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ORIGINAL ARTICLE

A LMI-based approach to global asymptotic stability of neural networks with time varying delays

Wenwu Yu

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Abstract In this paper, the asymptotic stability of neural networks with time varying delay is studied by using the nonsmooth analysis, Lyapunov functional method and linear matrix inequality (LMI) technique. It is noted that the proposed results do not require smoothness of the behaved function and activation function as well as boundedness of the activation function. Several sufficient conditions are presented to show the uniqueness and the global asymptotical stability of the equilibrium point. Also, a high-dimensional matrix condition to ensure the uniqueness and the global asymptotical stability of equilibrium point can be reduced to a low-dimensional condition. The obtained results are easy to apply and improve some earlier works. Finally, we give two simulations to justify the theoretical analysis in this paper.

Keywords Global asymptotical stability · Time varying delay · LMI technique · Lyapunov functional method · Nonsmooth analysis

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W. Yu (✉)
Department of Mathematics, Southeast University, Nanjing 210096, China
e-mail: yuwenwu2000@163.com

1 Introduction

The dynamical behaviors including stability [1–5, 10–21, 25–29], periodic bifurcation and chaos [6–9, 30–32] of neural networks have become a focal topic. Many researchers have made a lot of contributions to these subjects. Stability is a basic knowledge for dynamical systems and is useful in the application of the real systems.

Recently, there has been extensive interest in studying the effect of time delay on the collective dynamics of coupled models. It is well known that time delay is ubiquitous in most physical, chemical, biological, neural, and other natural system due to finite propagation speeds of signals, finite processing times in synapses, and finite reaction times.

It is well known that neural networks are complex and large-scale nonlinear systems, neural networks under study today have been dramatically simplified [6–9]. These investigations of simplified models are still very useful, since the dynamical characteristics found in simple models can be carried over to large-scale networks in some way. So in order to know much better of large-scale networks, we should study the simplified networks first. But there are inevitably some problems since simple models are carried over to large-scale networks, such as the complexity of the characteristic equation and the bifurcating periodic solutions. We must try to find the essence of dynamical systems.

In this paper, we will study the uniqueness and the global asymptotical stability of the equilibrium point.

The results we obtained improve and extend the earlier works.

The organization of this paper is as follows: In Section 2, model formulation and preliminaries are given for our main results. In Section 3, our main results are established. Several sufficient conditions are presented for the uniqueness and the global asymptotical stability of neural networks. We also give some remarks to show that our results improve and extend some earlier works. In Section 4, numerical simulations aimed at justifying the theoretical analysis will be reported. In Section 5, we give the conclusions.

2 Model formulation and preliminaries

In this paper, we consider the following delayed recurrent neural network model:

$$\dot{x}(t) = -b(x(t)) + Af(x(t)) + Bf(x(t-\tau(t))) + u, \tag{1}$$

or

$$\dot{x}_i(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + u_i, + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau(t))) \quad i = 1, 2, \ldots, n \tag{2}$$

where n denotes the number of units in a neural network, $\tau(t)$ is the time varying delay, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in R^n$ is the state vector associated with the neurons, $u = (u_1, u_2, \ldots, u_n)^T \in R^n$ is a constant external input vector, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^T \in R^n$ corresponds to the activation functions of neurons, $f(x(t-\tau(t))) = (f_1(x_1(t-\tau(t))), f_2(x_2(t-\tau(t))), \ldots, f_n(x_n(t-\tau(t))))^T \in R^n$, $b(x(t)) = (b_1(x_1(t)), b_2(x_2(t)), \ldots, b_n(x_n(t)))^T \in R^n$ are the behaved functions, $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively. Let I denote the identity matrix.

In this paper, the behaved functions and activation functions are assumed to satisfy the following properties:

A_1: The activation functions $f_i(x_i)(i = 1, 2, \ldots, n)$ are monotonically nondecreasing on R; $f_i(x_i)$ $(i = 1, 2, \ldots, n)$ are Lipschitz continuous, that is, there exist constants $\mu_i > 0$ such that

$$|f_i(\alpha_1) - f_i(\alpha_2)| \le \mu_i |\alpha_1 - \alpha_2|, \quad \forall \alpha_1, \alpha_2 \in R. \tag{3}$$

A_2: Each function $b_i \colon R \to R$ is locally Lipschitz and there exists $c_i > 0$ such that $b_i'(x) \ge c_i$ for all $x \in R$ at which $b_i(\cdot)$ is differentiable.

