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# Hopf bifurcation and stability of periodic solutions for van der Pol equation with time delay☆

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**Abstract**

In this paper, the van der Pol equation with a time delay is considered, where the time delay is regarded as a parameter. It is found that Hopf bifurcation occurs when this delay passes through a sequence of critical value. A formula for determining the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions is given by using the normal form method and center manifold theorem.


*Keywords:* Van der Pol equation; Time delay; Hopf bifurcation; Periodic solutions

## 1. Introduction

The well-known van der Pol equation, which describes the oscillations is the second-order nonlinear damped system governed by

$$\begin{cases} \dot{x}(t) = y(t) - f(x(t)), \\ \dot{y}(t) = -x(t). \end{cases} \tag{1.1}$$

This model is considered as one of the most intensely studied system in nonlinear system [9,12,23] and has served as a basic model in physics, electronics, biology, neurology and

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so on. Many efforts have been made to find its approximate solutions [1,10,24]. As we know, in ordinary differential equation, a non-constant periodic solution can arise a Hopf bifurcation. This occurs when a eigenvalue crosses the imaginary axis from left to right as a real parameter in the equation passes through a critical value [15,21]. In [10], the authors proposed a class of relaxation algorithms for finding the periodic steady-state solution of a van der Pol oscillation. In [1], the periodic solution of van der Pol equation is given in the form of a series converging for all values of the damping parameter. Recently, dynamical systems with time delays have been found in neural networks [2–8,11,18,20]. It is worth noting that the dynamical characteristics (including stable, unstable, oscillatory, and chaotic behavior) of neural networks with time delays have become a subject of intense research activities ([2–8,11,18,20] and the references cited therein), and neural networks involving persistent oscillations such as limit cycle may be applied to pattern recognition and associative memory. Thus it is of great interest to understand the mechanism of neural networks that cause and sustain such periodic activities. However, neural networks are complex and large-scale nonlinear dynamical systems. For simplicity, many researchers have directed their attention to the study of simple systems. This is still useful since the complexity found may be carried over to large networks. In [22], Murakaimi introduced a discrete time delay into the van der Pol equation (1.1) and the following delayed differential equation was obtained:

$$\begin{cases} \dot{x}(t) = y(t-\tau) - f(x(t-\tau)), \\ \dot{y}(t) = -x(t-\tau). \end{cases} \tag{1.2}$$

And the author discussed in detail the existence of periodic solution by using the center manifold approaches. However, the stability of bifurcating periodic solution was not discussed. Clearly, if $\tau=0$, (1.2) can reduce to (1.1). In [17], Liao studied Hopf bifurcation and stability of periodic solutions for van der Pol equation with distributed delay. Therefore, dynamical analysis of time-delay systems is an important topic in many fields [13,14,16,19,25].

In this paper, we will consider the van der Pol equation with a discrete delay, and study the existence of a Hopf bifurcation and the stability of bifurcating periodic solutions of Eq. (1.2). The obtained results find that both of them depend on the parameters $a$ and the delay $\tau$.

The organization of this paper is as follows: In Section 2, we will discuss the stability of the trivial solutions and the existence of Hopf bifurcation. In Section 3, a formula for determining the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions will be given by using the normal form method and center manifold theorem introduced by Hassard et al. [14]. In Section 4, numerical simulations aimed at justifying the theoretical analysis will be reported.

## 2. Stability of trivial solutions and existence of Hopf bifurcation

In this section, we consider the following delayed differential equation:

$$\begin{cases} \dot{x}(t) = y(t-\tau) - f(x(t-\tau)), \\ \dot{y}(t) = -x(t-\tau), \end{cases} \tag{2.1}$$

where $f(x) = ax + bx^2$, $a$ and $b$ are positive and real. The linear equation of (2.1) are as follows:

$$\begin{cases} \dot{x}(t) = -ax(t-\tau) + y(t-\tau), \\ \dot{y}(t) = -x(t-\tau). \end{cases} \tag{2.2}$$

Clearly, the point (0, 0) is the unique equilibria of (2.2).

The characteristic equation of the linearized system (2.2) is

$$det \begin{pmatrix} \lambda + a\mathrm{e}^{-\lambda\tau} & -\mathrm{e}^{-\lambda\tau} \\ \mathrm{e}^{-\lambda\tau} & \lambda \end{pmatrix} = 0. \tag{2.3}$$

By simple calculation, we obtain the following characteristic equation:

$$\lambda^2 + a\lambda\mathrm{e}^{-\lambda\tau} + \mathrm{e}^{-2\lambda\tau} = 0. \tag{2.4}$$

**Lemma 2.1.** (i) *If* $a > 2$ *holds, then* (2.4) *has a pair of purely imaginary roots* $\pm \mathrm{i}w_{0l}$ *when* $\tau = \tau_{nl}$ $(l = 1, 2; n = 0, 1, 2, \ldots)$, *where*

$$w_{0l} = \frac{a \pm \sqrt{a^2 - 4}}{2}, \tag{2.5}$$

$$\tau_{nl} = \frac{2n\pi + \pi/2}{w_{0l}} \quad (l = 1, 2; n = 0, 1, 2, \ldots). \tag{2.6}$$

(ii) *If* $0 < a \leqslant 2$ *holds, then* (2.4) *has a pair of purely imaginary roots* $\pm \mathrm{i}w_0$ *when* $\tau = \tau_j$ $(j = 0, 1, 2, \ldots)$, *where*

$$w_0 = 1, \tag{2.7}$$

$$\tau_j = \arcsin \frac{a}{2} + 2j\pi. \tag{2.8}$$

**Proof.** Suppose that $\lambda = \mathrm{i}w$, with $w > 0$, is a root of Eq. (2.4), then we obtain

$$-w^2 + a\mathrm{i}w(\cos(w\tau) - \mathrm{i}\sin(w\tau)) + (\cos(2w\tau) - \mathrm{i}\sin(2w\tau)) = 0.$$

Separating the real and imaginary parts, we have

$$\begin{cases} w^2 - aw\sin(w\tau) - \cos(2w\tau) = 0, \\ aw\cos(w\tau) - \sin(2w\tau) = 0. \end{cases} \tag{2.9}$$

Which can be rewritten as the following two equations:

$$\begin{cases} w^2 - aw\sin(w\tau) - \cos(2w\tau) = 0, \\ \cos(w\tau) = 0, \end{cases} \tag{2.10}$$

or

$$\begin{cases} w^2 - aw\ \sin(w\tau) - \cos(2w\tau) = 0, \\ \sin(w\tau) = \dfrac{aw}{2}. \end{cases} \tag{2.11}$$

Next we discuss it in two cases:

*Case* I: From $\cos(w\tau)=0$, it follows that $\cos(2w\tau)=-1$. If $\sin(w\tau)=-1$, then substituting this into the first equation of (2.10), we have $w^2 + aw + 1 = 0$. Suppose $w_1$ and $w_2$ are the two roots of this equation. Clearly,

$$w_1 + w_2 = -a < 0, \quad w_1 w_2 = 1 > 0.$$

The equation have two negative roots. Since we suppose $w > 0$, we choose $\sin(w\tau) = 1$. Substituting this into the first equation of (2.10), we obtain

$$w^2 - aw + 1 = 0. \tag{2.12}$$

For the case $a > 2$, we have two roots of Eq. (2.12): $w_{0l} = a \pm \sqrt{a^2 - 4}/2$, thus $\tau_{nl} = 2n\pi + \pi/2/w_{0l}$ $(l = 1, 2;\ n = 0, 1, 2, \ldots)$. The proof is completed.

*Case* II: From $\sin(w\tau) = aw/2$, we have $\cos(2w\tau) = 1 - 2\sin^2(w\tau) = 1 - a^2w^2/2$. Substituting these into the first equation of (2.11), from $0 < a \leqslant 2$, we obtain $w_0 = 1$ and $\tau_j = \arcsin\ a/2 + 2j\pi$. The proof is completed.

