.9 (a) Structuring element B "rolling" on the outer boundary of set A. (b) Heavy line is the outer boundary of the closing. (c) Complete closing (shaded).

Closing







(a) Closing of a binary image by a square B.

(b) Closing of a grey scale image with a line segment B of slope $-27/36$ (slope of the hatched lines).

Properties of Opening and Closing

Duality

$$\gamma_B = C\phi_B C$$

Increasingness

$$f \le g \quad \Rightarrow \quad \begin{cases} \gamma(f) \le \gamma(g) \\ \phi(f) \le \phi(g) \end{cases}$$

Idempotence

$$\gamma^{(n)} = \gamma$$

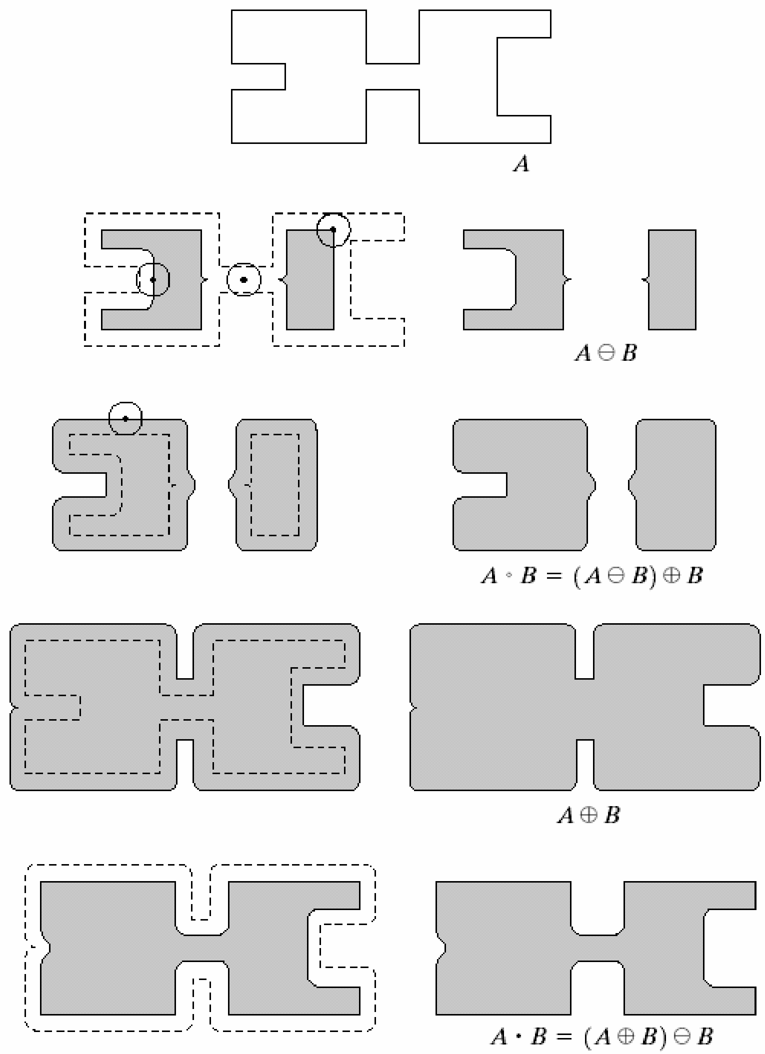
$$\phi^{(n)} = \phi$$

Sieving process: same sieve is not helpful using it more than once