A_3: $\tau(t)$ is a bounded differential function of time t, and the following condition is satisfied:

$$0 \le \dot{\tau}(t) \le h < 1,$$

where h is a positive constant.

Next, we give the definition of the Generalized Jacobian which are essential for conducting nonsmooth analysis on Lipschitz continuous functions as stated in [26, 27]. Let the function $F : R^n \to R^n$ be locally Lipschitz continuous. According to Rademacher's theorem [24, Th.9.60], F is differentiable almost everywhere. Let D_F denote the set of those points where F is differentiable and $F'(x)$ denote the Jacobian of F at $x \in D_F$. Then, the set D_F is dense in R^n. For any given $x \in R^n$ define

$$Lip_x F := \sup_{y \to x, x \ne y \in R^n} \frac{\|F(y) - F(x)\|}{\|y - x\|}.$$

Since F is locally Lipschitz continuous, the constant Lip_x is finite and we have $\|F'(x)\| \le Lip_x F$ for any $x \in D_F$. Now we are ready to define the generalized Jacobian in the sense of Clarke [24]:

Definition 1. For any $x \in R^n$, let $\partial F(x)$ be the set of the following collection of matrices

$$\partial F(x) = co\{W \mid \text{there exists a sequence of } \{x^k\} \subset D_F \text{ with } \lim_{x^k \to x} F'(x^k) = W\},$$

where $co\Omega$ denote the convex hull of the set Ω. We called $\partial F(x)$ as the generalized Jacobian.

It is easy to see that the above definition is well defined and $\|W\| \le Lip_x F$ for any $W \in \partial F(x)$. We say that $\partial F(x)$ is invertible if every element W in $\partial F(x)$

is nonsingular. The generalized Jacobian $\partial F(x)$ have many nice properties, but only a few of them need to be singled out for our purpose. For one thing, the collection $\partial F(x)$ reduces to a singleton $\{F'(x)\}$ whenever F is continuously differentiable at x. We stress that $\partial F(x)$ may contain other elements if F is only differentiable at x.

Lemma 1 (Lebourg Theorem). *[24, p. 41]: For any given $x, y \in R^n$, there exists an element W in the union $\bigcup_{z\in[x,y]} \partial F(z)$ such that*

$$F(y) - F(x) = W(y - x), \tag{4}$$

where $[x, y]$ denotes the segment connecting x and y.

For more discussion on the generalized Jacobian and its various applications, we refer to books [23, 24]. Now, we analyze (1) from the viewpoint of nonsmooth analysis.

It is known that the bounded activation functions always guarantee the existence of an equilibrium point for system (1).

We assume that the model (1) has an equilibrium $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ for a given u. To simplify the proofs, we will shift the equilibrium point x^* of (1) to the origin. Using the following transformation

$$y(t) = x(t) - x^*, \quad y(t - \tau(t)) = x(t - \tau(t)) - x^*.$$

The model (1) can be transformed into the following form:

$$\dot{y}(t) = -L(y(t)) + Ag(y(t)) + Bg(y(t - \tau(t))), \tag{5}$$

namely,

$$\dot{y}_i(t) = -l_i(y_i(t)) + \sum_{j=1}^{n} a_{ij} g_j(y_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(y_j(t - \tau(t))), \quad i = 1, 2, \ldots, n \tag{6}$$

where $g(y(t)) = (g_1(y_1(t)), g_2(y_2(t)), \ldots, g_n(y_n(t)))^T \in R^n$, $g_i(y_i(t)) = f_i(y_i(t) + x_i^*) - f_i(x_i^*)$, $g(0) = 0$, $L(y(t)) = (l_1(y_1(t)), l_2(y_2(t)), \ldots, l_n(y_n(t)))^T \in R^n$, $l_i(y_i(t)) = b_i(y_i(t) + x_i^*) - b_i(x_i^*)$, $b(0) = 0$. Moreover, from (3), we know that

$$|g_i(y_i)| \leq \mu_i |y_i| \quad \forall y_i \in R, \quad i = 1, 2, \ldots, n \tag{7}$$

To obtain our main results, we need the following two elementary lemmas:

Lemma 2. *For any vectors $x, y \in R^n$ and positive definite matrix $G \in R^{n\times n}$, the following matrix inequality holds:*

$$2x^T y \leq x^T G x + y^T G^{-1} y.$$

Lemma 3 (Schur complement [22]). *The following linear matrix inequality (LMI)*

$$\begin{pmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{pmatrix} > 0,$$

where $Q(x) = Q(x)^T$, $R(x) = R(x)^T$, is equivalent to one of the following conditions:

(i) $Q(x) > 0$, $R(x) - S(x)^T Q(x)^{-1} S(x) > 0$,

(ii) $R(x) > 0$, $Q(x) - S(x) R(x)^{-1} S(x)^T > 0$.