Denote

$$\lambda(\tau) = \alpha(\tau) + \mathrm{i}w(\tau),$$

is the root of Eq. (2.4) satisfying

$$\alpha(\tau_{nl}) = 0, \quad w(\tau_{nl}) = w_{0l},$$

where $\tau_{nl}$ is defined by (2.6). □

**Lemma 2.2.** *If $a > 2$ holds, then we have*

$$\frac{\mathrm{d}\alpha(\tau_{nl})}{\mathrm{d}\tau} > 0. \tag{2.13}$$

**Proof.** Taking the derivative of $\lambda$ with respect to $\tau$ in (2.4), we have

$$2\lambda\lambda' + a\lambda' \mathrm{e}^{-\lambda\tau} + a\lambda(-\lambda)\mathrm{e}^{-\lambda\tau} + a\lambda(-\lambda'\tau)\mathrm{e}^{-\lambda\tau} - 2(\lambda + \lambda'\tau)\mathrm{e}^{-2\lambda\tau} = 0,$$

it follows that:

$$\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau} = \frac{a\lambda^2\mathrm{e}^{-\lambda\tau} + 2\lambda\mathrm{e}^{-2\lambda\tau}}{2\lambda + a\mathrm{e}^{-\lambda\tau} - a\lambda\tau\mathrm{e}^{-\lambda\tau} - 2\tau\mathrm{e}^{-2\lambda\tau}}. \tag{2.14}$$

For the sake of simplicity, we denote $w_{0l}$ and $\tau_{nl}$ by $w$, $\tau$, respectively, then

$$\begin{aligned}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} &= \frac{-aw^2\mathrm{e}^{-\mathrm{i}w\tau} + 2\mathrm{i}w\mathrm{e}^{-2\mathrm{i}w\tau}}{2\mathrm{i}w + a\mathrm{e}^{-\mathrm{i}w\tau} - a\mathrm{i}w\tau\mathrm{e}^{-\mathrm{i}w\tau} - 2\tau\mathrm{e}^{-2\mathrm{i}w\tau}}\\
&= \frac{-aw^2[\cos(w\tau)-\mathrm{i}\ \sin(w\tau)]+2\mathrm{i}w[\cos(2w\tau)-\mathrm{i}\ \sin(2w\tau)]}{2\mathrm{i}w+a[\cos(w\tau)-\mathrm{i}\ \sin(w\tau)]-a\mathrm{i}w\tau[\cos(w\tau)-\mathrm{i}\ \sin(w\tau)]-2\tau[\cos(2w\tau)-\mathrm{i}\ \sin(2w\tau)]}\\
&= \frac{[-aw^2\ \cos(w\tau)+2w\ \sin(2w\tau)]+\mathrm{i}[aw^2\ \sin(w\tau)+2w\ \cos(2w\tau)]}{[a\ \cos(w\tau)-aw\tau\ \sin(w\tau)-2\tau\ \cos(2w\tau)]+\mathrm{i}[2w-a\ \sin(w\tau)-aw\tau\ \cos(w\tau)+2\tau\ \sin(2w\tau)]}.\end{aligned}$$

Let

$$\begin{aligned}Q &= [a\ \cos(w\tau) - aw\tau\ \sin(w\tau) - 2\tau\ \cos(2w\tau)]^2\\
&\quad + [2w - a\ \sin(w\tau) - aw\tau\ \cos(w\tau) + 2\tau\ \sin(2w\tau)]^2.\end{aligned}$$

$$\begin{aligned}Q\,\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right) &= [-aw^2\ \cos(w\tau) + 2w\ \sin(2w\tau)]\\
&\quad \times\ [a\ \cos(w\tau) - aw\tau\ \sin(w\tau) - 2\tau\ \cos(2w\tau)]\\
&\quad + [aw^2\ \sin(w\tau) + 2w\ \cos(2w\tau)]\\
&\quad \times\ [2w - a\ \sin(w\tau) - aw\tau\ \cos(w\tau) + 2\tau\ \sin(2w\tau)] \qquad (2.15)\end{aligned}$$

$$\begin{aligned}Q\,\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\bigg|_{\tau=\tau_{nl}} &= (aw^2 - 2w)(2w - a)\\
&= w(2aw^2 - a^2w - 4w + 2a)\\
&= w(2a^2w - 2a - a^2w - 4w + 2a)\\
&= w^2(a^2 - 4) > 0,\end{aligned}$$

this completes the proof.

Denote

$$\lambda(\tau) = \alpha(\tau) + \mathrm{i}w(\tau),$$

is the root of Eq. (2.4) satisfying

$$\alpha(\tau_j) = 0, \quad w(\tau_j) = w_0,$$

where $\tau_j$ is defined by (2.8). □

**Lemma 2.3.** *If* $0 < a \leqslant 2$ *holds, then*

$$\frac{\mathrm{d}\alpha(\tau_j)}{\mathrm{d}\tau} > 0. \qquad (2.16)$$

**Proof.** If $0 < a < 2$, from (2.15) we have

$$
\begin{aligned}
Q\,\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\Big|_{\tau=\tau_j} &= [-a\cos\tau + 2\sin(2\tau)][a\cos\tau - a^2\tau/2 - 2\tau(1-a^2)/2] \\
&\quad + [a^2/2 + 2 - a^2][2 - a^2/2 - a\tau\cos\tau + 2\tau\sin(2\tau)] \\
&= a\cos\tau(a\cos\tau + a^2\tau/2 - 2\tau) \\
&\quad + (2 - a^2/2)(2 - a^2/2 + a\tau\cos\tau) \\
&= a^2\cos^2\tau + a\tau\cos\tau(a^2/2 - 2) \\
&\quad + (2 - a^2/2)a\tau\cos\tau + (2 - a^2/2)^2 \\
&= a^2(1 - a^2/4) + (4 + a^4/4 - 2a^2) \\
&= a^2 - a^4/4 + 4 + a^4/4 - 2a^2 \\
&= 4 - a^2 > 0.
\end{aligned}
$$

If $a = 2$ holds, then (2.4) is equivalent to the following equation:

$$\lambda + \mathrm{e}^{-\lambda\tau} = 0. \tag{2.17}$$

Here, $\mathrm{w} = 1$, $\tau = \pi/2$. Using the same method as Lemma 2.2, we can obtain

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\Big|_{\tau=\tau_j} = 1 > 0.$$

This completes the proof. □

**Lemma 2.4.** *For Eq.* (2.4).

(I) *If* $a > 2$, *all the roots of Eq.* (2.4) *have strictly negative real parts for* $\tau \in [0, \tau_{01})$, *and Eq.* (2.4) *has a pair of imaginary roots* $\pm \mathrm{i}w_{01}$ *and all the other roots have strictly negative real parts when* $\tau = \tau_{01}$, *as well as Eq.* (2.4) *has at least a pair of roots with positive real parts when* $\tau > \tau_{01}$.

(II) *If* $0 < a \leqslant 2$, *all the roots of Eq.* (2.4) *have strictly negative real parts for* $\tau \in [0, \tau_0)$, *and Eq.* (2.4) *has a pair of imaginary roots* $\pm \mathrm{i}w_0$ *and all the other roots have strictly negative real parts when* $\tau = \tau_0$, *as well as Eq.* (2.4) *has at least a pair of roots with positive real parts when* $\tau > \tau_0$.

**Proof.** (I) Obviously, the roots of Eq. (2.4) continuously depend on the parameter $\tau$. When $\tau = 0$, we know that (2.4) has two roots $\lambda_1$ and $\lambda_2$, the product of the roots satisfy

$$\lambda_1 + \lambda_2 = -a(a > 0), \quad \lambda_1\lambda_2 = 1.$$

We obtain that $\lambda_1$ and $\lambda_2$ both have negative real parts. $\tau_{01}$ is the smallest positive value when Eq. (2.4) has a pair of purely imaginary roots. Since the roots of Eq. (2.4) continuously depend on the parameter $\tau$, we know that the roots of Eq. (2.4) have negative real parts when $\tau \in [0, \tau_{01})$.

Next we show that Eq. (2.4) has a pair of imaginary roots $\pm \mathrm{i}w_{01}$ and all the other roots have strictly negative real parts when $\tau = \tau_{01}$. Suppose, on the contrary, that $\lambda = u + \mathrm{i}v$ with $u > 0$ is a root of Eq. (2.4) with $\tau = \tau_{01}$. Since the roots of Eq. (2.4) continuously depend on the parameter $\tau$, there exists $\tau' \in (0, \tau_{01})$ such that (2.4) has a purely imaginary root at $\tau = \tau'$, which contradicts with the fact that $\tau_{01}$ is the smallest such $\tau$.