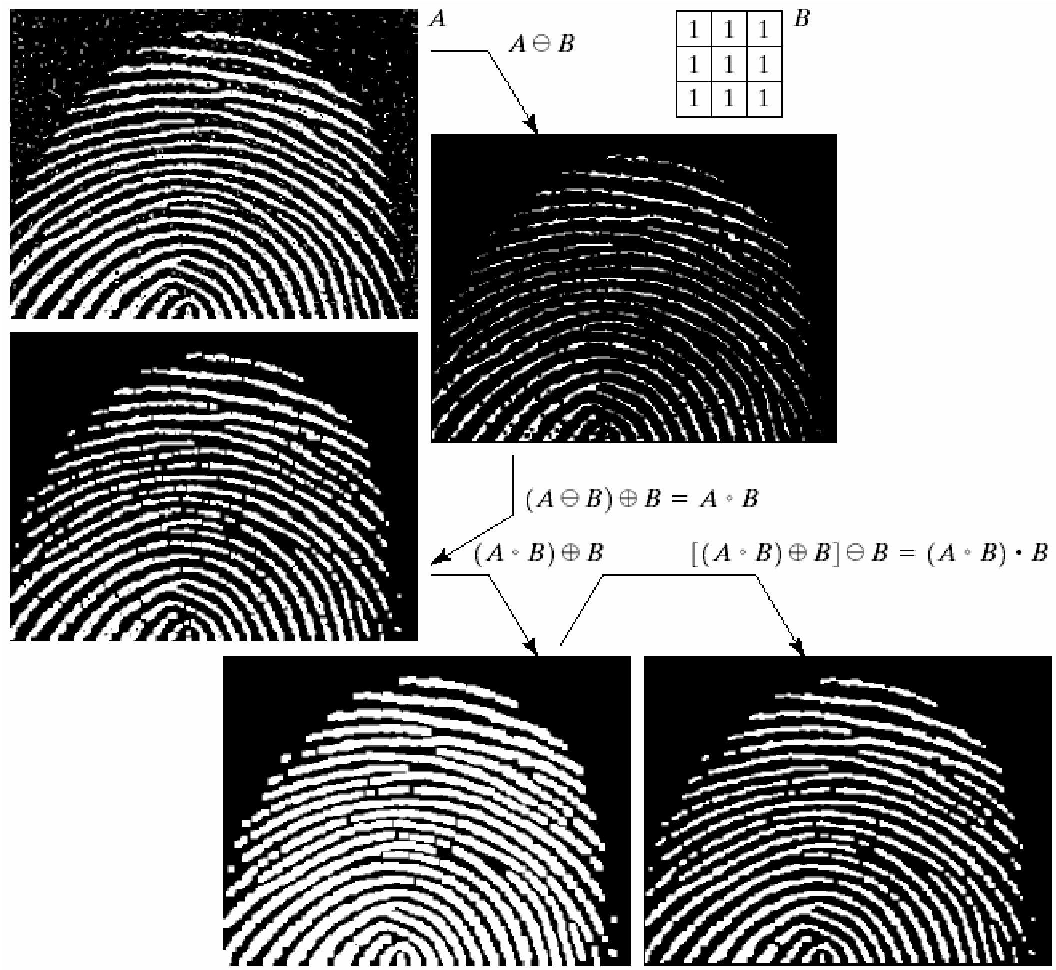
Opening and Closing: Example

a	
b	c
d	e
f	g
h	i

FIGURE 9.10 Morphological opening and closing. The structuring element is the small circle shown in various positions in (b). The dark dot is the center of the structuring element.



Opening and Closing: Example

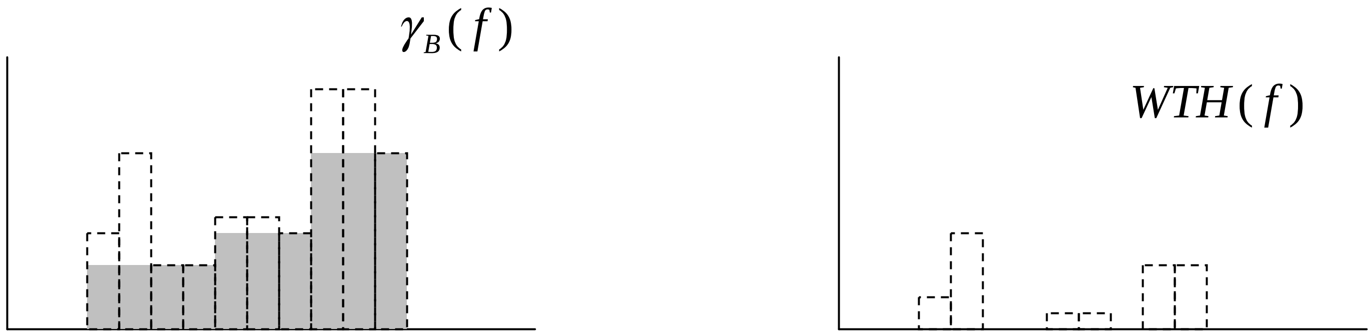


a b c d e f

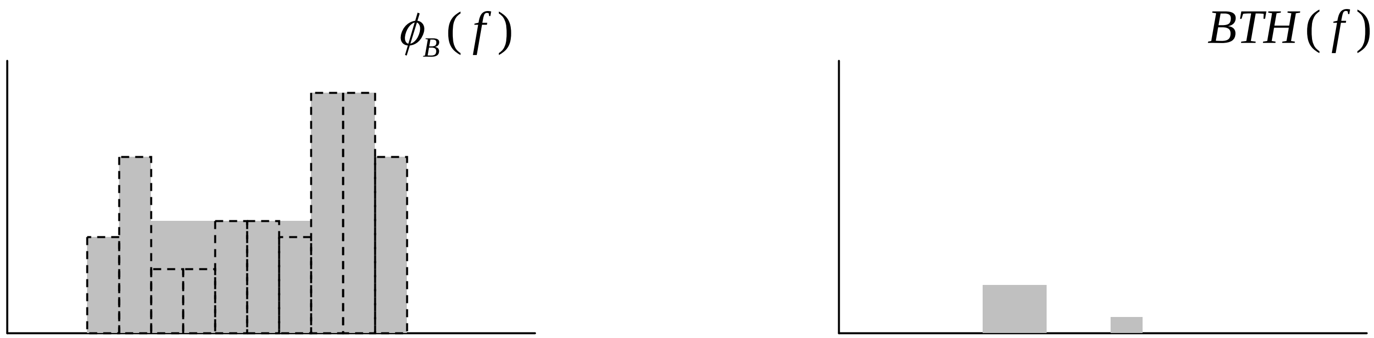
FIGURE 9.11 (a) Noisy image. (c) Eroded image. (d) Opening of A. (d) Dilation of the opening. (e) Closing of the opening. (Original image for this example courtesy of the National Institute of Standards and Technology.)

