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Stability and Hopf bifurcation analysis on a four-neuron BAM neural network with time delays ☆

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Abstract

In this Letter, a four-neuron BAM neural network with four time delays is considered, where the time delays are regarded as parameters. Its dynamics are studied in terms of local analysis and Hopf bifurcation analysis. By analyzing the associated characteristic equation, it is found that Hopf bifurcation occurs when these delays pass through a sequence of critical value. A formula for determining the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions is given by using the normal form method and center manifold theorem.

Keywords: BAM neural network; Time delay; Hopf bifurcation; Periodic solutions; Stability

1. Introduction

The dynamical characteristics (including stable, unstable, oscillatory, and chaotic behavior) of neural networks [1–14] have attracted attention of many researchers, and many results have been made to it. There has been increasing interest in investigating the dynamics of neural networks since Hopfield [15] constructed a simplified neural network model. Based on the Hopfield neural network model, Marcus and Westervelt [16] argued that time delays always occur in the signal transmission and proposed a neural network model with delay. Afterward, a variety of artificial models has been established to describe neural networks with delays [16–19].

Recently, there has been extensive interest in studying the effect of time delay on the collective dynamics of coupled models [1–3]. It is well known that time delay is ubiquitous in most physical, chemical, biological, neural, and other natural system due to finite propagation speeds of signals, finite processing times in synapses, and finite reaction times.

In [19–21], a class of two-layer associative networks, called bidirectional associative memory (BAM) neural networks with or without axonal signal transmission delays, has been proposed and applied in many fields such as pattern recognition and automatic control. The bidirectional associated memory neural networks with or without delays has been widely studied in [18,19,22–24]. However, most work focus on establishing the local and global stability. It is known to all that the stability property is only the dynamic behavior, and there are many other properties such as periodic oscillation, bifurcation, chaos and so on.

The delayed bidirectional associative memory neural network is described by the following system:

$$\begin{cases} \dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{m} c_{ij} f_i(y_j(t-\tau_{ij})), \\ \dot{y}_j(t) = -\nu_j y_j(t) + \sum_{i=1}^{n} d_{ji} g_j(x_i(t-\nu_{ij})), \end{cases} \tag{1.1}$$

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where c_{ij}, d_{ji} $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$ are the connection weights through neurons in two layers: the I-layer and J-layer; μ_i and ν_j describe the stability of internal neuron processes on the I-layer and J-layer, respectively. On the I-layer, the neurons whose states are denoted by $x_i(t)$ receive the inputs I_i and the inputs outputted by those neurons in the J layer via activation functions f_i, while on the J-layer, the neurons whose associated states denoted by $y_j(t)$ receive the inputs I_j and the inputs outputted by those neurons in the I-layer via activation functions g_j.

It is well known that neural networks are complex and large-scale non-linear systems, neural networks under study today have been dramatically simplified [1–8]. These investigation of simplified models are still very useful, since the dynamical characteristics found in simple models can be carried over to large-scale networks in some way. In order to know much better of large-scale networks, the simplified networks should be considered first. But there are inevitably some problems since simple models are carried over to large-scale networks, such as the complexity of the characteristic equation and the bifurcating periodic solutions. In this Letter, a four-neuron BAM neural network with four time delays has been considered, and it is a more general model.

The organization of this Letter is as follows: In Section 2, the stability of the trivial solutions and the existence of Hopf bifurcation is discussed. In Section 3, a formula for determining the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions will be given by using the normal form method and center manifold theorem introduced by Hassard at [9]. In Section 4, numerical simulations aimed at justifying the theoretical analysis will be reported.

2. Existence of Hopf bifurcation

The BAM neural networks with time delays considered in this Letter are described by the following differential equations with delay:

$$\begin{cases} \dot{x}_1(t) = -\mu_1 x_1(t) + c_{11} f_{11}(y_1(t-\tau_3)) + c_{12} f_{12}(y_2(t-\tau_3)), \\ \dot{x}_2(t) = -\mu_2 x_2(t) + c_{21} f_{21}(y_1(t-\tau_4)) + c_{22} f_{22}(y_2(t-\tau_4)), \\ \dot{y}_1(t) = -\mu_3 y_1(t) + d_{11} g_{11}(x_1(t-\tau_1)) + d_{12} g_{12}(x_2(t-\tau_2)), \\ \dot{y}_2(t) = -\mu_4 y_2(t) + d_{21} g_{21}(x_1(t-\tau_1)) + d_{22} g_{22}(x_2(t-\tau_2)). \end{cases} \tag{2.1}$$

To establish the main results for model (2.1), it is necessary to make the following assumptions:

(H$_1$) $f_{ij}, g_{ij} \in C^1$, $f_{ij}(0) = 0$, $g_{ij}(0) = 0$, for $i, j = 1, 2$;
(H$_2$) $\tau_1 + \tau_3 = \tau$, $\tau_2 + \tau_4 = \tau$.

Letting $u_1(t) = x_1(t-\tau_1)$, $u_2(t) = x_2(t-\tau_2)$, $u_3(t) = y_1(t)$, $u_4(t) = y_2(t)$, (2.1) can be written as the following equivalent system

$$\begin{cases} \dot{u}_1(t) = -\mu_1 u_1(t) + c_{11} f_{11}(u_3(t-\tau)) + c_{12} f_{12}(u_4(t-\tau)), \\ \dot{u}_2(t) = -\mu_2 u_2(t) + c_{21} f_{21}(u_3(t-\tau)) + c_{22} f_{22}(u_4(t-\tau)), \\ \dot{u}_3(t) = -\mu_3 u_3(t) + d_{11} g_{11}(u_1(t)) + d_{12} g_{12}(u_2(t)), \\ \dot{u}_4(t) = -\mu_4 u_4(t) + d_{21} g_{21}(u_1(t)) + d_{22} g_{22}(u_2(t)). \end{cases} \tag{2.2}$$

Under the hypothesis (H$_1$) and (H$_2$), the linear equation of (2.2) at $(0, 0, 0, 0)$ is as follows:

$$\begin{cases} \dot{u}_1(t) = -\mu_1 u_1(t) + \alpha_{11}(u_3(t-\tau)) + \alpha_{12}(u_4(t-\tau)), \\ \dot{u}_2(t) = -\mu_2 u_2(t) + \alpha_{21}(u_3(t-\tau)) + \alpha_{22}(u_4(t-\tau)), \\ \dot{u}_3(t) = -\mu_3 u_3(t) + \beta_{11}(u_1(t)) + \beta_{12}(u_2(t)), \\ \dot{u}_4(t) = -\mu_4 u_4(t) + \beta_{21}(u_1(t)) + \beta_{22}(u_2(t)), \end{cases} \tag{2.3}$$

where $\alpha_{ij} = c_{ij} f'_{ij}(0)$, $\beta_{ij} = d_{ij} g'_{ij}(0)$. The characteristic equation of the linearized system (2.3) is

$$\det \begin{pmatrix} \lambda + \mu_1 & 0 & -\alpha_{11} e^{-\lambda\tau} & -\alpha_{12} e^{-\lambda\tau} \\ 0 & \lambda + \mu_2 & -\alpha_{21} e^{-\lambda\tau} & -\alpha_{22} e^{-\lambda\tau} \\ -\beta_{11} & -\beta_{12} & \lambda + \mu_3 & 0 \\ -\beta_{21} & -\beta_{22} & 0 & \lambda + \mu_4 \end{pmatrix} = 0. \tag{2.4}$$

That the following four degree exponential polynomial equation is obtained

$$\lambda^4 + d_1\lambda^3 + d_2\lambda^2 + d_3\lambda + d_4 + \left(d_5\lambda^2 + d_6\lambda + d_7\right)e^{-\lambda\tau} + d_8 e^{-2\lambda\tau} = 0, \tag{2.5}$$

where

$$d_1 = \mu_1 + \mu_2 + \mu_3 + \mu_4, \tag{2.6}$$

$$d_2 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4, \tag{2.7}$$

$$d_3 = \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4, \tag{2.8}$$

$$d_4 = \mu_1\mu_2\mu_3\mu_4, \tag{2.9}$$
$$d_5 = -\alpha_{11}\beta_{11} - \alpha_{12}\beta_{21} - \alpha_{21}\beta_{12} - \alpha_{22}\beta_{22}, \tag{2.10}$$
$$d_6 = -\mu_1(\alpha_{22}\beta_{22} + \alpha_{21}\beta_{12}) - \mu_2(\alpha_{11}\beta_{11} + \alpha_{12}\beta_{21}) - \mu_3(\alpha_{12}\beta_{21} + \alpha_{22}\beta_{22}) - \mu_4(\alpha_{21}\beta_{12} + \alpha_{11}\beta_{11}), \tag{2.11}$$
$$d_7 = -\mu_1\mu_3\alpha_{22}\beta_{22} - \mu_2\mu_3\alpha_{12}\beta_{21} - \mu_1\mu_4\alpha_{21}\beta_{12} - \mu_2\mu_4\alpha_{11}\beta_{11}, \tag{2.12}$$
$$d_8 = (\beta_{11}\beta_{22} - \beta_{12}\beta_{21})(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}). \tag{2.13}$$

Multiplying $e^{s\tau}$ on both sides of (2.5), it is obvious to obtain

$$\big(\lambda^4 + d_1\lambda^3 + d_2\lambda^2 + d_3\lambda + d_4\big)e^{\lambda\tau} + \big(d_5\lambda^2 + d_6\lambda + d_7\big) + d_8 e^{-\lambda\tau} = 0. \tag{2.14}$$