Thirdly, we will show Eq. (2.4) has at least a pair of roots with positive real parts when $\tau > \tau_{01}$. From Lemma 2.2, we have $\mathrm{d}\alpha(\tau_{01})/\mathrm{d}\tau > 0$. The proof is completed.

(II) The approach is the same as (I), we omit it. □

**Theorem 2.1.** *For Eq.* (2.1).

(I) *If* $a > 2$ *and from Lemma* 2.4(I), *we obtain its zero solution is asymptotically stable for* $\tau \in [0, \tau_{01})$, *and unstable for* $\tau > \tau_{01}$, *and Eq.* (2.1) *undergoes a Hopf bifurcation at the origin when* $\tau = \tau_{01}$. *That is, system* (2.1) *has a branch of periodic solutions bifurcating from the zero solution near* $\tau = \tau_{01}$.

(II) *If* $0 < a \leqslant 2$ *and from Lemma* 2.4(II), *we obtain its zero solution is asymptotically stable for* $\tau \in [0, \tau_0)$, *and unstable for* $\tau > \tau_0$, *and Eq.* (2.1) *undergoes a Hopf bifurcation at the origin when* $\tau = \tau_0$. *That is, system* (2.1) *has a branch of periodic solutions bifurcating from the zero solution near* $\tau = \tau_0$.

**Remark.** In [17], it is simpler than our models since the characteristic equation of ours is transcendental equation corresponding to polynomial equation of Liao. So he discussed the local stability and existence of Hopf bifurcation using Routh–Hurwitz criteria. The stability and existence of Hopf bifurcation which is studied in our paper are not as simple as his.

## 3. Stability of bifurcating periodic solutions

In this section, formulae for determining the direction of Hopf bifurcation and stability of bifurcating periodic solutions of system (2.1) at $\tau_0$ shall be presented by employing the normal form method and center manifold theorem introduced by Hassard et al. [14].

For convenience, let $t = s\tau$, $x(s\tau) = x_1(s)$, $y(s\tau) = x_2(s)$ and $\tau = \tau_0 + \mu$, $\mu \in R$. Denote $t = s$, then system (2.1) is equivalent to the system:

$$\begin{cases} \dot{x}_1(t) = (\tau_0 + \mu)(-ax_1(t-1) + x_2(t-1) - bx_1^2(t-1)), \\ \dot{x}_2(t) = -(\tau_0 + \mu)x_1(t-1). \end{cases} \tag{3.1}$$

Its linear part is given by

$$\begin{cases} \dot{x}_1(t) = (\tau_0 + \mu)(-ax_1(t-1) + x_2(t-1)), \\ \dot{x}_2(t) = -(\tau_0 + \mu)x_1(t-1). \end{cases} \tag{3.2}$$

The nonlinear part of (3.1) is

$$f(\mu, u_t) = (\tau_0 + \mu)\begin{pmatrix} -bx_1^2(t-1) \\ 0 \end{pmatrix}. \tag{3.3}$$

Denote $C^k[-1,0]=\{\varphi|\varphi:[-1,0]\to R^2$, each component of $\varphi$ has $k$ order continuous derivative$\}$. For convenience, denote $C[-1,0]$ by $C^0[-1,0]$. The solutions map continuous initial data into $R^2$. We are interested in periodic solutions. For $\phi(\theta)=(\phi_1(\theta)\ \phi_2(\theta))^{\mathrm{T}}\in C[-1,0]$, define an operator

$$L_\mu\phi=(\tau_0+\mu)\begin{pmatrix}-a & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\phi_1(-1)\\ \phi_2(-1)\end{pmatrix}, \tag{3.4}$$

where $L_\mu$ is a one-parameter family of bounded linear operators in $C[-1,0]\to R^2$. By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta,\mu)$ in $[-1,0]\to R^2$, such that

$$L_\mu\phi=\int_{-1}^{0}\mathrm{d}\eta(\theta,\mu)\phi(\theta).$$

In fact, we choose

$$\eta(\theta,\mu)=(\tau_0+\mu)\begin{pmatrix}-a & 1\\ -1 & 0\end{pmatrix}\delta(\theta+1) \tag{3.5}$$

(where $\delta(\theta)$ is Dirac function), then (3.4) is satisfied.

For $\phi\in C^1[-1,0]$, define

$$A(\mu)\phi=\begin{cases}\dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & -1\leqslant\theta<0,\\ \int_{-1}^{0}\mathrm{d}\eta(\theta,\mu)\phi(\theta), & \theta=0\end{cases} \tag{3.6}$$

and

$$R(\mu)\phi=\begin{cases}\begin{pmatrix}0\\0\end{pmatrix}, & -1\leqslant\theta<0,\\ (\tau_0+\mu)\begin{pmatrix}-b\phi_1^2(-1)\\ 0\end{pmatrix}, & \theta=0.\end{cases} \tag{3.7}$$

In order to conveniently study Hopf bifurcation problem, we transform system (3.1) into a operator equation of the form:

$$\dot{u_t}=A(\mu)u_t+Ru_t, \tag{3.8}$$

where $u=(x_1,x_2)^{\mathrm{T}}$. As in [13], $u_t=u(t+\theta)$, $\theta\in(-1,0]$.

The adjoint operator $A^*$ of A is defined by

$$A^*(\mu)\psi=\begin{cases}-\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & 0<s\leqslant 1,\\ \int_{-1}^{0}\mathrm{d}\eta^{\mathrm{T}}(s,\mu)\psi(-s), & s=0\end{cases} \tag{3.9}$$

where $\eta^{\mathrm{T}}$ is the transpose of the matrix $\eta$.

The domains of $A$ and $A^*$ are $C^1[-1,0]$ and $C^1[0,1]$, respectively. For $\phi \in C[-1,0]$ and $\psi \in C[0,1]$. In order to normalize the eigenvectors of operator $A$ and adjoint operator $A^*$, we need to introduce the following bilinear form:

$$\langle \psi, \phi \rangle = \bar{\psi}(0) \cdot \phi(0) - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}^{\mathrm{T}}(\xi - \theta)\, \mathrm{d}\eta(\theta)\phi(\xi)\, \mathrm{d}\xi. \tag{3.10}$$

Here $\eta(\theta) = \eta(\theta, 0)$, $C^2$ is complex plane. And for $c$ and $d$ in $C^2$, $c \cdot d$ means $\sum_{i=1}^{2} c_i d_i$, where $c_i$ and $d_i$ are components of $c$ and $d$, respectively. Then, as usual,

$$\langle \psi, A\phi \rangle = \langle A^*\psi, \phi \rangle \tag{3.11}$$

for $(\phi, \psi) \in D(A) \times D(A^*)$. We normalize $q$ and $q^*$ by the condition

$$\langle q^*, q \rangle = 1, \quad \langle q^*, \bar{q} \rangle = 0.$$

By discussion in Section 2 and transformation $t = s\tau$, we know that $\pm \mathrm{i}\tau_0 w_0$ are eigenvalues of $A(0)$ and other eigenvalues have strictly negative real parts. Thus they are also eigenvalues of $A^*$. Next we calculate the eigenvector $q$ of $A$ belonging to the eigenvalue $\mathrm{i}\tau_0 w_0$ and the eigenvector $q^*$ of $A^*$ belonging to the eigenvalue $-\mathrm{i}\tau_0 w_0$. Let

$$q(\theta) = \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \mathrm{e}^{\mathrm{i}\tau_0 w_0 \theta}, \quad -1 < \theta \leqslant 0. \tag{3.12}$$

From the above discussion, we know that

$$Aq(0) = \mathrm{i}\tau_0 w_0 q(0),$$

$$\tau_0 \begin{pmatrix} -a & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\tau_0 w_0} \\ \alpha \mathrm{e}^{-\mathrm{i}\tau_0 w_0} \end{pmatrix} = \mathrm{i}\tau_0 w_0 \begin{pmatrix} 1 \\ \alpha \end{pmatrix},$$

i.e.,

$$\begin{cases} (-a + \alpha)\mathrm{e}^{-\mathrm{i}\tau_0 w_0} = \mathrm{i}w_0, \\ -\mathrm{e}^{-\mathrm{i}\tau_0 w_0} = \mathrm{i}\alpha w_0. \end{cases}$$