Top Hat transform

$$WTH(f) = f - \gamma(f)$$



$$BTH(f) = \phi(f) - f$$



Top Hat transform

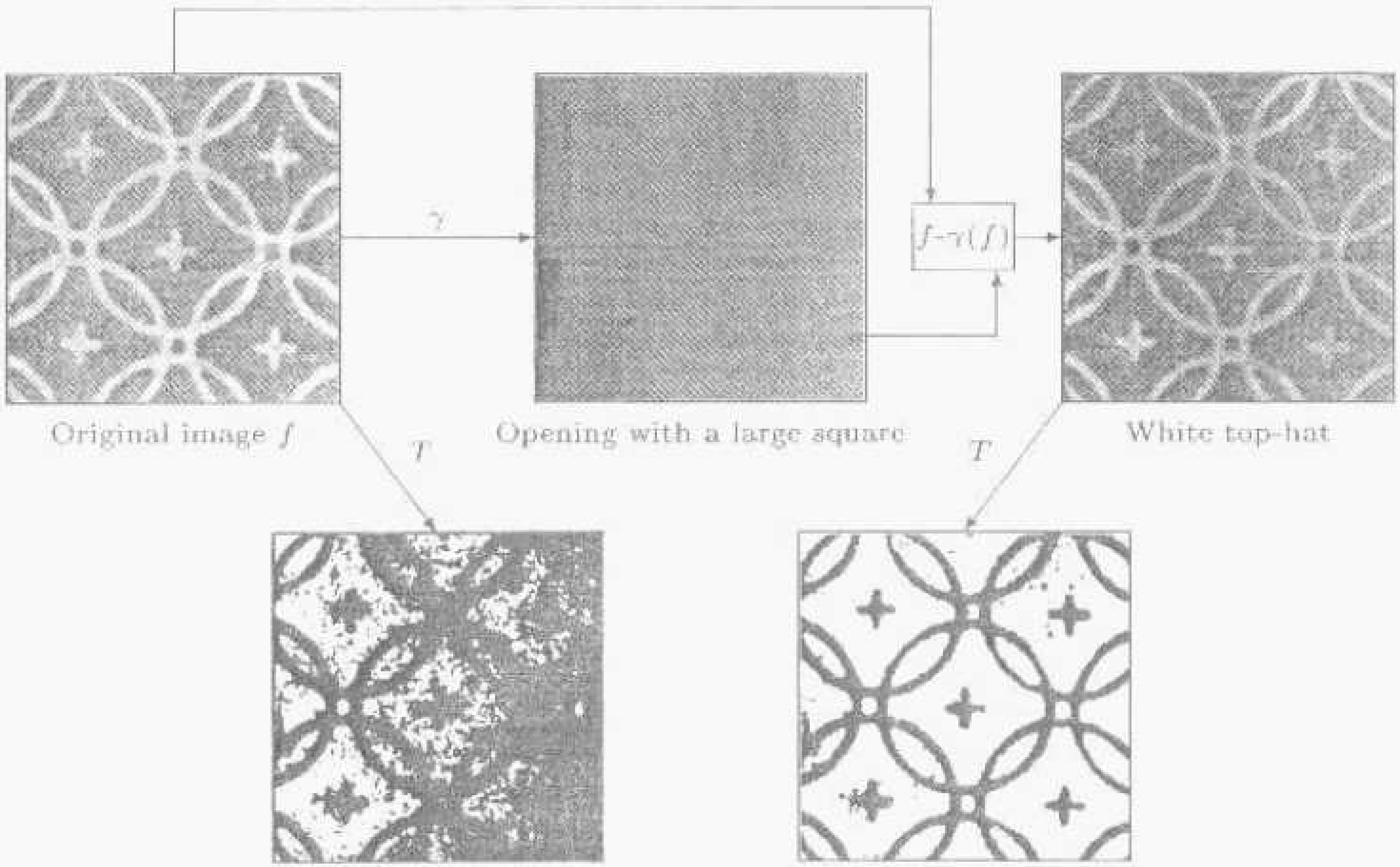
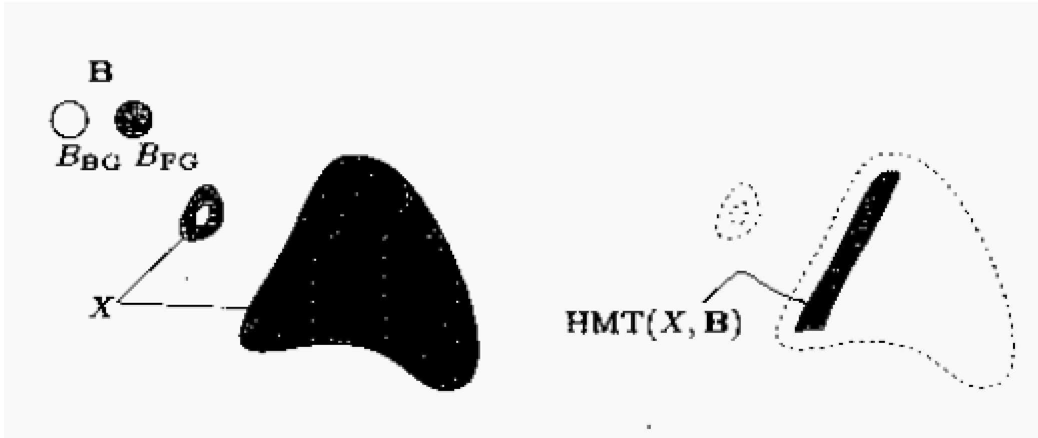


Fig. 4.18. Use of top-hat for mitigating inhomogeneous illumination. The performance of this technique is illustrated by the thresholds on the original and top-hat images.