Let $s = i\omega_0$, $\tau = \tau_0$, and substituting this into (2.14), for the sake of simplicity, denote ω_0 and τ_0 by ω, τ, respectively, then (2.14) becomes

$$\big(\omega^4 - id_1\omega^3 - d_2\omega^2 + d_3 i\omega + d_4\big)\big(\cos(\omega\tau) + i\sin(\omega\tau)\big) - d_5\omega^2 + id_6\omega + d_7 + d_8\big(\cos(\omega\tau) - i\sin(\omega\tau)\big) = 0. \tag{2.15}$$

Separating the real and imaginary parts, it is easy to obtain

$$\begin{cases} (\omega^4 - d_2\omega^2 + d_4 + d_8)\cos(\omega\tau) + (d_1\omega^3 - d_3\omega)\sin(\omega\tau) = d_5\omega^2 - d_7, \\ (-d_1\omega^3 + d_3\omega)\cos(\omega\tau) + (\omega^4 - d_2\omega^2 + d_4 - d_8)\sin(\omega\tau) = -d_6\omega. \end{cases} \tag{2.16}$$

By simple calculation, the following equations are obtained

$$\sin(\omega\tau) = \frac{\omega[(d_1d_5 - d_6)\omega^4 + (d_2d_6 - d_1d_7 - d_3d_5)\omega^2 + d_3d_7 - d_4d_6 - d_6d_8]}{\omega^8 + (d_1^2 - 2d_2)\omega^6 + (d_2^2 + 2d_4 - 2d_1d_3)\omega^4 + (d_3^2 - 2d_2d_4)\omega^2 + d_4^2 - d_8^2}, \tag{2.17}$$

and

$$\cos(\omega\tau) = \frac{d_5\omega^6 + (d_1d_6 - d_2d_5 - d_7)\omega^4 + (d_4d_5 - d_5d_8 + d_2d_7 - d_3d_6)\omega^2 + d_7d_8 - d_4d_7}{\omega^8 + (d_1^2 - 2d_2)\omega^6 + (d_2^2 + 2d_4 - 2d_1d_3)\omega^4 + (d_3^2 - 2d_2d_4)\omega^2 + d_4^2 - d_8^2}. \tag{2.18}$$

Let $e_1 = d_1^2 - 2d_2$, $e_2 = d_2^2 + 2d_4 - 2d_1d_3$, $e_3 = d_3^2 - 2d_2d_4$, $e_4 = d_4^2 - d_8^2$, $e_5 = d_1d_5 - d_6$, $e_6 = d_2d_6 - d_1d_7 - d_3d_5$, $e_7 = d_3d_7 - d_4d_6 - d_6d_8$, $e_8 = d_5$, $e_9 = d_1d_6 - d_2d_5 - d_7$, $e_{10} = d_4d_5 - d_5d_8 + d_2d_7 - d_3d_6$, $e_{11} = d_7d_8 - d_4d_7$, and $\sin(\omega\tau), \cos(\omega\tau)$ can be written as

$$\sin(\omega\tau) = \frac{\omega(e_5\omega^4 + e_6\omega^2 + e_7)}{\omega^8 + e_1\omega^6 + e_2\omega^4 + e_3\omega^2 + e_4}, \tag{2.19}$$

and

$$\cos(\omega\tau) = \frac{e_8\omega^6 + e_9\omega^4 + e_{10}\omega^2 + e_{11}}{\omega^8 + e_1\omega^6 + e_2\omega^4 + e_3\omega^2 + e_4}. \tag{2.20}$$

As is known to all that $\sin^2(\omega\tau) + \cos^2(\omega\tau) = 1$, we have

$$\omega^{16} + f_7\omega^{14} + f_6\omega^{12} + f_5\omega^{10} + f_4\omega^8 + f_3\omega^6 + f_2\omega^4 + f_1\omega^2 + f_0 = 0, \tag{2.21}$$

where $f_7 = 2e_1$, $f_6 = e_1^2 + 2e_2 - e_8^2$, $f_5 = 2e_3 + 2e_1e_2 - 2e_8e_9 - e_5^2$, $f_4 = e_2^2 + 2e_4 + 2e_1e_3 - 2e_5e_6 - e_9^2 - 2e_8e_{10}$, $f_3 = 2e_1e_4 + 2e_2e_3 - e_6^2 - 2e_5e_7 - 2e_8e_{11} - 2e_9e_{10}$, $f_2 = e_3^2 + 2e_2e_4 - 2e_6e_7 - e_{10}^2 - 2e_9e_{11}$, $f_1 = 2e_3e_4 - e_7^2 - 2e_{10}e_{11}$, $f_0 = e_4^2 - e_{11}^2$. Denote $z = \omega^2$, (2.21) becomes

$$z^8 + f_7z^7 + f_6z^6 + f_5z^5 + f_4z^4 + f_3z^3 + f_2z^2 + f_1z + f_0 = 0. \tag{2.22}$$

Let

$$l(z) = z^8 + f_7z^7 + f_6z^6 + f_5z^5 + f_4z^4 + f_3z^3 + f_2z^2 + f_1z + f_0. \tag{2.23}$$

Suppose

(H_3) (2.22) has at least one positive real root.

If μ_k $(k = 1, 2, 3, 4)$, $c_{ij}, d_{ij}, f_{ij}, g_{ij}$ $(i, j = 1, 2)$ of the system (2.1) are given, it is easy to use computer to calculate the roots of (2.22). Since $\lim_{z\to\infty} l(z) = +\infty$, we conclude that if $f_0 < 0$, then (2.22) has at least one positive real root.

Without loss of generality, assuming that it have eight positive real roots, defined by $z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8$, respectively. Then (2.22) have eight positive roots

$$\omega_1 = \sqrt{z_1}, \quad \omega_2 = \sqrt{z_2}, \quad \omega_3 = \sqrt{z_3}, \quad \omega_4 = \sqrt{z_4}, \quad \omega_5 = \sqrt{z_5}, \quad \omega_6 = \sqrt{z_6}, \quad \omega_7 = \sqrt{z_7}, \quad \omega_8 = \sqrt{z_8}.$$

By (2.20), we have

$$\cos(\omega_k\tau)=\frac{e_8\omega_k^6+e_9\omega_k^4+e_{10}\omega_k^2+e_{11}}{\omega_k^8+e_1\omega_k^6+e_2\omega_k^4+e_3\omega_k^2+e_4}. \tag{2.24}$$

Thus, denoting

$$\tau_k^j=\frac{1}{\omega_k}\left\{\arccos\left(\frac{e_8\omega_k^6+e_9\omega_k^4+e_{10}\omega_k^2+e_{11}}{\omega_k^8+e_1\omega_k^6+e_2\omega_k^4+e_3\omega_k^2+e_4}\right)+2j\pi\right\}, \tag{2.25}$$

where $k=1,\dots,8;\ j=0,1,\dots,$ then $\pm i\omega_k$ is a pair of purely imaginary roots of (2.5) with τ_k^j. Define

$$\tau_0=\tau_{k_0}^0=\min_{k\in\{k=1,\dots,8\}}\{\tau_k^0\},\qquad \omega_0=\omega_{k_0}. \tag{2.26}$$

Note that when $\tau=0$, (2.5) becomes

$$\lambda^4+d_1\lambda^3+(d_2+d_5)\lambda^2+(d_3+d_6)\lambda+d_4+d_7+d_8=0. \tag{2.27}$$

A set of necessary and sufficient conditions for all roots of (2.27) to have a negative real part is given by the well-known Routh–Hurwitz criteria in the following form:

$$D_1=d_1>0, \tag{2.28}$$

$$D_2=\det\begin{pmatrix} d_1 & d_3+d_6\\ 1 & d_2+d_5\end{pmatrix}=d_1(d_2+d_5)-(d_3+d_6)>0, \tag{2.29}$$

$$D_3=\det\begin{pmatrix} d_1 & d_3+d_6 & 0\\ 1 & d_2+d_5 & d_4+d_7+d_8\\ 0 & d_1 & d_3+d_6\end{pmatrix}=d_1\big[(d_2+d_5)(d_3+d_6)-d_1(d_4+d_7+d_8)\big]-(d_3+d_6)^2>0, \tag{2.30}$$

$$D_4=\det\begin{pmatrix} d_1 & d_3+d_6 & 0 & 0\\ 1 & d_2+d_5 & d_4+d_7+d_8 & 0\\ 0 & d_1 & d_3+d_6 & 0\\ 0 & 1 & d_2+d_5 & d_4+d_7+d_8\end{pmatrix}=(d_4+d_7+d_8)D_3>0. \tag{2.31}$$

In order to give the main results in this Letter, it is necessary to make the following assumptions:

(H_4) If (2.28)–(2.31) holds, (2.27) have four roots with negative real parts and when $\tau=0$, system (2.1) is stable near the equilibrium.

(H_5) $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_0}\neq 0$.