Hence, we obtain

$$\alpha = a + \mathrm{i}w_0 \mathrm{e}^{\mathrm{i}\tau_0 w_0} \quad \text{or} \quad \alpha = \frac{\mathrm{i}}{w_0} \mathrm{e}^{-\mathrm{i}\tau_0 w_0}. \tag{3.13}$$

Suppose that the eigenvector $q^*$ of $A^*$ is

$$q^*(s) = \frac{1}{\rho} \begin{pmatrix} 1 \\ \beta \end{pmatrix} \mathrm{e}^{\mathrm{i}\tau_0 w_0 s}, \quad 0 \leqslant s < 1. \tag{3.14}$$

Then we have the following relationship:

$$A^* q^*(0) = -\mathrm{i}\tau_0 w_0 q^*(0).$$

$$\tau_0 \begin{pmatrix} -a & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\tau_0 w_0} \\ \beta \mathrm{e}^{\mathrm{i}\tau_0 w_0} \end{pmatrix} = -\mathrm{i}\tau_0 w_0 \begin{pmatrix} 1 \\ \beta \end{pmatrix},$$

i.e.,

$$\begin{cases} (-a-\beta)\mathrm{e}^{\mathrm{i}\tau_0 w_0} = -\mathrm{i}w_0, \\ \mathrm{e}^{\mathrm{i}\tau_0 w_0} = -\mathrm{i}\beta w_0. \end{cases}$$

Hence, we obtain

$$\beta = -a + \mathrm{i}w_0 \mathrm{e}^{-\mathrm{i}\tau_0 w_0} \quad \text{or} \quad \beta = \frac{\mathrm{i}}{w_0}\mathrm{e}^{\mathrm{i}\tau_0 w_0}. \tag{3.15}$$

Let

$$\langle q^*, q\rangle = 1.$$

One can obtain $\rho$,

$$\begin{aligned} \langle q^*, q\rangle &= \bar{q}^*(0)\cdot q(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \bar{q}^{*\mathrm{T}}(\xi-\theta)\,\mathrm{d}\eta(\theta)q(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{\bar{\rho}}(1+\bar{\beta}\alpha) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \frac{1}{\bar{\rho}}(1\ \ \bar{\beta})\mathrm{e}^{-\mathrm{i}\tau_0 w_0(\xi-\theta)}\,\mathrm{d}\eta(\theta)\begin{pmatrix}1\\ \alpha\end{pmatrix}\mathrm{e}^{\mathrm{i}\tau_0 w_0 \xi}\,\mathrm{d}\xi \\ &= \frac{1}{\bar{\rho}}(1+\bar{\beta}\alpha) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \tau_0\frac{1}{\bar{\rho}}(1\ \ \bar{\beta})\begin{pmatrix}-a & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1\\ \alpha\end{pmatrix} \\ &\quad\times\ \mathrm{e}^{\mathrm{i}\tau_0 w_0\theta}\delta(\theta+1)\,\mathrm{d}\xi\,\mathrm{d}\theta \\ &= \frac{1}{\bar{\rho}}(1+\bar{\beta}\alpha) - \frac{1}{\bar{\rho}}\tau_0(a+\bar{\beta}-\alpha)\mathrm{e}^{-\mathrm{i}\tau_0 w_0} \\ &= 1. \end{aligned}$$

Hence, we have

$$\bar{\rho} = (1+\bar{\beta}\alpha) - \tau_0(a+\bar{\beta}-\alpha)\mathrm{e}^{-\mathrm{i}\tau_0 w_0}. \tag{3.16}$$

Note that $\bar{\beta} = -\alpha$, $\bar{\alpha} = -\beta$, using the same method it is easy to proof $\langle q^*, \bar{q}\rangle = 0$, we omit it. Now we obtain $q$ and $q^*$.

Next, we study the stability of bifurcating periodic solutions. As in [14], the bifurcating periodic solutions $Z(t, \mu(\varepsilon))$ has amplitude $\mathrm{O}(\varepsilon)$ and non-zero Floquet exponent $\beta(\varepsilon)$ with $\beta(0) = 0$. Under our hypotheses, $\mu$, $\beta$ are given by

$$\begin{cases} \mu = \mu_2\varepsilon^2 + \mu_4\varepsilon^4 + \cdots, \\ \beta = \beta_2\varepsilon^2 + \beta_4\varepsilon^4 + \cdots. \end{cases} \tag{3.17}$$

The sign of $\mu_2$ indicates the direction of bifurcation while that $\beta_2$ determines the stability of $Z(t, \mu(\varepsilon))$. $Z(t, \mu(\varepsilon))$ is stable if $\beta_2 < 0$ and unstable if $\beta_2 > 0$. In the following, we will show how to derive the coefficients in this expansions, but we compute $\mu_2$ and $\beta_2$ only.

We first construct the coordinates to describe a center manifold $\Omega_0$ near $\mu = 0$, which is a local invariant, attracting a two-dimensional manifold [14]. Let

$$z(t) = \langle q^*, u_t\rangle \tag{3.18}$$

and

$$W(t,\theta)=u_t-2\,\mathrm{Re}[z(t)q(\theta)]. \tag{3.19}$$

Where $u_t$ is a solution of (3.8). On the manifold $\Omega_0 : w(t,\theta)=w(z(t),\bar{z}(t),\theta)$, where

$$W(z,\bar{z},\theta)=W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\bar{z}+W_{02}(\theta)\frac{\bar{z}^2}{2}+\cdots. \tag{3.20}$$

In fact, $z$ and $\bar{z}$ are local coordinates of center manifold $\Omega_0$ in the direction of $q$ and $q^*$, respectively.

The existence of center manifold $\Omega_0$ enables us to reduce (3.8) to an ordinary differential equation in a single complex variable on $\Omega_0$. For the solution $u_t \in \Omega_0$ of (3.8), since $\mu=0$,

$$\begin{aligned}\dot{z}(t)&=\langle q^*,\dot{u}_t\rangle\\&=\langle q^*,Au_t+Ru_t\rangle\\&=\langle q^*,Au_t\rangle+\langle q^*,Ru_t\rangle\\&=\langle A^*q^*,u_t\rangle+\langle q^*,Ru_t\rangle\\&=\mathrm{i}\tau_0w_0z+\bar{q}^*(0)\cdot f(0,W(t,0)+2\,\mathrm{Re}[z(t)q(0)]).\end{aligned} \tag{3.21}$$

Rewrite (3.21) as

$$\dot{z}(t)=\mathrm{i}\tau_0w_0z+g(z,\bar{z}), \tag{3.22}$$

where

$$g(z,\bar{z})=g_{20}\frac{z^2}{2}+g_{11}z\bar{z}+g_{02}\frac{\bar{z}^2}{2}+g_{21}\frac{z^2\bar{z}}{2}+\cdots. \tag{3.23}$$

In the following, our motivation is to expand $g$ in powers of $z$ and $\bar{z}$ and then obtain, from the coefficients of this expansion, the values of $\mu_2$ and $\beta_2$ using algorithm presented by Hassard et al. [14]. Substituting (3.8) and (3.21) into

$$\dot{W}=\dot{u}_t-\dot{z}q-\dot{\bar{z}}\bar{q},$$

we have

$$\begin{aligned}\dot{W}&=\dot{u}_t-\dot{z}q-\dot{\bar{z}}\bar{q}\\&=Au_t+Ru_t-[\mathrm{i}\tau_0w_0z+\bar{q}^*(0)\cdot f(z,\bar{z})]q-[-\mathrm{i}\tau_0w_0\bar{z}+q^*(0)\cdot\bar{f}(z,\bar{z})]\bar{q}\\&=AW+2A\,\mathrm{Re}(zq)+Ru_t-2\,\mathrm{Re}[\bar{q}^*(0)\cdot f(z,\bar{z})q(\theta)]-2\,\mathrm{Re}[\mathrm{i}\tau_0w_0zq(\theta)]\\&=AW-2\,\mathrm{Re}[\bar{q}^*(0)\cdot f(z,\bar{z})q(\theta)]+Ru_t\\&=\begin{cases}AW-2\,\mathrm{Re}[\bar{q}^*(0)\cdot f(z,\bar{z})q(\theta)], & -1\leqslant\theta<0,\\ AW-2\,\mathrm{Re}[\bar{q}^*(0)\cdot f(z,\bar{z})q(\theta)]+f, & \theta=0.\end{cases}\end{aligned} \tag{3.24}$$