Hit-or-Miss



$$HMT_B(X) = \{x \mid (B_{FG})_x \subseteq X, (B_{BG})_x \subseteq X^c\}$$

$$HMT_B(X) = \varepsilon_{B_{FG}}(X) \bigcap \varepsilon_{B_{BG}}(X^c)$$

- Property: $$HMT_B(X) = HMT_{B^c}(X^c)$$

where, $$B = (B_1, B_2)$$
$$B^c = (B_2, B_1)$$

Thinning and Thickening

$$THIN_B(f) = f - HMT_B(f)$$

$$THICK_B(f) = f + HMT_B(f)$$

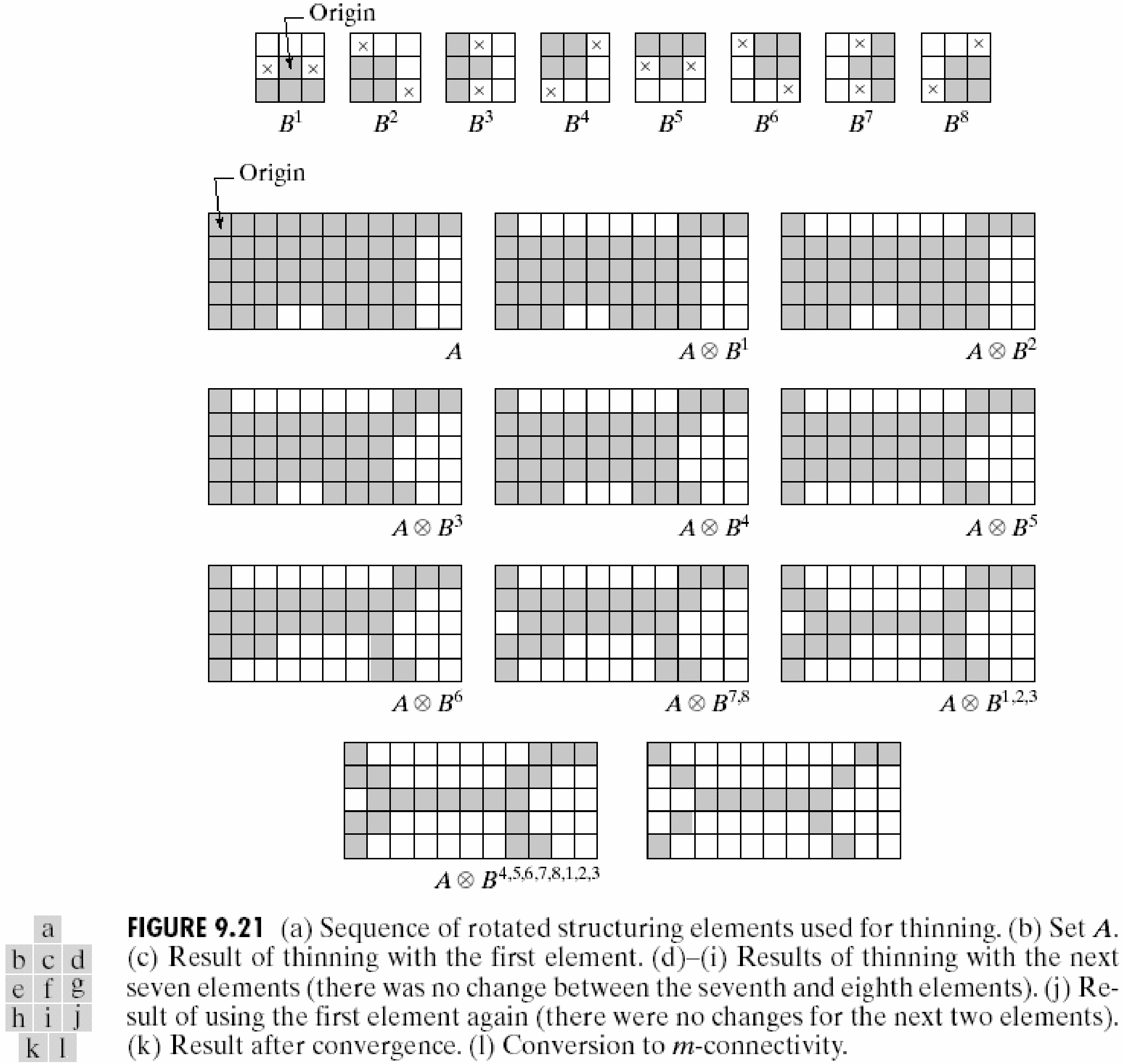
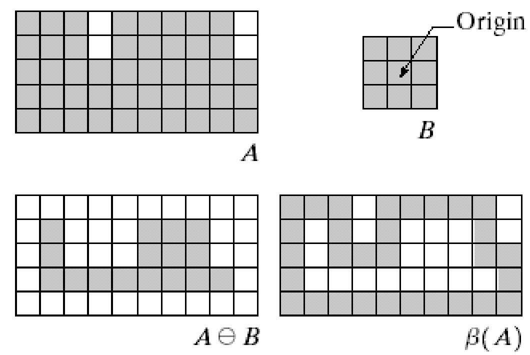


FIGURE 9.21 (a) Sequence of rotated structuring elements used for thinning. (b) Set A. (c) Result of thinning with the first element. (d)–(i) Results of thinning with the next seven elements (there was no change between the seventh and eighth elements). (j) Result of using the first element again (there were no changes for the next two elements). (k) Result after convergence. (l) Conversion to m-connectivity.