Taking the derivative of λ with respect to τ in (2.14), it is easy to obtain:

$$\begin{aligned}&\left(4\lambda^3+3d_1\lambda^2+2d_2\lambda+d_3\right)e^{\lambda\tau}\frac{d\lambda}{d\tau}+\left(\lambda+\tau\frac{d\lambda}{d\tau}\right)\left(\lambda^4+d_1\lambda^3+d_2\lambda^2+d_3\lambda+d_4\right)e^{\lambda\tau}\\&\quad+2d_5\lambda\frac{d\lambda}{d\tau}+d_6\frac{d\lambda}{d\tau}-d_8e^{-\lambda\tau}\left(\lambda+\tau\frac{d\lambda}{d\tau}\right)=0,\end{aligned}$$

it follows that:

$$\frac{d\lambda(\tau)}{d\tau}=\frac{-\lambda(\lambda^4+d_1\lambda^3+d_2\lambda^2+d_3\lambda+d_4)e^{\lambda\tau}+d_8\lambda e^{-\lambda\tau}}{(4\lambda^3+3d_1\lambda^2+2d_2\lambda+d_3)e^{\lambda\tau}+(\lambda^4+d_1\lambda^3+d_2\lambda^2+d_3\lambda+d_4)\tau e^{\lambda\tau}+2d_5\lambda+d_6-d_8\tau e^{-\lambda\tau}}. \tag{2.32}$$

For the sake of simplicity, denoting w_0 and τ_n by w, τ respectively, then

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1}&=\frac{(4\lambda^3+3d_1\lambda^2+2d_2\lambda+d_3)e^{\lambda\tau}+2d_5\lambda+d_6}{-\lambda(\lambda^4+d_1\lambda^3+d_2\lambda^2+d_3\lambda+d_4)e^{\lambda\tau}+d_8\lambda e^{-\lambda\tau}}-\frac{\tau}{\lambda}\\&=\frac{(4\lambda^3+3d_1\lambda^2+2d_2\lambda+d_3)e^{\lambda\tau}+2d_5\lambda+d_6}{d_5\lambda^3+d_6\lambda^2+d_7\lambda+2d_8\lambda e^{-\lambda\tau}}-\frac{\tau}{\lambda}\\&=\frac{(-4i\omega^3-3d_1\omega^2+2d_2i\omega+d_3)[\cos(\omega\tau)+i\sin(\omega\tau)]+2d_5i\omega+d_6}{-d_5i\omega^3-d_6\omega^2+d_7i\omega+2d_8i\omega[\cos(\omega\tau)-i\sin(\omega\tau)]}-\frac{\tau}{i\omega}\end{aligned}$$

$$
\begin{aligned}
&= \frac{[(-3d_1\omega^2+d_3)\cos(\omega\tau)+(4\omega^3-2d_2\omega)\sin(\omega\tau)+d_6]}{[-d_6\omega^2+2d_8\sin(\omega\tau)]+i[-d_5\omega^3+d_7\omega+2d_8\omega\cos(\omega\tau)]}\\
&\quad + i\frac{[(-4\omega^3+2d_2\omega)\cos(\omega\tau)+(-3d_1\omega^2+d_3)\sin(\omega\tau)+2d_5\omega]}{[-d_6\omega^2+2d_8\sin(\omega\tau)]+i[-d_5\omega^3+d_7\omega+2d_8\omega\cos(\omega\tau)]}-\frac{\tau}{i\omega}.
\end{aligned} \tag{2.33}
$$

Let

$$
\begin{aligned}
&Q=\left[-d_6\omega^2+2d_8\sin(\omega\tau)\right]^2+\left[-d_5\omega^3+d_7\omega+2d_8\omega\cos(\omega\tau)\right]^2>0,\\
&Q\,\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}=\left[\left(-3d_1\omega^2+d_3\right)\cos(\omega\tau)+\left(4\omega^3-2d_2\omega\right)\sin(\omega\tau)+d_6\right]\left[-d_6\omega^2+2d_8\sin(\omega\tau)\right]\\
&\qquad\qquad +\left[\left(-4\omega^3+2d_2\omega\right)\cos(\omega\tau)+\left(-3d_1\omega^2+d_3\right)\sin(\omega\tau)+2d_5\omega\right]\left[-d_5\omega^3+d_7\omega+2d_8\omega\cos(\omega\tau)\right],
\end{aligned} \tag{2.34}
$$

noticing that

$$
\operatorname{sign}\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)\bigg|_{\tau=\tau_0}\right]=\operatorname{sign}\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\bigg|_{\tau=\tau_0}\right]. \tag{2.35}
$$

Till now, we can employ a result from Ruan and Wei [1] to analyze (2.5), which is, for the convenience of the reader, stated as follows.

Lemma 2.1 [1]. *Consider the exponential polynomial*

$$
\begin{aligned}
P\left(\lambda,e^{-\lambda\tau_1},\dots,e^{-\lambda\tau_m}\right)&=\lambda^n+p_1^{(0)}\lambda^{n-1}+\dots+p_{n-1}^{(0)}\lambda+p_n^{(0)}+\left[p_1^{(1)}\lambda^{n-1}+\dots+p_{n-1}^{(1)}\lambda+p_n^{(1)}\right]e^{-\lambda\tau_1}\\
&\quad+\dots+\left[p_1^{(m)}\lambda^{n-1}+\dots+p_{n-1}^{(m)}\lambda+p_n^{(m)}\right]e^{-\lambda\tau_m},
\end{aligned}
$$

where $\tau_i\geqslant 0$ $(i=1,2,\dots,m)$ *and* $p_j^{(i)}$ $(i=0,1,\dots,m;\ j=1,2,\dots,n)$ *are constants. As* $(\tau_1,\tau_2,\dots,\tau_m)$ *vary, the sum of the order of the zeros of* $P(\lambda,e^{-\lambda\tau_1},\dots,e^{-\lambda\tau_m})$ *on the open right half plane can change only if a zero appears on or crosses the imaginary axis.*

From Lemma 2.1, it is easy to obtain the following theorem:

Theorem 2.2. *Suppose that* (H_3), (H_4) *and* (H_5) *holds, then the following results hold*:

(I) *For Eq.* (2.1), *its zero solution is asymptotically stable for* $\tau\in[0,\tau_0)$;
(II) *Eq.* (2.1) *undergoes a Hopf bifurcation at the origin when* $\tau=\tau_0$. *That is, system* (2.1) *has a branch of periodic solutions bifurcating from the zero solution near* $\tau=\tau_0$.

Remark 2.3. In [2], Yu and Cao study a van der Pol equation, and the characteristic equation is

$$\lambda^2+a\lambda e^{-\lambda\tau}+e^{-2\lambda\tau}=0.$$

In [3], Guo and Huang study a two-neuron network model with three delays, the coefficient of the system must satisfy some given conditions, and the characteristic equation discussed in [3] is

$$\left[\lambda+1-\beta e^{-\lambda\tau}\right]^2-\bar{a_{12}}\bar{a_{21}}e^{-2\lambda\tau}=0.$$

In [6], Song and Wei study a delayed predator-prey system, the characteristic equation is

$$\lambda^2+p\lambda+r+(s\lambda+q)e^{-s\tau}=0,$$

clearly, the characteristic equations they have discussed are two degree exponential polynomial equation, and can be solved easily.

Remark 2.4. In [5], Song, Han and Wei study a simplified BAM neural network with three delays, but through a simple transformation the model can be changed into one time delay since the BAM neural network do not have self-connections. By the method studied in [4], Song, Han and Wei study the following characteristic equation

$$\lambda^3+a_2\lambda^2+a_1\lambda+a_0+(b_1\lambda+b_0)e^{-\lambda\tau}=0,$$

and it is a special case in our model. A four degree exponential polynomial equation has been discussed.

The coefficients given in the above characteristic equations, the reader may refer to the references. In this Letter, a method to solve characteristic Eq. (2.5) is proposed.

Remark 2.5. In [7,8], it is simpler than our models since the characteristic equation of ours is transcendental equation corresponding to polynomial equation of Liao. So he discussed the local stability and existence of Hopf bifurcation using Routh–Hurwitz criteria. The stability and existence of Hopf bifurcation which is studied in our Letter is not as simple as his.

3. Stability of bifurcating periodic solutions

In this section, formulae for determining the direction of Hopf bifurcation and stability of bifurcating periodic solutions of system (2.2) at τ_0 shall be presented by employing the normal form method and center manifold theorem introduced by Hassard at [9].

For convenience, let $t = s\tau$, $x_i(t) = u_i(\tau t)$ and $\tau = \tau_0 + \mu$, $\mu \in R$. Then system (2.2) is equivalent to the system:

$$\begin{cases} \dot{x}_1(t) = (\tau_0+\mu)[-\mu_1 x_1(t) + c_{11} f_{11}(x_3(t-1)) + c_{12} f_{12}(x_4(t-1))], \\ \dot{x}_2(t) = (\tau_0+\mu)[-\mu_2 x_2(t) + c_{21} f_{21}(x_3(t-1)) + c_{22} f_{22}(x_4(t-1))], \\ \dot{x}_3(t) = (\tau_0+\mu)[-\mu_3 x_3(t) + d_{11} g_{11}(x_1(t)) + d_{12} g_{12}(x_2(t))], \\ \dot{x}_4(t) = (\tau_0+\mu)[-\mu_4 x_4(t) + d_{21} g_{21}(x_1(t)) + d_{22} g_{22}(x_2(t))]. \end{cases} \tag{3.1}$$

Its linear part is given by

$$\begin{cases} \dot{x}_1(t) = (\tau_0+\mu)[-\mu_1 x_1(t) + \alpha_{11}(x_3(t-1)) + \alpha_{12}(x_4(t-1))], \\ \dot{x}_2(t) = (\tau_0+\mu)[-\mu_2 x_2(t) + \alpha_{21}(x_3(t-1)) + \alpha_{22}(x_4(t-1))], \\ \dot{x}_3(t) = (\tau_0+\mu)[-\mu_3 x_3(t) + \beta_{11}(x_1(t)) + \beta_{12}(x_2(t))], \\ \dot{x}_4(t) = (\tau_0+\mu)[-\mu_4 x_4(t) + \beta_{21}(x_1(t)) + \beta_{22}(x_2(t))], \end{cases} \tag{3.2}$$