3 Global asymptotical stability criteria

In this section, new criteria are presented for the global asymptotical stability of the equilibrium point of the neural network defined by (5). Its proof is based on a new Lyapunov functional method, nonsmooth analysis and linear matrix inequality (LMI) approach.

Theorem 1. *Under the assumptions A_1 and A_2, the origin is the unique equilibrium of (5) if there are positive definite diagonal matrix $D = diag(d_1, d_2, \ldots, d_n) > 0$ and positive definite matrix $H = (h_{ij})_{n\times n}$, such that any one of the following conditions hold:*

(i) $$N = \begin{pmatrix} 2DC\Lambda^{-1} - DA - A^T D - H & -DB \\ -B^T D & H \end{pmatrix} > 0, \tag{8}$$

or equivalently,

$$\text{(ii)}\ \ 2DC\Lambda^{-1} - DA - A^T D - H \\ -DBH^{-1}B^T D > 0, \tag{9}$$

where $\Lambda = diag(\mu_1, \mu_2, \ldots, \mu_n)$ *and* $C = diag(c_1, c_2, \ldots, c_n)$.

Proof: By the LMI approach from Lemma 2, we can easily prove that the condition (i) is equivalent to the condition (ii) Here we take $Q(x) = 2DC\Lambda^{-1} - DA - A^T D - H$, $S(x) = -DB$ and $R(x) = H$. In the following, we shall prove that this theorem is true if (ii) is satisfied. □

We will prove the uniqueness of the equilibrium point by using the method of contradiction. Consider the equilibrium equation of (5)

$$L(y^*) - Ag(y^*) - Bg(y^*) = 0. \tag{10}$$

It is evident to see that if $g(y^*) = 0$, then $y^* = 0$. Now let $g(y^*) \neq 0$. Multiplying both sides of (10) by $2g^T(y^*)D$, yields

$$2g^T(y^*)DL(y^*) - 2g^T(y^*)DAg(y^*) \\ -2g^T(y^*)DBg(y^*) = 0, \tag{11}$$

which can be rewritten as

$$2g^T(y^*)DL(y^*) - g^T(y^*)DAg(y^*) \\ - g^T(y^*)A^T Dg(y^*) \\ - 2g^T(y^*)DBg(y^*) = 0. \tag{12}$$

From Lemma 1, we have that

$$L(y^*) = b(y^* + x^*) - b(x^*) \\ = My^*,\ M \in \bigcup_{y\in[x^*, y^*+x^*]} \partial b(y).$$

Form the definition of b, matrix M is diagonal, and we denote $M = \text{diag}(m_1, m_2, \ldots, m_n)$.

From (7) and the assumptions A_1 and A_2, we get

$$y_i g_i(y_i) \geq 0\ \forall y_i \in R, \quad i = 1, 2, \ldots, n \tag{13}$$

and

$$g_i^2(y_i) \leq \mu_i y_i g_i(y_i) \forall y_i \in R, \quad i = 1, 2, \ldots, n \tag{14}$$

According to (13) and (14), we can obtain the following inequality

$$g^T(y^*)DL(y^*) = \sum_{i=1}^{n} g_i(y_i^*)d_i m_i y_i^* \\ \geq \sum_{i=1}^{n} \frac{d_i c_i}{\mu_i} g_i^2(y_i^*) \\ = g^T(y^*)DC\Lambda^{-1}g(y^*), \tag{15}$$

in which $\Lambda = \text{diag}(\mu_1, \mu_2, \ldots, \mu_n)$ and $C = \text{diag}(c_1, c_2, \ldots, c_n)$.