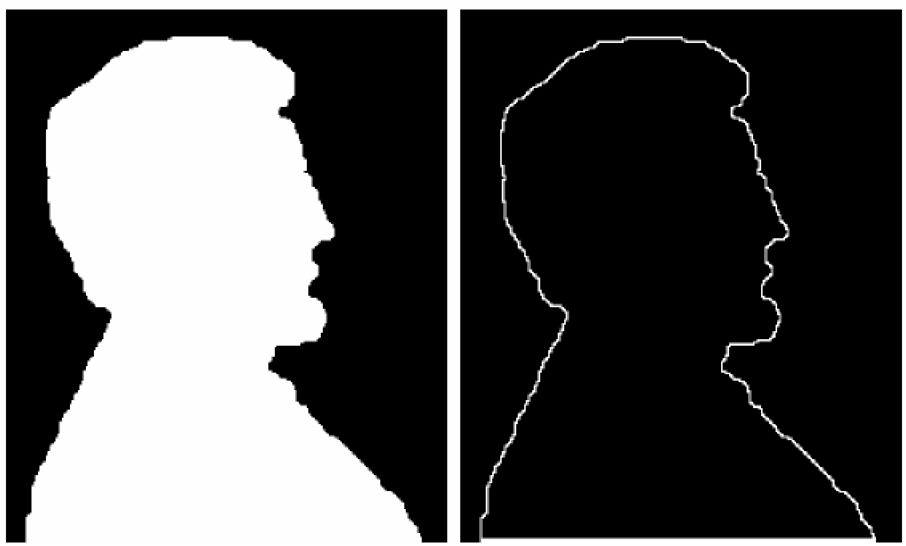
Example Applications: Boundary Extraction

$$\beta(f) = f - \varepsilon_B(f)$$



a b
c d

FIGURE 9.13 (a) Set A. (b) Structuring element B. (c) A eroded by B. (d) Boundary, given by the set difference between A and its erosion.



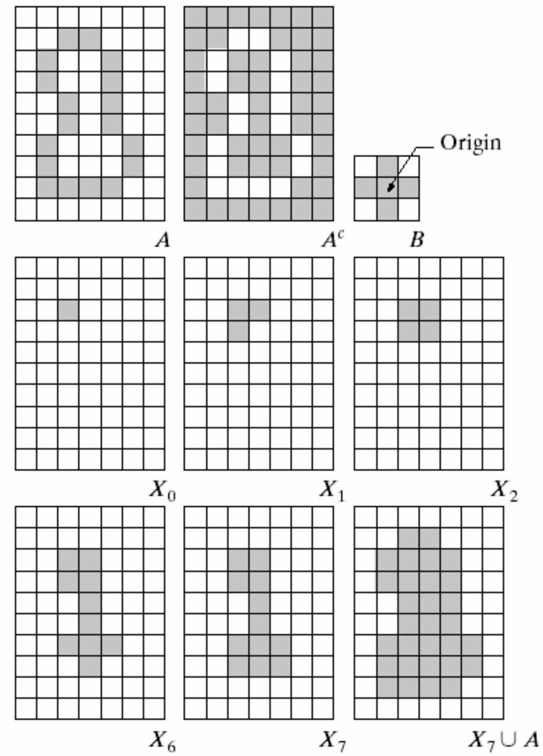
a b

FIGURE 9.14 (a) A simple binary image, with 1's represented in white. (b) Result of using Eq. (9.5-1) with the structuring element in Fig. 9.13(b).

Example Applications: Region Filling

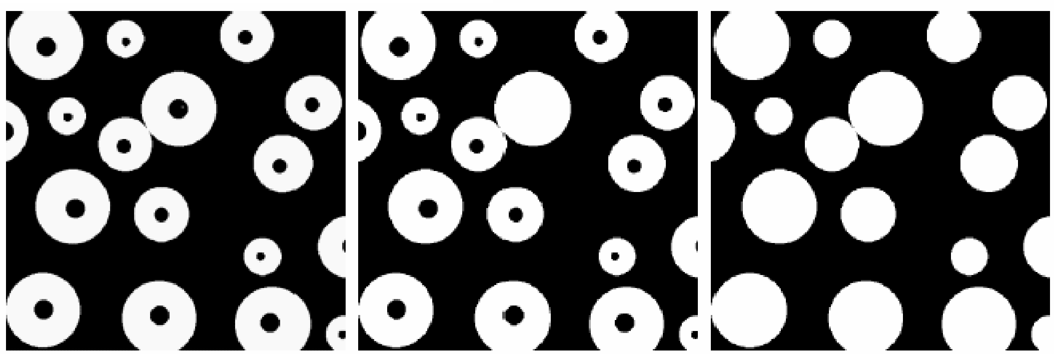
$$X_k = \delta_B(X_{k-1}) \cap A^c \qquad k = 1,2,3,\cdots$$