Let

$$\dot{W}=AW+H(z,\bar{z},\theta), \tag{3.25}$$

where

$$H(z,\bar{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\bar{z}+H_{02}(\theta)\frac{\bar{z}^2}{2}+\cdots. \tag{3.26}$$

Taking the derivative of $W$ with respect to $t$ in (3.20), we have

$$\dot{W} = W_z \dot{z} + W_{\bar{z}} \dot{\bar{z}}. \tag{3.27}$$

Substituting (3.20) and (3.22) into (3.27), we obtain

$$\begin{aligned}\dot{W} &= (W_{20} z + W_{11} \bar{z} + \cdots)(\mathrm{i}\tau_0 w_0 z + g) \\ &\quad + (W_{11} z + W_{02} \bar{z} \cdots)(-\mathrm{i}\tau_0 w_0 \bar{z} + \bar{g}).\end{aligned} \tag{3.28}$$

Then substituting (3.20) and (3.26) into (3.25), we have the following results:

$$\dot{W} = (AW_{20} + H_{20}) \frac{z^2}{2} + (AW_{11} + H_{11}) z\bar{z} + (AW_{02} + H_{02}) \frac{\bar{z}^2}{2} + \cdots. \tag{3.29}$$

Comparing the coefficients of (3.28) with (3.29),

$$(A - 2\mathrm{i}\tau_0 w_0) W_{20}(\theta) = -H_{20}(\theta), \tag{3.30}$$

$$AW_{11}(\theta) = -H_{11}(\theta) \tag{3.31}$$

hold.

According to (3.21) and (3.22), we know

$$g(z, \bar{z}) = \bar{q}^*(0) \cdot f(z, \bar{z}) = \frac{\tau_0}{\bar{\rho}} \begin{pmatrix} 1 \\ \bar{\beta} \end{pmatrix} \cdot \begin{pmatrix} -bx_1^2(t-1) \\ 0 \end{pmatrix}, \tag{3.32}$$

where

$$x_1(t + \theta) = W^{(1)}(t, \theta) + z(t) q^{(1)}(\theta) + \bar{z}(t) \bar{q}^{(1)}(\theta),$$

$$x_2(t + \theta) = W^{(2)}(t, \theta) + z(t) q^{(2)}(\theta) + \bar{z}(t) \bar{q}^{(2)}(\theta).$$

From (3.32) and (3.21), we have

$$\begin{aligned} g(z, \bar{z}) &= -\frac{\tau_0 b}{\bar{\rho}} x_1^2(t-1) \\ &= -\frac{\tau_0 b}{\bar{\rho}} [W^{(1)}(t, \theta) + z(t) q^{(1)}(\theta) + \bar{z}(t) \bar{q}^{(1)}(\theta)]^2 \\ &= -\frac{\tau_0 b}{\bar{\rho}} [W_{20}^{(1)}(-1) \frac{z^2}{2} + W_{11}^{(1)}(-1) z\bar{z} + W_{02}^{(1)}(-1) \frac{\bar{z}^2}{2} \\ &\quad + z(t) q^{(1)}(-1) + \bar{z}(t) \bar{q}^{(1)}(-1)]^2. \end{aligned} \tag{3.33}$$

Comparing the coefficients in (3.23) with those in (3.33), it follows that:

$$\begin{aligned} g_{20} &= -\frac{2\tau_0 b}{\bar{\rho}} [q^{(1)}(-1)]^2, \\ g_{11} &= -\frac{2\tau_0 b}{\bar{\rho}} q^{(1)}(-1) \bar{q}^{(1)}(-1), \\ g_{02} &= -\frac{2\tau_0 b}{\bar{\rho}} [\bar{q}^{(1)}(-1)]^2, \\ g_{21} &= -\frac{2\tau_0 b}{\bar{\rho}} W_{20}^{(1)}(-1) \bar{q}^{(1)}(-1). \end{aligned} \tag{3.34}$$

In the following, we focus on the computation of $W_{20}(\theta)$. Eqs. (3.24) and (3.25) imply that

$$\begin{aligned}H(z,\bar{z},\theta) &= -2\,\mathrm{Re}(\bar{q}^*(0)\cdot f(z,\bar{z})q(\theta)) + Ru_t\\ &= -gq(\theta) - \bar{g}\bar{q}(\theta) + Ru_t\\ &= -\left(\frac{1}{2}g_{20}z^2 + g_{11}z\bar{z} + \frac{1}{2}g_{02}\bar{z}^2 + \cdots\right)q(\theta)\\ &\quad -\left(\frac{1}{2}\bar{g}_{20}\bar{z}^2 + \bar{g}_{11}z\bar{z} + \frac{1}{2}\bar{g}_{02}z^2 + \cdots\right)\bar{q}(\theta) + Ru_t.\end{aligned} \tag{3.35}$$

From (3.33), $Ru_t = \tau_0\begin{pmatrix}-bx_1^2(t-1)\\0\end{pmatrix} = \begin{pmatrix}\bar{\rho}g\\0\end{pmatrix}$, when $\theta=0$ at $\mu=0$. Comparing the coefficients in (3.26) with those in (3.35), we can obtain that

$$H_{20}(\theta) = \begin{cases}-g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), & -1\leqslant\theta<0,\\ -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + \begin{pmatrix}\bar{\rho}g_{20}\\0\end{pmatrix}, & \theta=0\end{cases} \tag{3.36}$$

and

$$H_{11}(\theta) = \begin{cases}-g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta), & -1\leqslant\theta<0,\\ -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \begin{pmatrix}\bar{\rho}g_{11}\\0\end{pmatrix}, & \theta=0.\end{cases} \tag{3.37}$$

Substituting (3.36) into (3.30) and (3.37) into (3.31) respectively, it follows that:

$$\begin{cases}\dot{W}_{20}(\theta) = 2\mathrm{i}\tau_0 w_0 W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta),\\ \dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta).\end{cases} \tag{3.38}$$

We can easily obtain the solutions of (3.38):

$$\begin{cases}W_{20}(\theta) = \dfrac{\mathrm{i}g_{20}}{\tau_0 w_0}q(0)\mathrm{e}^{\mathrm{i}\tau_0 w_0\theta} - \dfrac{\bar{g}_{02}}{3\mathrm{i}\tau_0 w_0}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\tau_0 w_0\theta} + E_1\mathrm{e}^{2\mathrm{i}\tau_0 w_0\theta},\\ W_{11}(\theta) = \dfrac{g_{11}}{\mathrm{i}\tau_0 w_0}q(0)\mathrm{e}^{\mathrm{i}\tau_0 w_0\theta} - \dfrac{\bar{g}_{11}}{\mathrm{i}\tau_0 w_0}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\tau_0 w_0\theta} + E_2.\end{cases} \tag{3.39}$$

Next we focus on the computation of $E_1$, from (3.30), we have

$$AW_{20}(0) = 2\mathrm{i}\tau_0 w_0 W_{20}(0) - H_{20}(0),$$

then

$$\tau_0\begin{pmatrix}-a & 1\\ -1 & 0\end{pmatrix}W_{20}(-1) = 2\mathrm{i}\tau_0 w_0 W_{20}(0) - H_{20}(0). \tag{3.40}$$