According to Lemma 1, we have

$$2g^T(y^*)DBg(y^*) \leq g^T(y^*)Hg(y^*) \\ + g^T(y^*)DBH^{-1}B^T Dg(y^*). \tag{16}$$

Substituting (15) and (16) into (12), we obtain

$$g^T(y^*)[2DC\Lambda^{-1} - DA - A^T D - H \\ - DBH^{-1}B^T D]g(y^*) \leq 0, \tag{17}$$

obviously, (17) contradicts with the condition (ii) which in turn implies that at the equilibrium point $g(y^*) = 0$, as well as $y^* = 0$. Thus, we proved that the origin of model (5) is a unique equilibrium point.

First, we consider the constant time delay where $\tau(t) = \tau$.

Theorem 2. *Under the assumptions* A_1 *and* A_2, *the equilibrium point of model (5) is globally asymptotically stable if there are positive definite diagonal matrix* $D = diag(d_1, d_2, \ldots, d_n) > 0$, *positive definite matrix* $H = (h_{ij})_{n\times n}$, $P = (p_{ij})_{n\times n}$ *and positive constant* $\alpha > 0$, *such that*

$$M = \begin{pmatrix} 2PC & -PA & -PB \\ -A^T P & \alpha(2DC\Lambda^{-1} - DA - A^T D - H) & -\alpha DB \\ -B^T P & -\alpha B^T D & \alpha H \end{pmatrix} > 0. \tag{18}$$

Proof: We now prove that the condition given in (18) also imply the global stability of the origin of (5). Consider the Lyapunov functional

$$V(y(t)) = y^T P y + 2\alpha \sum_{i=1}^{n} d_i \int_0^{y_i(t)} g_i(s)\,ds + \alpha \int_{t-\tau}^{t} g^T(y(s)) H g(y(s))\,ds, \qquad (19)$$

where $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is positive definite diagonal matrix, $H = (h_{ij})_{n\times n}$ and $P = (p_{ij})_{n\times n}$ are positive definite matrices.

Taking the derivative of $V(y)$ along the trajectories of (5), we obtain

$$\begin{aligned}
\dot V(y(t))|_{(5)} &= 2y^T P\dot y + 2\alpha g^T(y(t)) D\dot y(t) \\
&\quad + \alpha g^T(y(t)) H g(y(t)) \\
&\quad - \alpha g^T(y(t-\tau)) H g(y(t-\tau)) \\
&= -2y^T(t) P L(y(t)) + 2y^T(t) P A g(y(t)) \\
&\quad + 2y^T P B g(y(t-\tau)) \\
&\quad - 2\alpha g^T(y(t)) D L(y(t)) \\
&\quad + 2\alpha g^T(y(t)) D A g(y(t)) \\
&\quad + 2\alpha g^T(y(t)) D B g(y(t-\tau)) \\
&\quad + \alpha g^T(y(t)) H g(y(t)) \\
&\quad - \alpha g^T(y(t-\tau)) H g(y(t-\tau)) \\
&= -(y^T(t)\ g^T(y(t)\ g^T(y(t-\tau))) \\
&\quad \times M \begin{pmatrix} y(t) \\ g(y(t) \\ g(y(t-\tau)) \end{pmatrix} \\
&\quad -2y^T(t) P[L(y(t)) - C y(t)] \\
&\quad -2\alpha g^T(y(t)) D[L(y(t)) - C\Lambda^{-1} g(y(t))].
\end{aligned} \qquad (20)$$

From Lemma 1, we have

$$\begin{aligned}
L(y(t)) &= b(y(t)+x^*) - b(x^*) \\
&= \widehat{M} y(t),\ \widehat{M} \in \bigcup_{z\in[x^*, y+x^*]} \partial b(z),
\end{aligned} \qquad (21)$$

where $\widehat{M} = \mathrm{diag}(\widehat{m}_1, \widehat{m}_2, \ldots, \widehat{m}_n)$. It is obvious to see $\widehat{m}_i \geq l_i$ for $i = 1, 2, \ldots, n$. Together with (13) and (14), we obtain

$$\begin{aligned}
y^T(t) P L(y(t)) &= \sum_{i=1}^{n} y_i(t) p_i \widehat{m}_i y_i(t) \\
&\geq \sum_{i=1}^{n} p_i c_i y_i^2(t) \\
&= y^T(t) P C y(t).
\end{aligned} \qquad (22)$$

$$\begin{aligned}
g^T(y(t)) D L(y(t)) &= \sum_{i=1}^{n} g_i(y_i(t)) d_i \widehat{m}_i y_i(t) \\
&\geq \sum_{i=1}^{n} \frac{d_i c_i}{\mu_i} g_i^2(y_i(t)) \\
&= g^T(y(t)) D C \Lambda^{-1} g(y(t)).
\end{aligned} \qquad (23)$$