- start with $X_0=p$
- stop when $X_k=X_{k-1}$



a b c
d e f
g h i

FIGURE 9.15 Region filling. (a) Set *A*. (b) Complement of *A*. (c) Structuring element *B*. (d) Initial point inside the boundary. (e)–(h) Various steps of Eq. (9.5-2). (i) Final result [union of (a) and (h)].



a b c

FIGURE 9.16 (a) Binary image (the white dot inside one of the regions is the starting point for the region-filling algorithm). (b) Result of filling that region (c) Result of filling all regions.

Example Applications: Connected component extraction

$$X_k = \delta_B(X_{k-1}) \cap A \qquad k = 1,2,3,\cdots$$

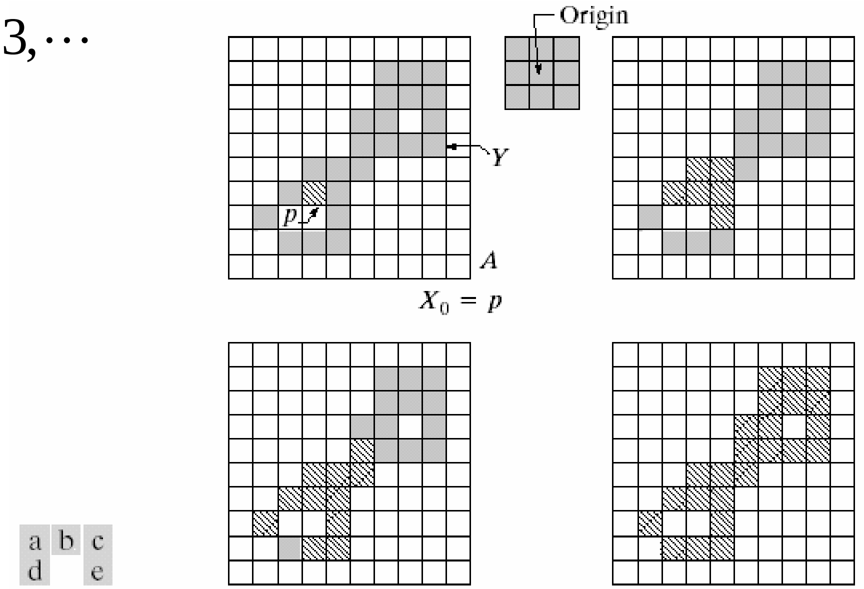


FIGURE 9.17 (a) Set A showing initial point p (all shaded points are valued 1, but are shown different from p to indicate that they have not yet been found by the algorithm). (b) Structuring element. (c) Result of first iterative step. (d) Result of second step. (e) Final result.

Example Applications: Convex Hull

$$X_k^i = HMT_{B^i}\left(X_{k-1}\right) \cup A \quad i = 1,2,3,4 \quad k = 1,2,3,\cdots$$