The non-linear part of (3.1) is

$$f(\mu, x_t) = (\tau_0+\mu)\begin{pmatrix} l_{11}x_3^2(t-1) + l_{12}x_4^2(t-1) + l'_{11}x_3^3(t-1) + l'_{12}x_4^3(t-1) + \text{h.o.t.} \\ l_{21}x_3^2(t-1) + l_{22}x_4^2(t-1) + l'_{21}x_3^3(t-1) + l'_{22}x_4^3(t-1) + \text{h.o.t.} \\ m_{11}x_1^2(t) + m_{12}x_2^2(t) + m'_{11}x_1^3(t) + m'_{12}x_2^3(t) + \text{h.o.t.} \\ m_{21}x_1^2(t) + m_{22}x_2^2(t) + m'_{21}x_1^3(t) + m'_{22}x_2^3(t) + \text{h.o.t.} \end{pmatrix}, \tag{3.3}$$

where $x_t(\theta) = \begin{pmatrix} x_{1t}(\theta) \\ x_{2t}(\theta) \\ x_{3t}(\theta) \\ x_{4t}(\theta) \end{pmatrix} = \begin{pmatrix} x_1(t+\theta) \\ x_2(t+\theta) \\ x_3(t+\theta) \\ x_4(t+\theta) \end{pmatrix}$, $l_{ij} = c_{ij} f''_{ij}(0)/2!$, $l'_{ij} = c_{ij} f'''_{ij}(0)/3!$, $m_{ij} = d_{ij} g''_{ij}(0)/2!$, $m'_{ij} = d_{ij} g'''_{ij}(0)/3!$ $(i, j = 1, 2)$. Denote $\mathrm{C}^k[-1,0] = \{\varphi \mid \varphi : [-1,0] \to \mathrm{R}^4$, each component of φ has k order continuous derivative$\}$. For convenience, denote $\mathrm{C}[-1,0]$ by $\mathrm{C}^0[-1,0]$. The solutions map continuous initial data into R^4. We are interested in periodic solutions. For $\phi(\theta) = (\phi_1(\theta)\ \phi_2(\theta)\ \phi_3(\theta)\ \phi_4(\theta))^T \in \mathrm{C}[-1,0]$, define an operator

$$L_\mu\phi = (\tau_0+\mu)\begin{pmatrix} -\mu_1 & 0 & 0 & 0 \\ 0 & -\mu_2 & 0 & 0 \\ \beta_{11} & \beta_{12} & -\mu_3 & 0 \\ \beta_{21} & \beta_{22} & 0 & -\mu_4 \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \\ \phi_4(0) \end{pmatrix} + (\tau_0+\mu)\begin{pmatrix} 0 & 0 & \alpha_{11} & \alpha_{12} \\ 0 & 0 & \alpha_{21} & \alpha_{21} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \\ \phi_4(-1) \end{pmatrix}, \tag{3.4}$$

where L_μ is a one-parameter family of bounded linear operators in $\mathrm{C}[-1,0] \to \mathrm{R}^4$. By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta,\mu)$ in $[-1,0] \to \mathrm{R}^4$, such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta,\mu)\,\phi(\theta).$$

In fact, choosing

$$\eta(\theta,\mu) = (\tau_0+\mu)\begin{pmatrix} -\mu_1 & 0 & 0 & 0 \\ 0 & -\mu_2 & 0 & 0 \\ \beta_{11} & \beta_{12} & -\mu_3 & 0 \\ \beta_{21} & \beta_{22} & 0 & -\mu_4 \end{pmatrix}\delta(\theta) + (\tau_0+\mu)\begin{pmatrix} 0 & 0 & \alpha_{11} & \alpha_{12} \\ 0 & 0 & \alpha_{21} & \alpha_{21} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\delta(\theta+1) \tag{3.5}$$

(where $\delta(\theta)$ is Dirac function), then (3.4) is satisfied.

For $\phi \in \mathrm{C}^1[-10]$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \leqslant \theta < 0, \\ \int_{-1}^{0} d\eta(\theta,\mu)\,\phi(\theta), & \theta = 0, \end{cases} \tag{3.6}$$

and

$$R(\mu)\phi = \begin{cases} \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix}, & -1 \leqslant \theta < 0, \\ f(\mu, \phi), & \theta = 0. \end{cases} \tag{3.7}$$

In order to conveniently study Hopf bifurcation problem, we transform system (3.1) into an operator equation of the form:

$$\dot{x}_t = A(\mu)x_t + Rx_t, \tag{3.8}$$

where $x = (x_1, x_2, x_3, x_4)^T$. As in [9], $x_t = x(t+\theta)$, $\theta \in (-1, 0]$.

The adjoint operator A^* of A is defined by

$$A^*(\mu)\psi = \begin{cases} -\frac{d\psi(s)}{ds}, & 0 < s \leqslant 1, \\ \int_{-1}^{0} d\eta^T(s, \mu)\,\psi(-s), & s = 0, \end{cases} \tag{3.9}$$

where η^T is the transpose of the matrix η.

The domains of A and A^* are $C^1[-1, 0]$ and $C^1[0, 1]$, respectively. For $\phi \in C[-1, 0]$ and $\psi \in C[0, 1]$. In order to normalize the eigenvectors of operator A and adjoint operator A^*, the following bilinear form is needed to introduce:

$$\langle \psi, \phi \rangle = \bar{\psi}(0) \cdot \phi(0) - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}^T(\xi - \theta)\, d\eta(\theta)\, \phi(\xi)\, d\xi, \tag{3.10}$$

here $\eta(\theta) = \eta(\theta, 0)$, C^2 is complex plane. And for c and d in C^2, $c \cdot d$ means $\sum_{i=1}^{4} c_i d_i$, where c_i and d_i are components of c and d, respectively. Then, as usual,

$$\langle \psi, A\phi \rangle = \langle A^*\psi, \phi \rangle, \tag{3.11}$$

for $(\phi, \psi) \in D(A) \times D(A^*)$. Normalizing q and q^* by the condition

$$\langle q^*, q \rangle = 1, \qquad \langle q^*, \bar{q} \rangle = 0.$$

By discussion in Section 2 and transformation $t = s\tau$, we know that $\pm i\tau_0 w_0$ are eigenvalues of $A(0)$ and other eigenvalues have strictly negative real parts. Thus they are also eigenvalues of A^*. Next we calculate the eigenvector q of A belonging to the eigenvalue $i\tau_0 w_0$ and the eigenvector q^* of A^* belonging to the eigenvalue $-i\tau_0 w_0$. Let

$$q(\theta) = \begin{pmatrix} 1\\ \alpha\\ \beta\\ \gamma \end{pmatrix} e^{i\tau_0\omega_0\theta}, \quad -1 < \theta \leqslant 0. \tag{3.12}$$

From the above discussion, it is easy to know that

$$Aq(0) = i\tau_0\omega_0 q(0), \qquad \tau_0 \begin{pmatrix} i\omega_0 + \mu_1 & 0 & -\alpha_{11}e^{-i\omega_0\tau_0} & -\alpha_{12}e^{-i\omega_0\tau_0} \\ 0 & i\omega_0 + \mu_2 & -\alpha_{21}e^{-i\omega_0\tau_0} & -\alpha_{22}e^{-i\omega_0\tau_0} \\ -\beta_{11} & -\beta_{12} & i\omega_0 + \mu_3 & 0 \\ -\beta_{21} & -\beta_{22} & 0 & i\omega_0 + \mu_4 \end{pmatrix} q(0) = \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix}. \tag{3.13}$$

Hence, we obtain

$$\alpha = \frac{-\alpha_{11}\beta_{11}\beta_{22}(i\omega_0 + \mu_4)e^{-i\omega_0\tau_0} - \alpha_{12}\beta_{21}\beta_{22}(i\omega_0 + \mu_3)e^{-i\omega_0\tau_0} + \beta_{22}(i\omega_0 + \mu_1)(i\omega_0 + \mu_3)(i\omega_0 + \mu_4)}{\alpha_{11}\beta_{12}\beta_{22}(i\omega_0 + \mu_4)e^{-i\omega_0\tau_0} + \alpha_{12}\beta_{22}\beta_{22}(i\omega_0 + \mu_3)e^{-i\omega_0\tau_0}}, \tag{3.14}$$

$$\beta = \frac{\beta_{12}(i\omega_0 + \mu_1)(i\omega_0 + \mu_4) + \alpha_{12}(\beta_{11}\beta_{22} - \beta_{12}\beta_{21})e^{-i\omega_0\tau_0}}{\alpha_{11}\beta_{12}(i\omega_0 + \mu_4)e^{-i\omega_0\tau_0} + \alpha_{12}\beta_{22}(i\omega_0 + \mu_3)e^{-i\omega_0\tau_0}}, \tag{3.15}$$

$$\gamma = \frac{\beta_{22}(i\omega_0 + \mu_1)(i\omega_0 + \mu_3) - \alpha_{11}(\beta_{11}\beta_{22} - \beta_{12}\beta_{21})e^{-i\omega_0\tau_0}}{\alpha_{11}\beta_{12}(i\omega_0 + \mu_4)e^{-i\omega_0\tau_0} + \alpha_{12}\beta_{22}(i\omega_0 + \mu_3)e^{-i\omega_0\tau_0}}. \tag{3.16}$$