Substituting (3.36) and (3.39) into (3.40), we have the following relationship:

$$\begin{aligned}&\tau_0\begin{pmatrix}-a & 1\\ -1 & 0\end{pmatrix}\left(\frac{\mathrm{i}g_{20}}{\tau_0 w_0}q(0)\mathrm{e}^{-\mathrm{i}\tau_0 w_0} - \frac{\bar{g}_{02}}{3\mathrm{i}\tau_0 w_0}\bar{q}(0)\mathrm{e}^{\mathrm{i}\tau_0 w_0} + E_1\mathrm{e}^{-2\mathrm{i}\tau_0 w_0}\right)\\ &\quad = -2g_{20}q(0) - \frac{2}{3}\bar{g}_{02}\bar{q}(0) + 2\mathrm{i}\tau_0 w_0 E_1 + g_{20}q(0) + \bar{g}_{02}\bar{q}(0) - \begin{pmatrix}\bar{\rho}g_{20}\\0\end{pmatrix},\end{aligned}$$

$$\begin{pmatrix} 2\mathrm{i}\tau_0 w_0 + \tau_0 a \mathrm{e}^{-2\mathrm{i}\tau_0 w_0} & -\tau_0 \mathrm{e}^{-2\mathrm{i}\tau_0 w_0} \\ \tau_0 \mathrm{e}^{-2\mathrm{i}\tau_0 w_0} & 2\mathrm{i}\tau_0 w_0 \end{pmatrix} E_1$$
$$= \tau_0 \begin{pmatrix} -a & 1 \\ -1 & 0 \end{pmatrix} \left( \frac{\mathrm{i}g_{20}}{\tau_0 w_0} q(0)\mathrm{e}^{-\mathrm{i}\tau_0 w_0} - \frac{\bar{g}_{02}}{3\mathrm{i}\tau_0 w_0} \bar{q}(0)\mathrm{e}^{\mathrm{i}\tau_0 w_0} \right) + g_{20} q(0)$$
$$- \frac{1}{3}\bar{g}_{02}\bar{q}(0) + \begin{pmatrix} \bar{\rho} g_{20} \\ 0 \end{pmatrix}.$$

$$E_1 = \begin{pmatrix} 2\mathrm{i}\tau_0 w_0 + \tau_0 a \mathrm{e}^{-2\mathrm{i}\tau_0 w_0} & -\tau_0 \mathrm{e}^{-2\mathrm{i}\tau_0 w_0} \\ \tau_0 \mathrm{e}^{-2\mathrm{i}\tau_0 w_0} & 2\mathrm{i}\tau_0 w_0 \end{pmatrix}^{-1}$$
$$\times \left( \begin{pmatrix} -a & 1 \\ -1 & 0 \end{pmatrix} \left( \frac{\mathrm{i}g_{20}}{w_0} q(0)\mathrm{e}^{-\mathrm{i}\tau_0 w_0} - \frac{\bar{g}_{02}}{3\mathrm{i}w_0} \bar{q}(0)\mathrm{e}^{\mathrm{i}\tau_0 w_0} \right) + g_{20} q(0) \right.$$
$$\left. -\frac{1}{3}\bar{g}_{02}\bar{q}(0) + \begin{pmatrix} \bar{\rho} g_{20} \\ 0 \end{pmatrix} \right). \tag{3.41}$$

Hence, we know $W_{20}$ and then we can obtain $g_{21}$. The following parameters can be calculated:

$$C_1(0) = \frac{\mathrm{i}}{2w}\left( g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{g_{21}}{2}, \tag{3.42}$$

$$\mu_2 = -\frac{\operatorname{Re} C_1(0)}{\operatorname{Re} \lambda'(\tau_0)}, \tag{3.43}$$

$$\beta_2 = 2\operatorname{Re} C_1(0). \tag{3.44}$$

If you want to know the detail, see appendix. As in [14], we have the following result:

**Theorem 3.1.** *Under the condition of Theorem* 2.1,

(I) $\mu = 0$ *is Hopf bifurcation value of system* (3.1).

(II) *the direction of Hopf bifurcation is determined by the sign of* $\mu_2$: *if* $\mu_2 > 0$, *the Hopf bifurcation is supercritical*; *if* $\mu_2 < 0$, *the Hopf bifurcation is subcritical*.

(III) *The stability of bifurcating periodic solutions is determined by* $\beta_2$: *if* $\beta_2 < 0$, *the periodic solutions are stable*; *if* $\beta_2 > 0$, *they are unstable*.

## 4. Numerical examples

In this section, some numerical results of simulating system (2.1) are presented at different data of $a$, $b$ and $\tau$.

First, let $b = 0$. Then system (2.1) is linear, we investigate the Hopf bifurcation at $\tau_0$. We fix $a = 1.5$, then $\tau_0 = \arcsin(0.75)$. So we choose $\tau = 0.8 < \tau_0$, $\tau_0$ and $0.9 > \tau_0$, respectively. The corresponding waveform and phase plots are shown in Figs. 1–3. By Theorem 2.1, we know in Fig. 1 its zero solution is asymptotically stable, in Fig. 2 undergoes a Hopf bifurcation at the origin, and in Fig. 3 is unstable.

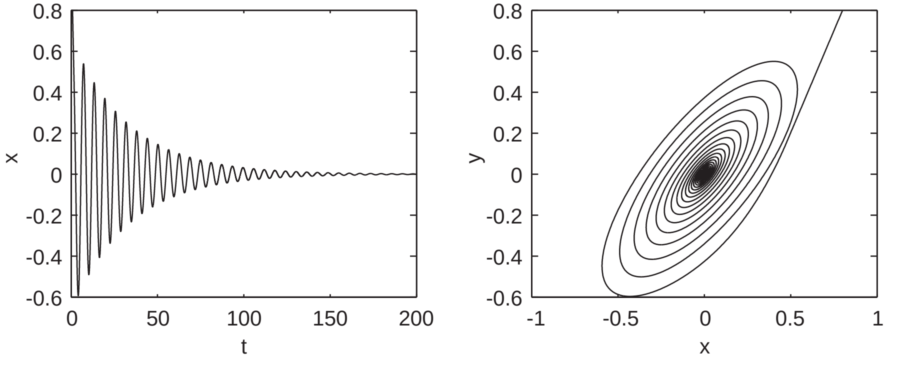


Fig. 1. $a = 1.5, \;\; b = 0, \;\; \tau < \tau_0$.

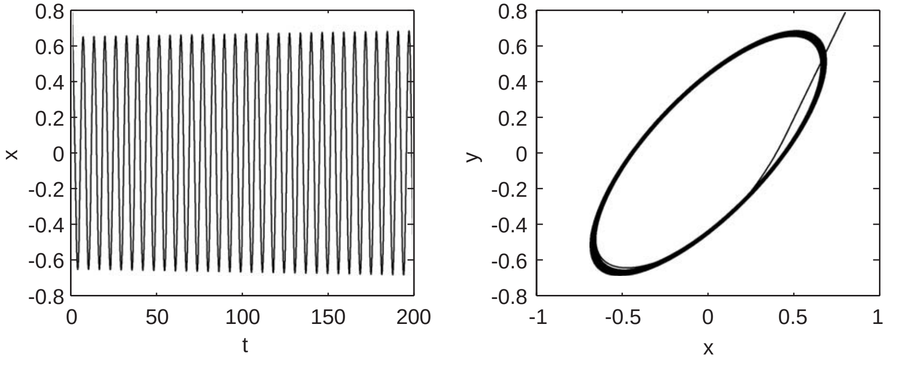


Fig. 2. $a = 1.5, \;\; b = 0, \;\; \tau = \tau_0$.

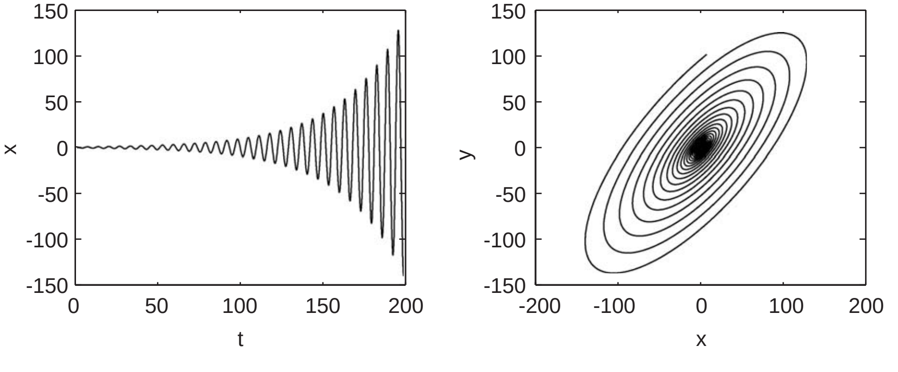


Fig. 3. $a = 1.5, \;\; b = 0, \;\; \tau > \tau_0$.