Therefore, From (20), we know that under the given condition (18), (22) and (23), $\dot V(y(t)) = 0$ if and only if $y(t) = g(y(t)) = g(y(t-\tau)) = 0$, otherwise $\dot V(y(t)) \leq 0$. Moreover, on the other hand, $V(y)$ is radially unbounded since $V(y(t)) \to \infty$ as $\|y(t)\| \to \infty$. We have proved that the equilibrium of (5) is globally asymptotically stable. This completes the proof. □

Theorem 3. *Under the assumptions A_1 and A_2, the origin is the unique equilibrium of (5) and it is globally asymptotically stable if there are positive definite diagonal matrix $D = diag(d_1, d_2, \ldots, d_n) > 0$ and positive definite matrix $H = (h_{ij})_{n\times n}$, such that any one of the condition of Theorem 1 is satisfied.*

Proof: Obviously, according to Theorem 1, we know that origin is the unique equilibrium of (5). Next, we prove that under the condition of Theorem 1, the condition in Theorem 2 is satisfied.

Let $W = [-PA\ \ -PB]$, $N = \begin{pmatrix} 2DC\Lambda^{-1} - DA - A^T D - H & -DB \\ -B^T D & H \end{pmatrix}$, and (18) is equivalent to

$$M = \begin{pmatrix} 2PC & W \\ W^T & \alpha N \end{pmatrix} > 0. \qquad (24)$$

According to Lemma 2, we have

$$PC > 0, \qquad \alpha N - \frac{1}{2} W^T (PC)^{-1} W > 0.$$

From the condition given in Theorem 1, we know $N > 0$, we choose α large enough, we can see that (18) is satisfied. For example, we choose $\alpha > \frac{\gamma_1}{\gamma_2}$, where γ_1 denotes the maximum eigenvalue of $\frac{1}{2} W^T (PC)^{-1} W$ and γ_2 denotes the minimum eigenvalue of N. This completes the proof. □

Corollary 1. *Under the assumptions A_1 and A_2, the origin is the unique equilibrium of (5) and it is globally asymptotically stable if there is a positive definite matrix $H = (h_{ij})_{n\times n}$, such that any one of the following conditions hold:*

$$(i) \quad \begin{pmatrix} 2C\Lambda^{-1} - A - A^T - H & -B \\ -B^T & H \end{pmatrix} > 0,$$

or equivalently,

$$(ii) \quad 2C\Lambda^{-1} - A - A^T - H - BH^{-1}B^T > 0,$$

where $\Lambda = diag(\mu_1, \mu_2, \ldots, \mu_n)$.

Proof: In Theorem 3, let $D = I$, its proof is obvious and here we omitted. □

Corollary 2. *Under the assumptions A_1 and A_2, the origin is the unique equilibrium of (5) and it is globally asymptotically stable if any one of the following conditions hold:*

$$\text{(i)} \quad \begin{pmatrix} 2C\Lambda^{-1} - A - A^T - I & -B \\ -B^T & I \end{pmatrix} > 0,$$

or equivalently,

$$\text{(ii)} \quad 2C\Lambda^{-1} - A - A^T - I - BB^T > 0,$$

where $\Lambda = diag(\mu_1, \mu_2, \ldots, \mu_n)$.

Proof: In Theorem 3, let $D = H = I$, Corollary 2 is a direct result of Theorem 3, here we omitted. □

Corollary 3. *Under the assumptions A_1 and A_2, the equilibrium point model (5) is globally asymptotically stable if there are positive definite matrix $H = (h_{ij})_{n\times n}$, $P = (p_{ij})_{n\times n}$ and positive constant $\alpha > 0$, such that*

$$\begin{pmatrix} 2PC & -PA & -PB \\ -A^TP & \alpha(2C\Lambda^{-1} - A - A^T - H) & -\alpha B \\ -B^TP & -\alpha B^T & \alpha H \end{pmatrix} > 0.$$

Proof: If we take $D = I$, Corollary 3 is a direct result of Theorem 2, here we omitted. □

Corollary 4. *Under the assumptions A_1 and A_2, the equilibrium point model (5) is globally asymptotically stable if there is a positive definite matrix $P = (p_{ij})_{n\times n}$, such that*

$$\begin{pmatrix} 2PC & -PA & -PB \\ -A^TP & (2C\Lambda^{-1} - A - A^T - I) & -B \\ -B^TP & -B^T & I \end{pmatrix} > 0.$$

Proof: If we take $D = H = I$ and $\alpha = 1$, Corollary 3 is a direct result of Theorem 2, here we omitted. □

Next, we consider time varying delay $\tau(t)$.