Suppose that the eigenvector q^* of A^* is

$$q^*(s) = \frac{1}{\rho} \begin{pmatrix} 1\\ \alpha^*\\ \beta^*\\ \gamma^* \end{pmatrix} e^{i\tau_0\omega_0 s}, \quad 0 \leqslant s < 1. \tag{3.17}$$

Then the following relationship is obtained:

$$A^*q^*(0) = -i\tau_0\omega_0 q^*(0), \qquad \tau_0 \begin{pmatrix} -i\omega_0+\mu_1 & 0 & -\beta_{11} & -\beta_{21} \\ 0 & -i\omega_0+\mu_2 & -\beta_{12} & -\beta_{22} \\ -\alpha_{11}e^{i\omega_0\tau_0} & -\alpha_{21}e^{i\omega_0\tau_0} & i\omega_0+\mu_3 & 0 \\ -\alpha_{12}e^{i\omega_0\tau_0} & -\alpha_{22}e^{i\omega_0\tau_0} & 0 & -i\omega_0+\mu_4 \end{pmatrix} q^*(0) = \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix}. \tag{3.18}$$

Hence, we obtain

$$\alpha^* = \frac{-\alpha_{11}\alpha_{22}\beta_{11}(-i\omega_0+\mu_4)e^{i\omega_0\tau_0} - \alpha_{12}\alpha_{22}\beta_{21}(-i\omega_0+\mu_3)e^{i\omega_0\tau_0} + \alpha_{22}(-i\omega_0+\mu_1)(-i\omega_0+\mu_3)(-i\omega_0+\mu_4)}{(\alpha_{22}e^{i\omega_0\tau_0})[\alpha_{21}\beta_{11}(-i\omega_0+\mu_4)+\alpha_{22}\beta_{21}(-i\omega_0+\mu_3)]}, \tag{3.19}$$

$$\beta^* = \frac{\alpha_{21}(-i\omega_0+\mu_1)(-i\omega_0+\mu_4) + \beta_{21}(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21})e^{i\omega_0\tau_0}}{\alpha_{21}\beta_{11}(-i\omega_0+\mu_4)+\alpha_{22}\beta_{21}(-i\omega_0+\mu_3)}, \tag{3.20}$$

$$\gamma^* = \frac{\alpha_{22}(-i\omega_0+\mu_1)(-i\omega_0+\mu_3) - \beta_{11}(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21})e^{i\omega_0\tau_0}}{\alpha_{21}\beta_{11}(-i\omega_0+\mu_4)+\alpha_{22}\beta_{21}(-i\omega_0+\mu_3)}. \tag{3.21}$$

Let

$$\langle q^*, q\rangle = 1.$$

One can obtain ρ,

$$\begin{aligned}
\langle q^*, q\rangle &= \bar{q}^*(0)\cdot q(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \bar{q}^{*T}(\xi-\theta)\,d\eta(\theta)\,q(\xi)\,d\xi \\
&= \frac{1}{\bar\rho}(1+\alpha\bar\alpha^*+\beta\bar\beta^*+\gamma\bar\gamma^*) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \frac{1}{\bar\rho}\begin{pmatrix}1 & \bar\alpha^* & \bar\beta^* & \bar\gamma^*\end{pmatrix} e^{-i\tau_0\omega_0(\xi-\theta)}\,d\eta(\theta)\begin{pmatrix}1\\ \alpha\\ \beta\\ \gamma\end{pmatrix} e^{i\tau_0\omega_0\xi}\,d\xi \\
&= \frac{1}{\bar\rho}(1+\alpha\bar\alpha^*+\beta\bar\beta^*+\gamma\bar\gamma^*) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \tau_0\frac{1}{\bar\rho}\begin{pmatrix}1 & \bar\alpha^* & \bar\beta^* & \bar\gamma^*\end{pmatrix}\left[\begin{pmatrix} -\mu_1 & 0 & 0 & 0\\ 0 & -\mu_2 & 0 & 0\\ \beta_{11} & \beta_{12} & -\mu_3 & 0\\ \beta_{21} & \beta_{22} & 0 & -\mu_4\end{pmatrix}\delta(\theta) \right.\\
&\quad \left. + \begin{pmatrix} 0 & 0 & \alpha_{11} & \alpha_{12}\\ 0 & 0 & \alpha_{21} & \alpha_{21}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\delta(\theta+1)\right]\begin{pmatrix}1\\ \alpha\\ \beta\\ \gamma\end{pmatrix} e^{i\tau_0\omega_0\theta}\,d\xi\,d\theta \\
&= \frac{1}{\bar\rho}(1+\alpha\bar\alpha^*+\beta\bar\beta^*+\gamma\bar\gamma^*) + \frac{1}{\bar\rho}\tau_0\big[\beta(\alpha_{11}+\alpha_{21}\bar\alpha^*)+\gamma(\alpha_{12}+\alpha_{22}\bar\alpha^*)\big]e^{-i\tau_0\omega_0} \\
&= 1.
\end{aligned}$$

Hence, we have

$$\bar\rho = (1+\alpha\bar\alpha^*+\beta\bar\beta^*+\gamma\bar\gamma^*) + \tau_0\big[\beta(\alpha_{11}+\alpha_{21}\bar\alpha^*)+\gamma(\alpha_{12}+\alpha_{22}\bar\alpha^*)\big]e^{-i\tau_0 w\omega_0}. \tag{3.22}$$

Using the same method it is easy to proof $\langle q^*, \bar q\rangle = 0$, we omit it. Now we obtain q and q^*.

Next, we study the stability of bifurcating periodic solutions. As in [9], the bifurcating periodic solutions $Z(t,\mu(\varepsilon))$ has amplitude $O(\varepsilon)$ and non-zero Floquet exponent $\beta(\varepsilon)$ with $\beta(0)=0$. Under the hypotheses, μ, β are given by

$$\begin{cases} \mu = \mu_2\varepsilon^2 + \mu_4\varepsilon^4 + \cdots, \\ \beta = \beta_2\varepsilon^2 + \beta_4\varepsilon^4 + \cdots. \end{cases} \tag{3.23}$$

The sign of μ_2 indicates the direction of bifurcation while that β_2 determines the stability of $Z(t,\mu(\varepsilon))$. $Z(t,\mu(\varepsilon))$ is stable if $\beta_2 < 0$ and unstable if $\beta_2 > 0$. In the following, we will show how to derive the coefficients in this expansions, but we compute μ_2 and β_2 only.

We first construct the coordinates to describe a center manifold Ω_0 near $\mu = 0$, which is a local invariant, attracting a two-dimensional manifold [9]. Let

$$z(t) = \langle q^*, x_t\rangle, \tag{3.24}$$

and

$$W(t,\theta) = x_t - 2\,\mathrm{Re}\big[z(t)q(\theta)\big]. \tag{3.25}$$

Where x_t is a solution of (3.8). On the manifold Ω_0: $W(t,\theta) = W(z(t), \bar{z}(t), \theta)$, where

$$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{3.26}$$

In fact, z and $\bar{z}$ are local coordinates of center manifold Ω_0 in the direction of q and q^*, respectively.

The existence of center manifold Ω_0 enables us to reduce (3.8) to an ordinary differential equation in a single complex variable on Ω_0. For the solution $x_t \in \Omega_0$ of (3.8), since $\mu = 0$,

$$\begin{aligned}\dot{z}(t) &= \langle q^*, \dot{x}_t\rangle = \langle q^*, Ax_t + Rx_t\rangle = \langle q^*, Ax_t\rangle + \langle q^*, Rx_t\rangle = \langle A^*q^*, x_t\rangle + \langle q^*, Rx_t\rangle \\ &= i\tau_0\omega_0 z + \bar{q}^*(0)\cdot f\big(0, W(t,0) + 2\,\mathrm{Re}\big[z(t)q(0)\big]\big).\end{aligned} \tag{3.27}$$

Rewrite (3.27) as

$$\dot{z}(t) = i\tau_0\omega_0 z + g(z,\bar{z}), \tag{3.28}$$

where

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{3.29}$$

In the following, the motivation is to expand g in powers of z and $\bar{z}$ and then obtain, from the coefficients of this expansion, the values of μ_2 and β_2 using algorithm presented by Hassard at [9]. Substituting (3.8) and (3.27) into

$$\dot{W} = \dot{x}_t - \dot{z}q - \dot{\bar{z}}\bar{q},$$

we have

$$\begin{aligned}\dot{W} &= \dot{x}_t - \dot{z}q - \dot{\bar{z}}\bar{q} \\ &= Ax_t + Rx_t - \big[i\tau_0 w_0 z + \bar{q}^*(0)\cdot f(z,\bar{z})\big]q - \big[-i\tau_0 w_0\bar{z} + q^*(0)\cdot\bar{f}(z,\bar{z})\big]\bar{q} \\ &= AW + 2A\,\mathrm{Re}(zq) + Rx_t - 2\,\mathrm{Re}\big[\bar{q}^*(0)\cdot f(z,\bar{z})q(\theta)\big] - 2\,\mathrm{Re}\big[i\tau_0 w_0 z q(\theta)\big] \\ &= AW - 2\,\mathrm{Re}\big[\bar{q}^*(0)\cdot f(z,\bar{z})q(\theta)\big] + Rx_t \\ &= \begin{cases} AW - 2\,\mathrm{Re}[\bar{q}^*(0)\cdot f(z,\bar{z})q(\theta)], & -1 \leqslant \theta < 0, \\ AW - 2\,\mathrm{Re}[\bar{q}^*(0)\cdot f(z,\bar{z})q(\theta)] + f, & \theta = 0.\end{cases}\end{aligned} \tag{3.30}$$

Let

$$\dot{W} = AW + H(z,\bar{z},\theta), \tag{3.31}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{3.32}$$

Taking the derivative of W with respect to t in (3.26), we have

$$\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}. \tag{3.33}$$

Substituting (3.26) and (3.28) into (3.33), we obtain

$$\dot{W} = (W_{20}z + W_{11}\bar{z} + \cdots)(i\tau_0 w_0 z + g) + (W_{11}z + W_{02}\bar{z} + \cdots)(-i\tau_0 w_0\bar{z} + \bar{g}). \tag{3.34}$$

Then substituting (3.26) and (3.32) into (3.31), the following results is obtained:

$$\dot{W} = (AW_{20} + H_{20})\frac{z^2}{2} + (AW_{11} + H_{11})z\bar{z} + (AW_{02} + H_{02})\frac{\bar{z}^2}{2} + \cdots. \tag{3.35}$$

Comparing the coefficients of (3.34) with (3.35),

$$(A - 2i\tau_0 w_0)W_{20}(\theta) = -H_{20}(\theta), \tag{3.36}$$

$$AW_{11}(\theta) = -H_{11}(\theta), \tag{3.37}$$

hold.