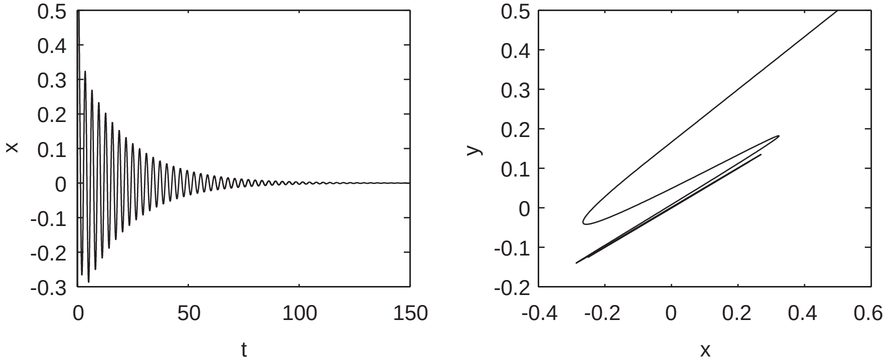


Fig. 4. $a = 2.5,\ \ b = 0,\ \ \tau < \tau_{01}$.

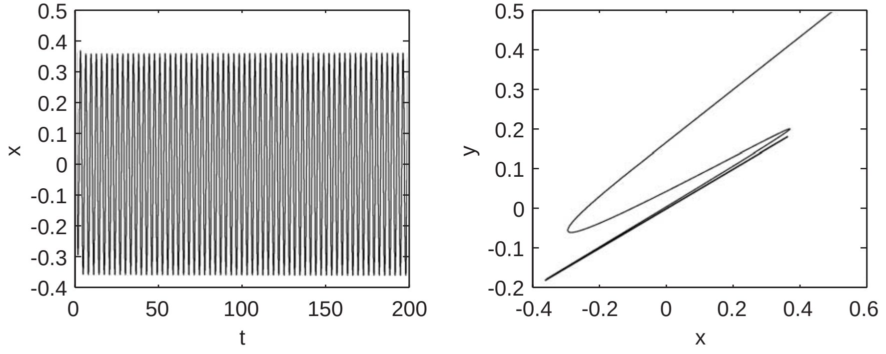


Fig. 5. $a = 2.5,\ \ b = 0,\ \ \tau = \tau_{01}$.

Next, let $b = 0$. Then system (2.1) is linear, we investigate the Hopf bifurcation at $\tau_{01}$. We fix $a = 2.5$, $\tau_{01} = \pi/(a + \sqrt{a^2 - 4})$. So we choose $\tau = 0.75 < \tau_{01}$, $\tau_{01}$ and $0.8 > \tau_{01}$, respectively. The corresponding waveform and phase plots are shown in Figs. 4–6. By Theorem 2.1, we know in Fig. 4 its zero solution is asymptotically stable, in Fig. 5 undergoes a Hopf bifurcation at the origin, and in Fig. 6 is unstable.

Finally, we fix $a = 1$ and $\tau = 0.55 > \tau_0 = \arcsin(a/2)$ and we choose $b = 1$ and 1.5, respectively. With these parameters, $\mu_2 > 0$. Hence, by Theorem 3.1, we know that the bifurcating point is supercritical. Correspondingly, $\beta_2 = -0.962$ and $-0.4416$, and so these bifurcating periodic solutions are stable, as shown in Figs. 7 and 8.

## 5. Conclusions

The van der Pol equation provides rich dynamical behavior. From the viewpoint of non-linear systems, their analysis are useful in solving problems of both theoretical and practical

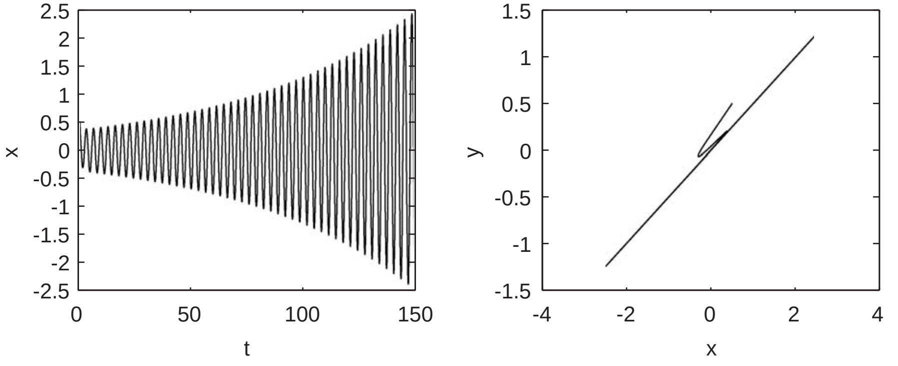


Fig. 6. $a = 2.5,\ \ b = 0,\ \ \tau > \tau_{01}$.

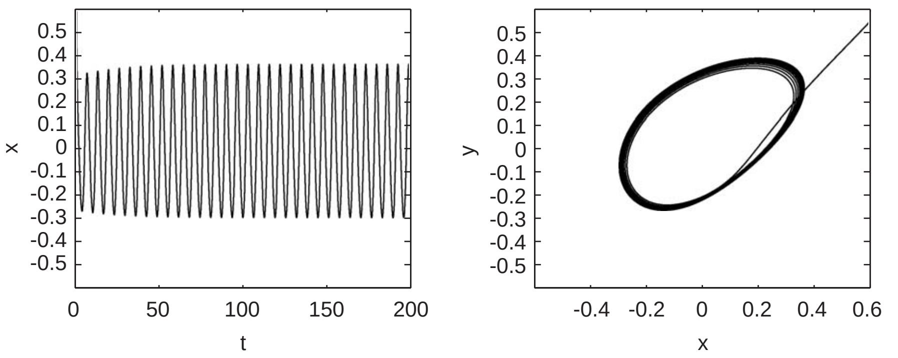


Fig. 7. $a = 1,\ \ b = 1,\ \ \tau = 0.55 > \tau_0$.

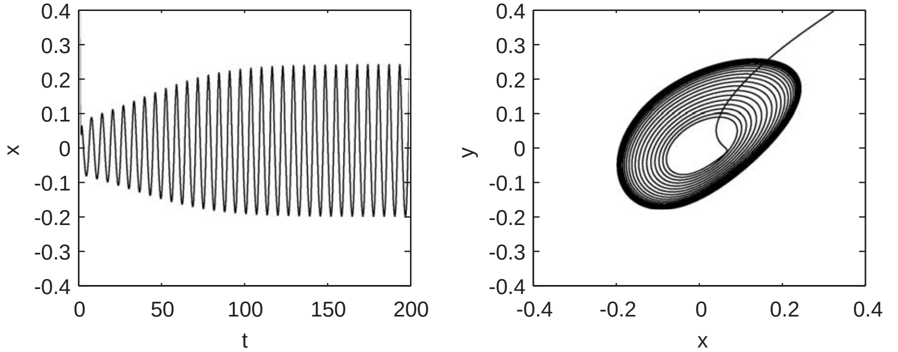


Fig. 8. $a = 1,\ \ b = 1.5,\ \ \tau = 0.55 > \tau_0$.

importance. Although the systems with time delay discussed above are quite simple, they are potentially useful as the complexity found might be carried over to a general van der Pol equation with time delays.

By calling the time delay as a parameter, we have shown that a Hopf bifurcation occurs when this parameter passes through a critical value. The direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are also discussed.

## Appendix A.

In this appendix, we want to derive formula (3.42), (3.43) and (3.44).

### *A.1. Poincaré normal form*

We assume we are given a $2 \times 2$ system in the following Poincaré norm form:

$$\begin{aligned} \dot{X} &= A(\mu)X + \sum_{j=1}^{[L/2]} B_j(\mu)X|X|^{2j} + \mathrm{o}(|X||(X,\mu)|^{L+1}), \\ &= F(X,\mu), \end{aligned} \tag{A.1}$$

where

$$A(\mu) = \begin{pmatrix} \alpha(\mu) & -\omega(\mu) \\ \omega(\mu) & \alpha(\mu) \end{pmatrix} \quad (\lambda(\mu) = \alpha(\mu) + \mathrm{i}\omega(\mu)), \tag{A.2}$$

$$B_j(\mu) = \begin{pmatrix} \operatorname{Re} c_j(\mu) & -\operatorname{Im} c_j(\mu) \\ \operatorname{Im} c_j(\mu) & \operatorname{Re} c_j(\mu) \end{pmatrix} \quad \left(1 \leqslant j \leqslant \left[\frac{L}{2}\right]\right), \tag{A.3}$$

and $F(X,\mu)$ is jointly $C^{L+2}$ in $X$ and $\mu$.