Theorem 4. *Under the assumptions $A_1 - A_3$, the equilibrium point of model (5) is globally asymptotically stable if there are positive definite diagonal matrix $D = diag(d_1, d_2, \ldots, d_n) > 0$, positive definite matrix $H = (h_{ij})_{n\times n}$, $P = (p_{ij})_{n\times n}$ and positive constant $\alpha > 0$, such that*

$$M_1 = \begin{pmatrix} 2PC & -PA & -PB \\ -A^TP & \alpha(2DC\Lambda^{-1} - DA - A^TD - H) & -\alpha DB \\ -B^TP & -\alpha B^TD & \alpha(1-h)H \end{pmatrix} > 0. \tag{25}$$

Proof: Consider the same Lyapunov functional the same as in (19):

$$V(y(t)) = y^TPy + 2\alpha\sum_{i=1}^{n} d_i \int_0^{y_i(t)} g_i(s)ds + \alpha\int_{t-\tau(t)}^{t} g^T(y(s))Hg(y(s))ds, \tag{26}$$

where $D = \text{diag}(d_1, d_2, \ldots, d_n)$ is positive definite diagonal matrix, $H = (h_{ij})_{n\times n}$ and $P = (p_{ij})_{n\times n}$ are positive definite matrices.

Taking the derivative of $V(y)$ along the trajectories of (5), we obtain

$$
\begin{aligned}
\dot V(y(t))|_{(5)} &= 2y^T P\dot y + 2\alpha g^T(y(t))D\dot y(t) \\
&\quad + \alpha g^T(y(t))Hg(y(t)) \\
&\quad - \alpha g^T(y(t-\tau))Hg(y(t-\tau)) \\
&= -2y^T(t)PL(y(t)) + 2y^T(t)PAg(y(t)) \\
&\quad + 2y^T PBg(y(t-\tau)) \\
&\quad - 2\alpha g^T(y(t))DL(y(t)) \\
&\quad + 2\alpha g^T(y(t))DAg(y(t)) \\
&\quad + 2\alpha g^T(y(t))DBg(y(t-\tau)) \\
&\quad + \alpha g^T(y(t))Hg(y(t)) \\
&\quad - \alpha(1-\dot\tau(t))g^T(y(t-\tau))Hg(y(t-\tau)) \\
&= -\left(y^T(t)\ g^T(y(t)\ g^T(y(t-\tau))\right) \\
&\quad \times M_1 \begin{pmatrix} y(t) \\ g(y(t) \\ g(y(t-\tau)) \end{pmatrix} \\
&\quad - 2y^T(t)P[L(y(t)) - Cy(t)] \\
&\quad - 2\alpha g^T(y(t))D[L(y(t)) \\
&\quad - C\Lambda^{-1}g(y(t))] \\
&\quad - \alpha(h-\dot\tau(t))g^T(y(t-\tau))Hg(y(t-\tau)).
\end{aligned}
\tag{27}
$$

The same as in Theorem 2, we can obtain the global asymptotical stability of equilibrium point of (5). □

Theorem 5. *Under the assumptions A_1–A_3, the origin is the unique equilibrium of (5) and it is globally asymptotically stable if there are positive definite diagonal matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n) > 0$ and positive definite matrix $H = (h_{ij})_{n\times n}$, such that any one of the following conditions hold:*

(i)
$$
N = \begin{pmatrix} 2DC\Lambda^{-1} - DA - A^T D - H & -DB \\ -B^T D & (1-h)H \end{pmatrix} > 0,
\tag{28}
$$

or equivalently,

(ii)
$$
2DC\Lambda^{-1} - DA - A^T D - H - \frac{1}{1-h}DBH^{-1}B^T D > 0,
\tag{29}
$$

where $\Lambda = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_n)$ and $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$.