According to (3.27) and (3.28), we know

$$g(z,\bar z)=\bar q^*(0)\cdot f(z,\bar z)=\frac{\tau_0}{\bar\rho}\begin{pmatrix}1 & \bar\alpha^* & \bar\beta^* & \bar\gamma^*\end{pmatrix}\begin{pmatrix} l_{11}x_{3t}^2(-1)+l_{12}x_{4t}^2(-1)+l'_{11}x_{3t}^3(-1)+l'_{12}x_{4t}^3(-1)+\text{h.o.t.}\\ l_{21}x_{3t}^2(-1)+l_{22}x_{4t}^2(-1)+l'_{21}x_{3t}^3(-1)+l'_{22}x_{4t}^3(-1)+\text{h.o.t.}\\ m_{11}x_{1t}^2(0)+m_{12}x_{2t}^2(0)+m'_{11}x_{1t}^3(0)+m'_{12}x_{2t}^3(0)+\text{h.o.t.}\\ m_{21}x_{1t}^2(0)+m_{22}x_{2t}^2(0)+m'_{21}x_{1t}^3(0)+m'_{22}x_{2t}^3(0)+\text{h.o.t.}\end{pmatrix},\tag{3.38}$$

where $x_t(\theta)=(x_{1t}(\theta)\ x_{2t}(\theta)\ x_{3t}(\theta)\ x_{4t}(\theta))^T=W(t,\theta)+zq(\theta)+\bar z\bar q(\theta)$ and $q(\theta)=(1\ \alpha\ \beta\ \gamma)^Te^{i\tau_0\omega_0\theta}$, and then we have

$$x_{1t}(0)=z+\bar z+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\bar z+W_{02}^{(1)}(0)\frac{\bar z^2}{2}+o\big(|(z,\bar z)|^3\big),$$
$$x_{2t}(0)=\alpha z+\bar\alpha\bar z+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\bar z+W_{02}^{(2)}(0)\frac{\bar z^2}{2}+o\big(|(z,\bar z)|^3\big),$$
$$x_{3t}(-1)=z\beta e^{-i\tau_0\omega_0}+\bar z\bar\beta e^{i\tau_0\omega_0}+W_{20}^{(3)}(-1)\frac{z^2}{2}+W_{11}^{(3)}(-1)z\bar z+W_{02}^{(3)}(-1)\frac{\bar z^2}{2}+o\big(|(z,\bar z)|^3\big),$$
$$x_{4t}(-1)=z\gamma e^{-i\tau_0\omega_0}+\bar z\bar\gamma e^{i\tau_0\omega_0}+W_{20}^{(4)}(-1)\frac{z^2}{2}+W_{11}^{(4)}(-1)z\bar z+W_{02}^{(4)}(-1)\frac{\bar z^2}{2}+o\big(|(z,\bar z)|^3\big).$$

It follows that

$$\begin{aligned}g(z,\bar z)=\frac{\tau_0}{\bar\rho}\big\{&\big[(l_{11}+\bar\alpha^*l_{21})\beta^2e^{-2i\tau_0\omega_0}+(l_{12}+\bar\alpha^*l_{22})\gamma^2e^{-2i\tau_0\omega_0}+(\bar\beta^*m_{11}+\bar\gamma^*m_{21})+\alpha^2(\bar\beta^*m_{12}+\bar\gamma^*m_{22})\big]z^2\\&+\big[(l_{11}+\bar\alpha^*l_{21})\bar\beta^2e^{2i\tau_0\omega_0}+(l_{12}+\bar\alpha^*l_{22})\bar\gamma^2e^{2i\tau_0\omega_0}+(\bar\beta^*m_{11}+\bar\gamma^*m_{21})+\bar\alpha^2(\bar\beta^*m_{12}+\bar\gamma^*m_{22})\big]\bar z^2\\&+2\big[(l_{11}+\bar\alpha^*l_{21})|\beta|^2+(l_{12}+\bar\alpha^*l_{22})|\gamma|^2+(\bar\beta^*m_{11}+\bar\gamma^*m_{21})+(\bar\beta^*m_{12}+\bar\gamma^*m_{22})|\alpha|^2\big]z\bar z\\&+\big[(l_{11}+\bar\alpha^*l_{21})\big(\bar\beta e^{i\tau_0\omega_0}W_{20}^{(3)}(-1)+2\beta e^{-i\tau_0\omega_0}W_{11}^{(3)}(-1)\big)\\&+(l_{12}+\bar\alpha^*l_{22})\big(\bar\gamma e^{i\tau_0\omega_0}W_{20}^{(4)}(-1)+2\gamma e^{-i\tau_0\omega_0}W_{11}^{(4)}(-1)\big)\\&+(\bar\beta^*m_{11}+\bar\gamma^*m_{21})\big(W_{20}^{(1)}(0)+2W_{11}^{(1)}(0)\big)+(\bar\beta^*m_{12}+\bar\gamma^*m_{22})\big(\bar\alpha W_{20}^{(2)}(0)+2\alpha W_{11}^{(2)}(0)\big)\\&+3(l'_{11}+\bar\alpha^*l'_{21})\beta^2\bar\beta e^{-i\tau_0\omega_0}+3(l'_{12}+\bar\alpha^*l'_{22})\gamma^2\bar\gamma e^{-i\tau_0\omega_0}+3(\bar\beta^*m'_{11}+\bar\gamma^*m'_{21})+3(\bar\beta^*m'_{12}+\bar\gamma^*m'_{22})\alpha^2\bar\alpha\big]z^2\bar z\big\}.\end{aligned}\tag{3.39}$$

Comparing the coefficients in (3.29) with those in (3.39), it follows that:

$$\begin{aligned}g_{20}&=\frac{\tau_0}{\bar\rho}\big[(l_{11}+\bar\alpha^*l_{21})\beta^2e^{-2i\tau_0\omega_0}+(l_{12}+\bar\alpha^*l_{22})\gamma^2e^{-2i\tau_0\omega_0}+(\bar\beta^*m_{11}+\bar\gamma^*m_{21})+\alpha^2(\bar\beta^*m_{12}+\bar\gamma^*m_{22})\big],\\g_{02}&=\frac{\tau_0}{\bar\rho}\big[(l_{11}+\bar\alpha^*l_{21})\bar\beta^2e^{2i\tau_0\omega_0}+(l_{12}+\bar\alpha^*l_{22})\bar\gamma^2e^{2i\tau_0\omega_0}+(\bar\beta^*m_{11}+\bar\gamma^*m_{21})+\bar\alpha^2(\bar\beta^*m_{12}+\bar\gamma^*m_{22})\big],\\g_{11}&=\frac{2\tau_0}{\bar\rho}\big[(l_{11}+\bar\alpha^*l_{21})|\beta|^2+(l_{12}+\bar\alpha^*l_{22})|\gamma|^2+(\bar\beta^*m_{11}+\bar\gamma^*m_{21})+(\bar\beta^*m_{12}+\bar\gamma^*m_{22})|\alpha|^2\big],\\g_{21}&=\frac{\tau_0}{\bar\rho}\big[(l_{11}+\bar\alpha^*l_{21})\big(\bar\beta e^{i\tau_0\omega_0}W_{20}^{(3)}(-1)+2\beta e^{-i\tau_0\omega_0}W_{11}^{(3)}(-1)\big)\\&\quad+(l_{12}+\bar\alpha^*l_{22})\big(\bar\gamma e^{i\tau_0\omega_0}W_{20}^{(4)}(-1)+2\gamma e^{-i\tau_0\omega_0}W_{11}^{(4)}(-1)\big)\\&\quad+(\bar\beta^*m_{11}+\bar\gamma^*m_{21})\big(W_{20}^{(1)}(0)+2W_{11}^{(1)}(0)\big)+(\bar\beta^*m_{12}+\bar\gamma^*m_{22})\big(\bar\alpha W_{20}^{(2)}(0)+2\alpha W_{11}^{(2)}(0)\big)\\&\quad+3(l'_{11}+\bar\alpha^*l'_{21})\beta^2\bar\beta e^{-i\tau_0\omega_0}+3(l'_{12}+\bar\alpha^*l'_{22})\gamma^2\bar\gamma e^{-i\tau_0\omega_0}+3(\bar\beta^*m'_{11}+\bar\gamma^*m'_{21})+3(\bar\beta^*m'_{12}+\bar\gamma^*m'_{22})\alpha^2\bar\alpha\big].\end{aligned}\tag{3.40}$$