$$X_0^i = A$$

$$D^i = X_k^i$$

$$C(A) = \bigcup_{i=1}^{4} D^i$$

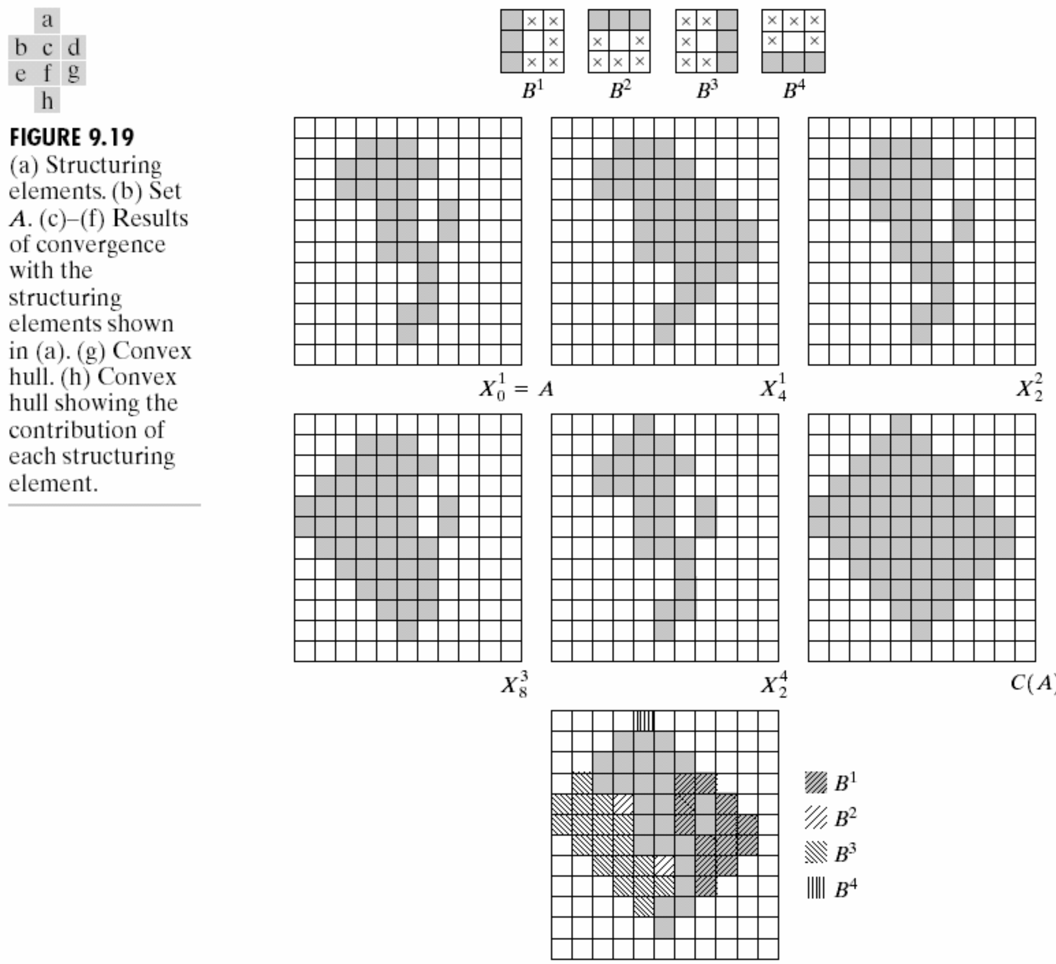


FIGURE 9.19 (a) Structuring elements. (b) Set A. (c)–(f) Results of convergence with the structuring elements shown in (a). (g) Convex hull. (h) Convex hull showing the contribution of each structuring element.

Skeletonization

$$S(A)=\bigcup_{k=0}^{K} S_k(A)$$

$$S_k(A)=\varepsilon_{kB}(A)-\phi_B\left(\varepsilon_{kB}(A)\right)$$

$$K=\max\left\{k \mid \varepsilon_{kB}(A)\neq null\right\}$$

$$A=\bigcup_{k=0}^{K}(\delta_{kB}(S_k(A)))$$

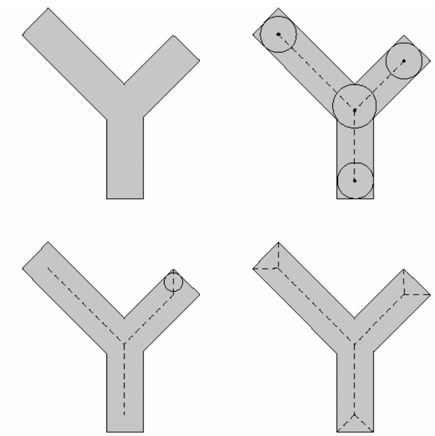


FIGURE 9.23 (a) Set A. (b) Various positions of maximum disks with centers on the skeleton of A. (c) Another maximum disk on a different segment of the skeleton of A. (d) Complete skeleton.

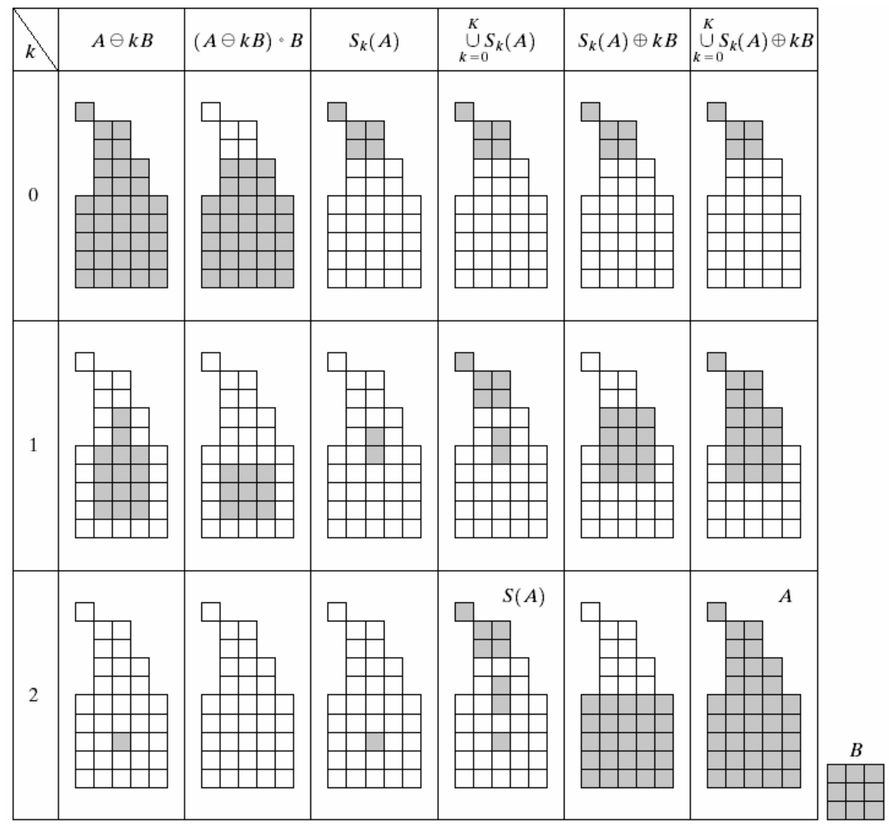


FIGURE 9.24 Implementation of Eqs. (9.5-11) through (9.5-15). The original set is at the top left, and its morphological skeleton is at the bottom of the fourth column. The reconstructed set is at the bottom of the sixth column.