Proof: From Lemma 3, it is easy to see that (25) is equivalent to (28). Since

$$
\frac{1}{1-h}DBH^{-1}B^T D \ge DBH^{-1}B^T D,
$$

we obtain

$$
\begin{aligned}
&2DC\Lambda^{-1} - DA - A^T D - H - DBH^{-1}B^T D \\
&\qquad \ge 2DC\Lambda^{-1} - DA - A^T D - H \\
&\qquad - \frac{1}{1-h}DBH^{-1}B^T D > 0.
\end{aligned}
$$

The condition (9) in Theorem 1 is also satisfied, the proof is completed by the same step as in Theorem 3. □

Corollary 5. *Under the assumptions $A_1 - A_3$, the origin is the unique equilibrium of (5) and it is globally asymptotically stable if*

$$
2C\Lambda^{-1} - A - A^T - I - \frac{1}{1-h}BB^T > 0.
\tag{30}
$$

Proof: In Theorem 5, let $D = H = I$, Corollary 5 is a direct result of Theorem 5, here we omitted. □

Theorem 6. *Under the assumptions $A_1 - A_3$, the equilibrium point of model (5) is globally asymptotically stable if there are positive definite diagonal matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n) > 0$, positive definite matrix $H = (h_{ij})_{n\times n}$, $P = (p_{ij})_{n\times n}$ and positive constant $\alpha > 0$, such that the matrix condition (25) or (28) is satisfied. Also, the two conditions are equivalent to each other.*

Proof: If (28) is satisfied, it is easy to see the condition (25) is satisfied by Theorem 5. Otherwise, if (25) is satisfied, we choose $P = \varepsilon I$ where ε is a sufficient small positive constant. From Lemma 3, the condition (28) is also satisfied. □

Remark 1. In [1], the authors have studied the global asymptotical stability of equilibrium point with constant delay, they only obtained Theorem 3. In this paper, the behaved functions studied are nonsmooth and also we study the time varying delay.

Remark 2. In [16], the global asymptotic stability of delayed cellular neural networks was investigated. If we choose $\alpha = 1$, the theorem stated in [16] is a special case in this paper.

Remark 3. In [11], the global robust stability of delayed neural networks was considered, if we choose $D = H = I$, it is also a special case in this paper.

Remark 4. In [25], the global asymptotic stability of a larger class of neural networks with constant time delay has been studied. However, in this paper, we note that our results do not require smoothness of the behaved function and activation function as well as boundedness of the activation function. If we choose $b(x(t)) = Cx(t)$, we have the same results as in [25]. So it is a special case in my proposed paper.

Remark 5. In [28], the global asymptotical stability analysis was investigated by Cao and Ho. The main theorem (Theorem 1) proposed in [28] is the same as Theorem 4 in this paper. However, we show the high-dimensional matrix condition (25) in Theorem 4 is equivalent to the low-dimensional matrix condition (28) in Theorem 5, which is an interesting phenomena demonstrated in Theorem 6. The high-dimensional matrix condition (25) can be reduced to the low-dimensional matrix condition (28), which are surprisingly equivalent. Furthermore, the function $b(x(t))$ in (1) is nonsmooth, which is more general than the smooth function in [28]. We have used the nonsmooth analysis method in this paper.

4 Numerical example

In this section, we will give two examples to justify the theoretical analysis in this paper.