In the follows, we focus on the computation of $W_{20}(\theta)$ and $W_{11}(\theta)$. (3.30) and (3.31) imply that

$$\begin{aligned}H(z,\bar z,\theta)&=-2\,\mathrm{Re}\big(\bar q^*(0)\cdot f(z,\bar z)q(\theta)\big)+Rx_t\\&=-gq(\theta)-\bar g\bar q(\theta)+Rx_t\\&=-\Big(\frac12 g_{20}z^2+g_{11}z\bar z+\frac12 g_{02}\bar z^2+\cdots\Big)q(\theta)-\Big(\frac12\bar g_{20}\bar z^2+\bar g_{11}z\bar z+\frac12\bar g_{02}z^2+\cdots\Big)\bar q(\theta)+Rx_t.\end{aligned}\tag{3.41}$$

Comparing the coefficients in (3.32) with those in (3.41), we can obtain that

$$H_{20}(\theta)=-g_{20}q(\theta)-\bar g_{02}\bar q(\theta),\quad -1\leqslant\theta<0,\tag{3.42}$$

and

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta), \quad -1 \leqslant \theta < 0. \tag{3.43}$$

Substituting (3.42) into (3.36) and (3.43) into (3.37) respectively, it follows that:

$$\begin{cases} \dot{W}_{20}(\theta) = 2i\tau_0 w\omega_0 W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta), \\ \dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta). \end{cases} \tag{3.44}$$

It is easy to obtain the solutions of (3.44):

$$\begin{cases} W_{20}(\theta) = \frac{ig_{20}}{\tau_0\omega_0}q(0)e^{i\tau_0\omega_0\theta} - \frac{\bar{g}_{02}}{3i\tau_0\omega_0}\bar{q}(0)e^{-i\tau_0\omega_0\theta} + E_1 e^{2i\tau_0\omega_0\theta}, \\ W_{11}(\theta) = \frac{g_{11}}{i\tau_0\omega_0}q(0)e^{i\tau_0\omega_0\theta} - \frac{\bar{g}_{11}}{i\tau_0\omega_0}\bar{q}(0)e^{-i\tau_0\omega_0\theta} + E_2. \end{cases} \tag{3.45}$$

Next we focus on the computation of E_1 and E_2, from (3.36) and (3.37), we have

$$AW_{20}(0) = 2i\tau_0\omega_0 W_{20}(0) - H_{20}(0), \tag{3.46}$$

and

$$AW_{11}(0) = -H_{11}(0). \tag{3.47}$$

From the definition of A in (3.6), we obtain

$$\int_{-1}^{0} d\eta(\theta)\, W_{20}(\theta) = 2i\tau_0\omega_0 W_{20}(0) - H_{20}(0), \tag{3.48}$$

and

$$\int_{-1}^{0} d\eta(\theta)\, W_{11}(\theta) = -H_{11}(0). \tag{3.49}$$

From (3.3) and (3.41)–(3.43), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + \tau_0 \begin{pmatrix} 2l_{11}\beta^2 e^{-2i\tau_0\omega_0} + 2l_{12}\gamma^2 e^{-2i\tau_0\omega_0} \\ 2l_{21}\beta^2 e^{-2i\tau_0\omega_0} + 2l_{22}\gamma^2 e^{-2i\tau_0\omega_0} \\ 2m_{11} + 2m_{12} \\ 2m_{21} + 2m_{22} \end{pmatrix}, \tag{3.50}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \tau_0 \begin{pmatrix} 2l_{11}|\beta|^2 + 2l_{12}|\gamma|^2 \\ 2l_{21}|\beta|^2 + 2l_{22}|\gamma|^2 \\ 2m_{11} + 2m_{12} \\ 2m_{21} + 2m_{22} \end{pmatrix}. \tag{3.51}$$

Substituting (3.45) and (3.50) into (3.48) and noticing that

$$\left(i\tau_0\omega_0 I - \int_{-1}^{0} e^{i\tau_0\omega_0\theta}\, d\eta(\theta)\right)q(0) = 0, \tag{3.52}$$

and

$$\left(-i\tau_0\omega_0 I - \int_{-1}^{0} e^{-i\tau_0\omega_0\theta}\, d\eta(\theta)\right)\bar{q}(0) = 0, \tag{3.53}$$

we obtain

$$\left(2i\tau_0\omega_0 I - \int_{-1}^{0} e^{2i\tau_0\omega_0\theta}\, d\eta(\theta)\right)E_1 = 2\tau_0 \begin{pmatrix} l_{11}\beta^2 e^{-2i\tau_0\omega_0} + l_{12}\gamma^2 e^{-2i\tau_0\omega_0} \\ l_{21}\beta^2 e^{-2i\tau_0\omega_0} + l_{22}\gamma^2 e^{-2i\tau_0\omega_0} \\ m_{11} + m_{12} \\ m_{21} + m_{22} \end{pmatrix}, \tag{3.54}$$

which leads to

$$\begin{pmatrix} 2i\omega_0+\mu_1 & 0 & -\alpha_{11}e^{-2i\omega_0\tau_0} & -\alpha_{12}e^{-2i\omega_0\tau_0} \\ 0 & 2i\omega_0+\mu_2 & -\alpha_{21}e^{-2i\omega_0\tau_0} & -\alpha_{22}e^{-2i\omega_0\tau_0} \\ -\beta_{11} & -\beta_{12} & 2i\omega_0+\mu_3 & 0 \\ -\beta_{21} & -\beta_{22} & 0 & 2i\omega_0+\mu_4 \end{pmatrix} E_1 = 2\begin{pmatrix} l_{11}\beta^2e^{-2i\tau_0\omega_0}+l_{12}\gamma^2e^{-2i\tau_0\omega_0} \\ l_{21}\beta^2e^{-2i\tau_0\omega_0}+l_{22}\gamma^2e^{-2i\tau_0\omega_0} \\ m_{11}+m_{12} \\ m_{21}+m_{22} \end{pmatrix}. \tag{3.55}$$

It follows that

$$E_1 = 2\begin{pmatrix} 2i\omega_0+\mu_1 & 0 & -\alpha_{11}e^{-2i\omega_0\tau_0} & -\alpha_{12}e^{-2i\omega_0\tau_0} \\ 0 & 2i\omega_0+\mu_2 & -\alpha_{21}e^{-2i\omega_0\tau_0} & -\alpha_{22}e^{-2i\omega_0\tau_0} \\ -\beta_{11} & -\beta_{12} & 2i\omega_0+\mu_3 & 0 \\ -\beta_{21} & -\beta_{22} & 0 & 2i\omega_0+\mu_4 \end{pmatrix}^{-1} \begin{pmatrix} l_{11}\beta^2e^{-2i\tau_0\omega_0}+l_{12}\gamma^2e^{-2i\tau_0\omega_0} \\ l_{21}\beta^2e^{-2i\tau_0\omega_0}+l_{22}\gamma^2e^{-2i\tau_0\omega_0} \\ m_{11}+m_{12} \\ m_{21}+m_{22} \end{pmatrix}. \tag{3.56}$$

Similarly, substituting (3.45) and (3.51) into (3.49), we can get

$$E_2 = 2\begin{pmatrix} \mu_1 & 0 & -\alpha_{11} & -\alpha_{12} \\ 0 & \mu_2 & -\alpha_{21} & -\alpha_{22} \\ -\beta_{11} & -\beta_{12} & \mu_3 & 0 \\ -\beta_{21} & -\beta_{22} & 0 & \mu_4 \end{pmatrix}^{-1} \begin{pmatrix} l_{11}|\beta|^2+l_{12}|\gamma|^2 \\ l_{21}|\beta|^2+l_{22}|\gamma|^2 \\ m_{11}+m_{12} \\ m_{21}+m_{22} \end{pmatrix}. \tag{3.57}$$

Hence, we know $W_{20}(\theta)$ and $W_{11}(\theta)$, then Eq. (3.40) can be obtained. The following parameters can be calculated:

$$C_1(0) = \frac{i}{2w}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \tag{3.58}$$

$$\mu_2 = -\frac{\operatorname{Re} C_1(0)}{\operatorname{Re}\lambda'(\tau_0)}, \tag{3.59}$$

$$\beta_2 = 2\operatorname{Re} C_1(0). \tag{3.60}$$

If you want to know the detail, see appendix in [2]. As in [9], the following result is established:

Theorem 3.1. *Under the condition of Theorem 2.2,*

(I) $\mu = 0$ *is Hopf bifurcation value of system (3.1).*
(II) *The direction of Hopf bifurcation is determined by the sign of* μ_2: *if* $\mu_2 > 0$, *the Hopf bifurcation is supercritical*; *if* $\mu_2 < 0$, *the Hopf bifurcation is subcritical.*
(III) *The stability of bifurcating periodic solutions is determined by* β_2: *if* $\beta_2 < 0$, *the periodic solutions are stable*; *if* $\beta_2 > 0$, *they are unstable.*