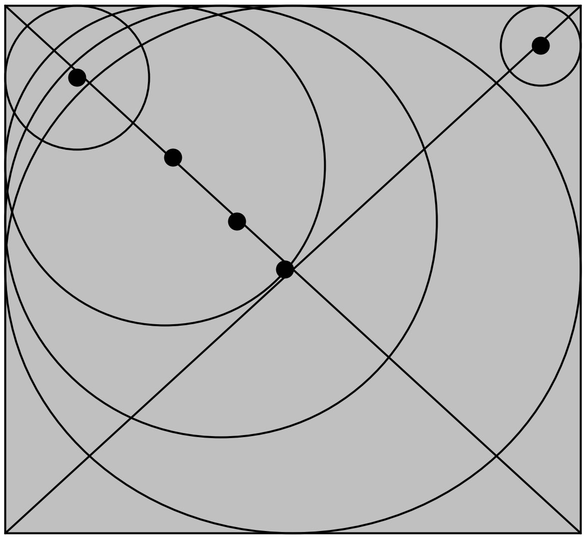
Skeletonization (Medial Axis Transform)

B is a "Maximal Disc" in set X if there are no other discs included in X and containing B

Skeleton is the loci of the centers of all "maximal discs"

$$S(X) = \bigcup_{k \geq 0} \{\varepsilon_{kB}(X) \setminus \gamma_B[\varepsilon_{kB}(X)]\}$$

Notion of "Maximal Disc"



Skeletonization

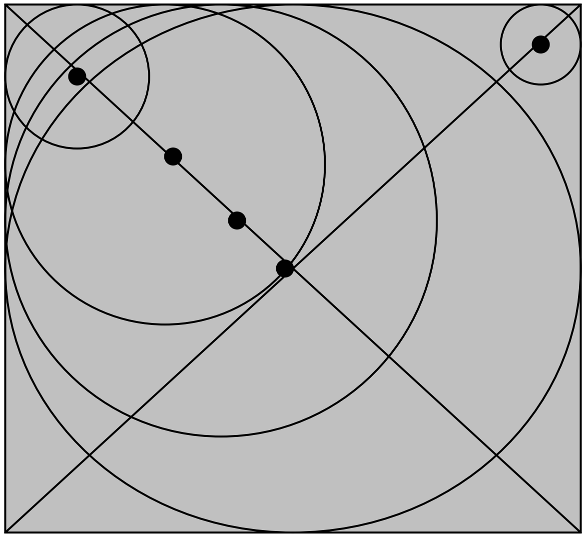
$$S(X) = \bigcup_{k=0}^{K} S_k(X)$$

$$S_k(X) = \varepsilon_{kB}(X) - \gamma_B(\varepsilon_{kB}(X))$$

$$\varepsilon_{kB}(X) = \varepsilon_B(\varepsilon_B(\cdots(\varepsilon_B(X))))$$

$$K = \max\{k \mid \varepsilon_{kB}(X) \neq \phi\}$$

Notion of "Maximal Disc"



Skeleton is the loci of the centers of all "maximal discs"

Reconstruction

$$X = \bigcup_{k=0}^{K} \delta_{kB}(S_k(X))$$

$$\delta_{kB}(X) = \delta_B(\delta_B(\cdots(\delta_B(S_k(X)))))$$

Matlab examples - dilation

```
originalBW = imread('text.png');
se = strel('line',11,90);
dilatedBW = imdilate(originalBW,se);
figure, imshow(originalBW), figure, imshow(dilatedBW)


originalI = imread('cameraman.tif');
se = strel('ball',5,5);
dilatedI = imdilate(originalI,se);
figure, imshow(originalI), figure, imshow(dilatedI)

se1 = strel('line',3,0);
se2 = strel('line',3,90);
composition = imdilate(1,[se1 se2],'full')
```

Matlab examples - erosion

```
originalBW = imread('text.png');
se = strel('line',11,90);
erodedBW = imerode(originalBW,se);
figure, imshow(originalBW)
figure, imshow(erodedBW)

originalI = imread('cameraman.tif');
se = strel('ball',5,5);
erodedI = imerode(originalI,se);
figure, imshow(originalI), figure, imshow(erodedI)
```

Matlab examples - closing

```
originalBW = imread('circles.png');
 figure, imshow(originalBW);
 se = strel('disk',10);
 closeBW = imclose(originalBW,se);
 figure, imshow(closeBW);
```

Matlab examples - opening

```
original = imread('snowflakes.png');
 se = strel('disk',5);
 afterOpening = imopen(original,se);
 figure, imshow(original), figure, imshow(afterOpening)
```

Matlab examples - HMT

```
bw=[0 0 0 0 0 0
    0 0 1 1 0 0
    0 1 1 1 1 0
    0 1 1 1 1 0
    0 0 1 1 0 0
    0 0 1 0 0 0]

interval = [0 -1 -1
            1  1 -1
            0  1  0];

bw2 = bwhitmiss(bw,interval)
```