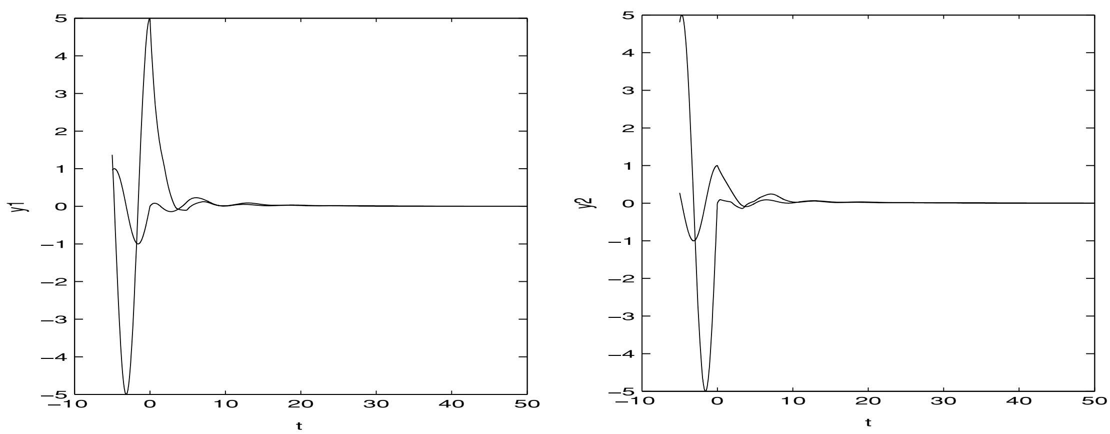

Fig. 1 Trajectories of state variables $x_1(t)$ and $x_2(t)$

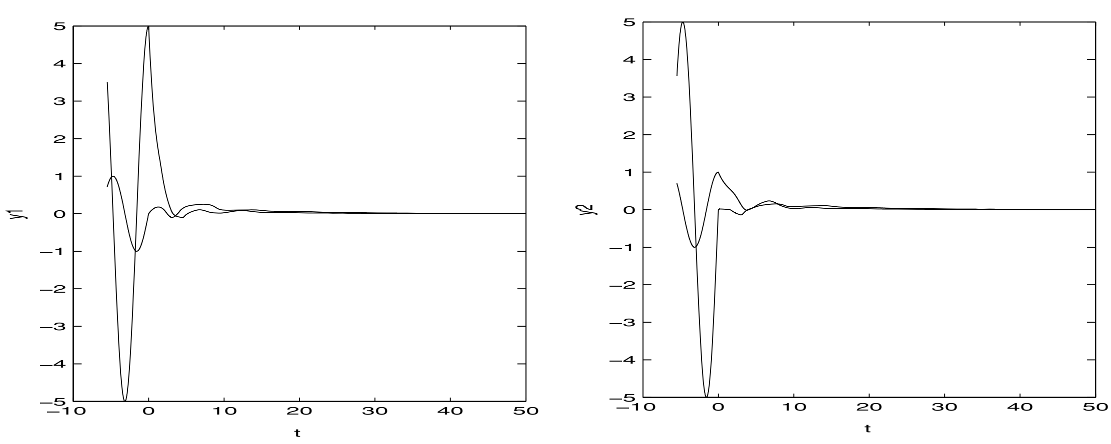

Fig. 2 Trajectories of state variables $x_1(t)$ and $x_2(t)$

Example 1. Consider a delayed neural network with the connection weight matrices

$$A = \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.3 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.1 & 0.3 \\ 0.2 & 0.1 \end{pmatrix},$$

where

$$b_i(u) = \begin{cases} u, & \text{if } u \geq 0, \\ 2u, & \text{if } u < 0, \end{cases}.$$

for $i = 1, 2$, $f_i(x) = (|x+1| - |x-1|)/2$ $(i = 1, 2)$, this implies that $\Lambda = C = I$. According to Corollary 2,

$$2C\Lambda^{-1} - A - A^T - I - BB^T = \begin{pmatrix} 0.5 & -0.25 \\ -0.25 & 0.35 \end{pmatrix} > 0,$$

the equilibrium of model (5) is globally asymptotically stable.

Next, we give a simulation to justify the theoretical analysis in the proposed paper, let $\tau = 5$, then we choose the initial functions

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$$

and

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 5\cos(t) \\ 5\sin(t) \end{pmatrix} (-\tau \leq t \leq 0),$$

respectively to show the global asymptotical stability of equilibrium point. From Fig. 1, we can know that the equilibrium of model (5) is globally asymptotically stable.

Example 2. Consider the same neural network model as above, but we choose $\tau = 5 + 0.5\sin(t)$. It is easy to see that $h = 0.5$. According to Corollary 5,

$$2C\Lambda^{-1} - A - A^T - I - \frac{1}{1-h}BB^T = \begin{pmatrix} 0.4 & -0.3 \\ -0.3 & 0.3 \end{pmatrix} > 0,$$

according to Corollary 5, the equilibrium of model (5) is globally asymptotically stable, which is illustrated in Fig. 2.

5 Conclusions

In this paper, we have studied global asymptotical stability of neural networks with time varying delays. We use nonsmooth analysis, Lyapunov functional method and LMI technique to solve this problem. Several sufficient conditions have been derived to ensure the existence of unique equilibrium point and global asymptotic stability for the delayed neural networks. Our results do not require smoothness of the behaved function and activation function as well as boundedness of the activation function. The obtained results improve and extend the earlier works. It is easy to apply these sufficient conditions to the real networks. Finally, we give two simulations to justify the obtained results.

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