4. Numerical examples

In this section, some numerical results of simulating system (2.1) are presented at justifying the theorem obtained above.
As an example, considering the following system:

$$\begin{cases} \dot{x}_1(t) = -2x_1(t) - 2\tanh(y_1(t-\tau_3)) - \tanh(y_2(t-\tau_3)), \\ \dot{x}_2(t) = -2x_2(t) - \tanh(y_1(t-\tau_4)) - 2\tanh(y_2(t-\tau_4)), \\ \dot{y}_1(t) = -2y_1(t) + 2\tanh(x_1(t-\tau_1)) + \tanh(x_2(t-\tau_2)), \\ \dot{y}_2(t) = -2y_2(t) + \tanh(x_1(t-\tau_1)) + 2\tanh(x_2(t-\tau_2)), \end{cases}$$

which has a unique steady state $(0\ 0\ 0\ 0)$. From (2.22), we have

$$z^8 + 32z^7 + 348z^6 + 1184z^5 - 4442z^4 - 44448z^3 - 120164z^2 - 128800z - 47775 = 0,$$

and it has only one positive real root 5. Also, (2.27) becomes

$$\lambda^4 + 8\lambda^3 + 34\lambda^2 + 72\lambda + 65 = 0,$$

and it have four roots with negative real parts, the hypothesis of (H_4) is clearly satisfied.
According to (2.25), we obtain

$$\tau_j = 0.6527 + 2.8099j \quad (j = 0, 1, \ldots).$$

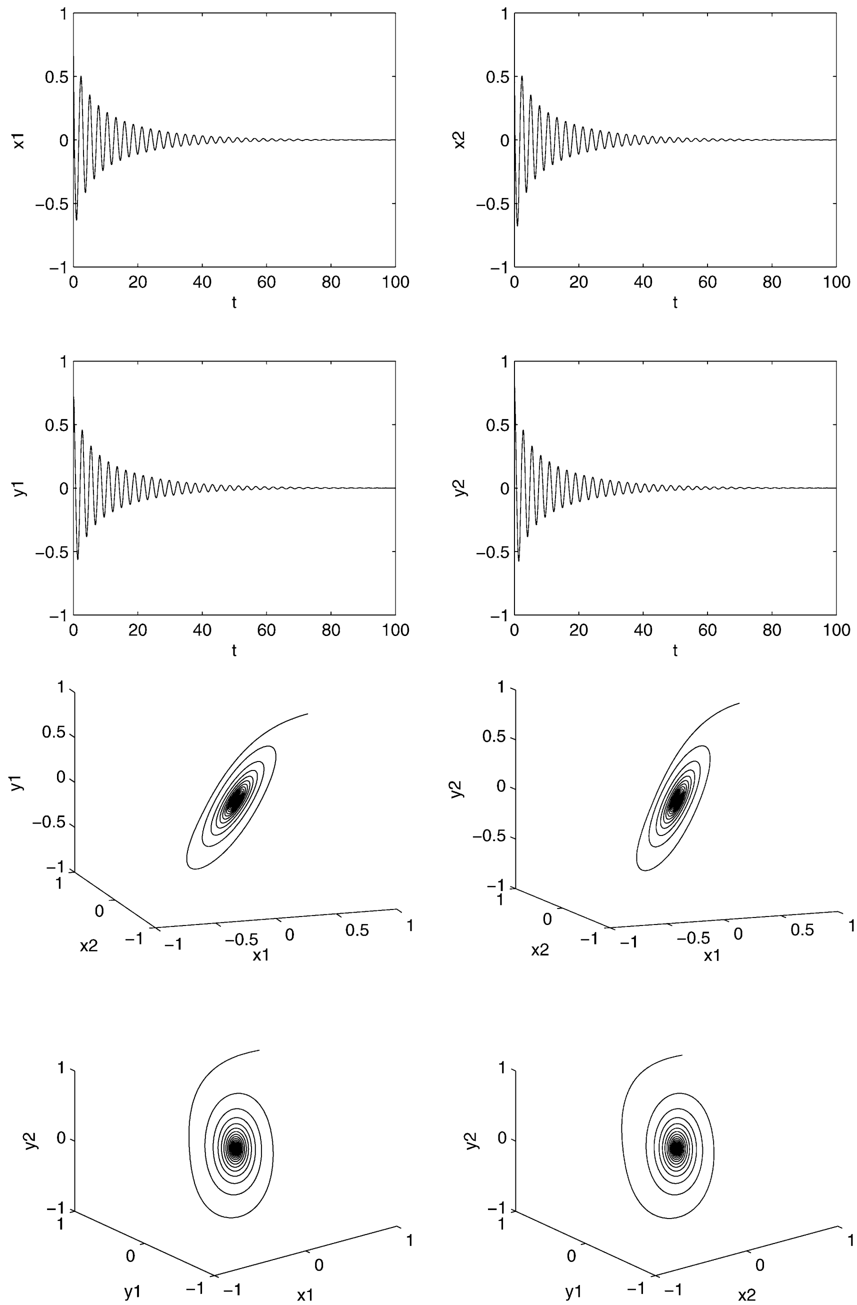

Fig. 1. $\tau = 0.6 < \tau_0$, the zero steady state is stable.

First, we choose $\tau = 0.6 < \tau_0$, the corresponding waveform and phase plots are shown in Fig. 1, by Theorem 2.2, we know its zero solution is asymptotically stable.

Finally, we choose $\tau = 0.7 > \tau_0$, the corresponding waveform and phase plots are shown in Fig. 2. It is easy to see that in Fig. 2 undergoes a Hopf bifurcation.

With these parameters, $\mu_2 > 0$. Hence, by Theorem 3.1, we know that the bifurcating point is supercritical. Correspondingly, $\beta_2 = -0.962$, and so these bifurcating periodic solutions are stable.

5. Conclusions

The bidirectional associative memory neural networks provide rich dynamical behaviors. From the viewpoint of non-linear systems, their analysis are useful in solving problems of both theoretical and practical importance. In this Letter, a four-neuron BAM neural network with four time delays has been studied, they are potentially useful as the complexity found might be carried over to a general BAM neural networks.

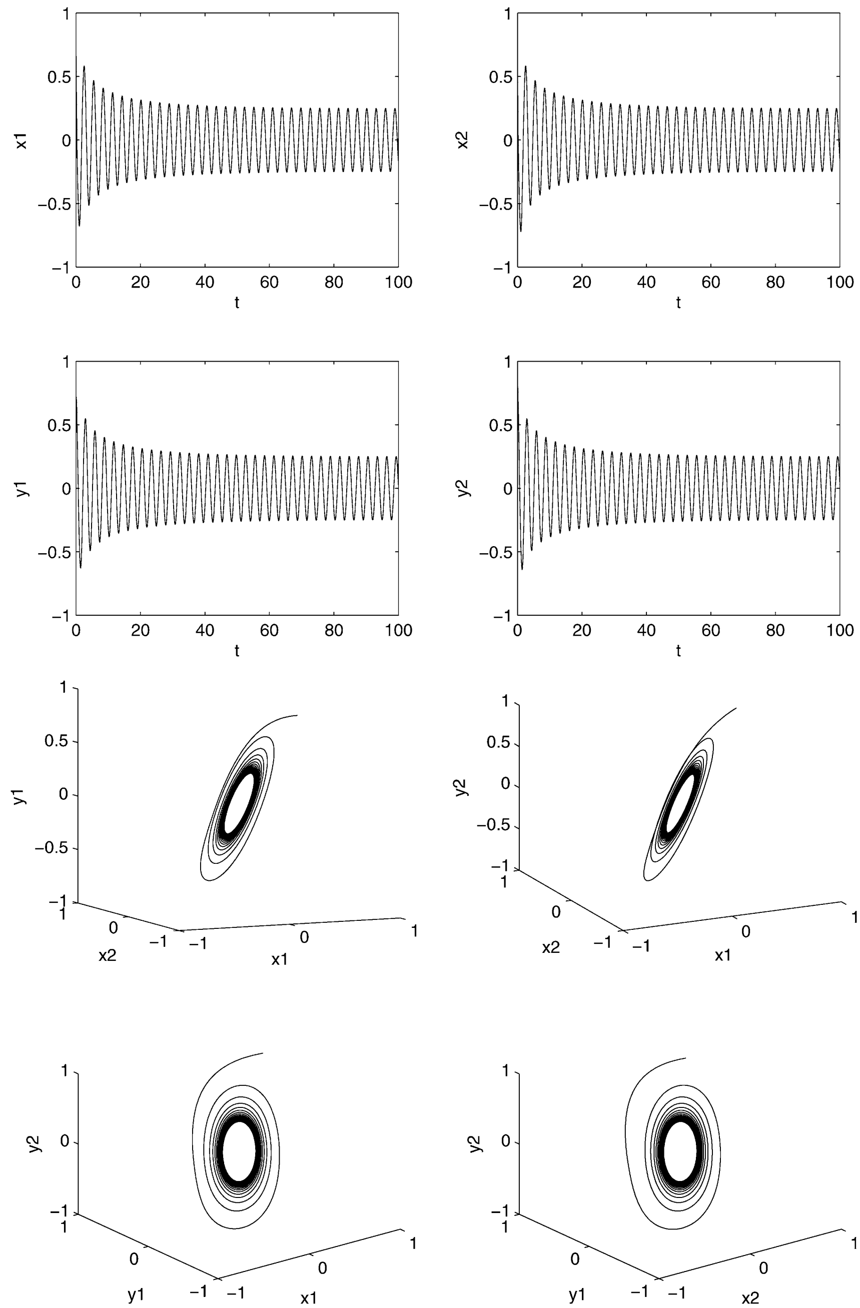

Fig. 2. $\tau = 0.7 > \tau_0$, the bifurcating periodic solution is stable.

By calling the time delay as a parameter, we have shown that a Hopf bifurcation occurs when this parameter passes through a critical value. The direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are also discussed.

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