Eq. (A.1) is equivalent to

$$\dot{\xi} = \lambda(\mu)\xi + \sum_{j=1}^{[L/2]} c_j(\mu)\xi|\xi|^{2j} + \mathrm{o}(|\xi||(\xi,\mu)|^{L+1}), \tag{A.4}$$

where $\xi = X_1 + \mathrm{i}X_2$.

We next derive the formula for the initial coefficients in the MacLaurin expansions of $\mu = \mu(\varepsilon)$ and $T = T(\varepsilon)$. We begin by rewrite the differential equation (A.4) as

$$\dot{\xi} = \lambda(\mu)\xi + \xi \sum_{j=1}^{M} c_j(\mu)(\xi\bar{\xi})^j, \tag{A.5}$$

where $\xi$ is a complex variable, $\lambda(0) = \mathrm{i}\omega_0$, $M \geqslant 1$ is arbitrary, and $c_j(\mu)$ are complex valued. This canonical is in Poincaré normal form. Observe that if $\xi$ is a solution, then so is $\xi \mathrm{e}^{\mathrm{i}\phi}$ for any real number $\phi$, and trajectories of (A.5) are circles with centers at $\xi = 0$. This simple geometry is reflected in efficient computation of MacLaurin expansions of $\mu(\varepsilon)$ and $T(\varepsilon)$.

Forming $\bar{\xi}\dot{\xi} + \dot{\bar{\xi}}\xi$ from (A.5), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\xi\bar{\xi}) = 2\xi\bar{\xi}\left\{\mathrm{Re}\,\lambda(\mu) + \sum_{j=1}^{M} \mathrm{Re}\,c_j(\mu)(\xi\bar{\xi})^j\right\}. \tag{A.6}$$

The right-hand side of (A.6) is zero if and only if $\xi = 0$ or

$$\mathrm{Re}\,\lambda(\mu) + \sum_{j=1}^{M} \mathrm{Re}\,c_j(\mu)(\xi\bar{\xi})^j = 0. \tag{A.7}$$

But if (A.7) holds, then (A.6) implies that $\xi\bar{\xi} = \varepsilon^2 \geqslant 0$, for some $\varepsilon \geqslant 0$. Setting $\xi\bar{\xi} = \varepsilon^2 \geqslant 0$ and $\mu = \mu(\varepsilon)$ in (A.7) and we obtain

$$\mathrm{Re}\,\lambda(\mu(\varepsilon)) + \sum_{j=1}^{M} \mathrm{Re}\,c_j(\mu)\varepsilon^{2j} = 0. \tag{A.8}$$

This equation determines the coefficients in the expansion

$$\mu = \sum_{j=1}^{M} \mu_j \varepsilon^j + \mathrm{O}(\varepsilon^{M+1}).$$

In the following analysis below of the case $\alpha'(0) \neq 0$, the coefficients $\mu_1, \mu_3, \mu_5, \ldots$ are shown to vanish, which is a priori obvious from (A.8). Expanding $\mu$ in powers of $\varepsilon$ in (A.8), we find that

$$\begin{aligned}&\alpha'(0)\sum_{j=1}^{M}\mu_j\varepsilon^j + \frac{\alpha''(0)}{2}\left(\sum_{j=1}^{M}\mu_j\varepsilon^j\right)^2 + \cdots + \mathrm{Re}\,c_1(0)\varepsilon^2\\ &\quad + \mathrm{Re}\,c_1'(0)\left[\sum_{j=1}^{M}\mu_j\varepsilon^j\right]\varepsilon^2 + \cdots + \mathrm{Re}\,c_2(0)\varepsilon^4 + \cdots = 0.\end{aligned} \tag{A.9}$$

At $\mathrm{O}(\varepsilon)$, (A.9) implies that $\alpha'(0)\mu_1 = 0$. Thus,

$$\mu_1 = 0, \tag{A.10}$$

since $\alpha'(0) \neq 0$ by hypothesis. Using this results in (A.9), we find that at $\mathrm{O}(\varepsilon^2)$,

$$\alpha'(0)\mu_2 + \mathrm{Re}\,c_1(0) = 0,$$

$$\mu_2 = -\frac{\mathrm{Re}\,c_1(0)}{\alpha'(0)}. \tag{A.11}$$

Given that (A.8) holds, we may rewrite (A.5) as

$$\dot{\xi} = \mathrm{i}\xi\,\mathrm{Im}\left[\lambda(\mu) + \sum_{j=1}^{M} c_j(\mu)\varepsilon^{2j}\right]. \tag{A.12}$$

Thus,

$$\xi = \varepsilon \mathrm{e}^{2\pi \mathrm{i} t/T(\varepsilon)},$$

where

$$\frac{2\pi}{T(\varepsilon)} = \mathrm{Im}\left[\lambda(\mu) + \sum_{j=1}^{M} c_j(\mu)\varepsilon^{2j}\right]. \tag{A.13}$$

From this equation the coefficients in the expansion

$$T(\varepsilon) = \frac{2\pi}{\omega_0} \sum_{j=0}^{M} \tau_i \varepsilon^{\mathrm{i}} + \mathrm{O}(\varepsilon^{M+1})$$

may be found. Explicitly at O(1), (A.13) yields

$$\tau_0 = 1 \tag{A.14}$$

and, to high order

$$\omega_0\left(-\sum_{i=1}^{4} \tau_i \varepsilon^{\mathrm{i}}\right) + \omega_0\left(\sum_{i=1}^{3} \tau_i \varepsilon^{\mathrm{i}}\right)^2 + \cdots = \omega'(0)(\mu_2 + \mu_4\varepsilon^2)\varepsilon^2 + \frac{\omega''(0)}{2}\mu_2^2\varepsilon^4$$
$$+ \mathrm{Im}\, c_1(0)\varepsilon^2 + [\mathrm{Im}\, c_1'(0)\mu_2 + \mathrm{Im}\, c_2(0)]\varepsilon^4 + \cdots. \tag{A.15}$$

Hence,

$$-\omega_0\tau_1 = 0,$$

thus

$$\tau_1 = 0. \tag{A.16}$$

since $\omega_0 > 0$ by hypothesis. Then at $\mathrm{O}(\varepsilon^2)$ (A.15) becomes

$$-\omega_0\tau_2 = \omega'(0)\mu_2 + \mathrm{Im}\, c_1(0)$$

then we obtain

$$\tau_2 = -\frac{1}{\omega_0}[\mathrm{Im}\, c_1(0) + \omega'(0)\mu_2]. \tag{A.17}$$

### *A.2. Stability criteria*

We shall now apply Floquet theorem to the real 2 by 2 system (A.1). We know that $\mu(\varepsilon)$ is $C^{L+1}$ in $\varepsilon$; hence $P_\varepsilon(t) = \varepsilon y(t, \varepsilon, \mu(\varepsilon))$ is $C^{L+1}$ in $t$ and $\varepsilon$. Since $X = P_\varepsilon(t)$ is a non-constant, $T(\varepsilon)$-periodic solution of (A.1), $\dot{P}_\varepsilon(t)$ is a non-trivial, $T(\varepsilon)$-periodic solution of variational system $\dot{y} = A(t, \varepsilon)y$, where $A(t, \varepsilon) = \partial F/\partial X$ at $(P_\varepsilon(t), \mu(\varepsilon))$. Following Floquet's theorem, one of the characteristic exponents associated with $P_\varepsilon$ is thus $0 \bmod 2\pi\mathrm{i}/T(\varepsilon)$. Hence, $P_\varepsilon$ has 0 and $\beta(\varepsilon)$ as a set of characteristic exponents, where we define

$$\beta(\varepsilon) = \frac{1}{T(\varepsilon)} \int_0^{T(\varepsilon)} tr\, A(s, \varepsilon)\, \mathrm{d}s, \tag{A.18}$$

since $T(\varepsilon) = T(\varepsilon, \mu(\varepsilon))$ is $C^{L+1}$ in $\varepsilon$ and $A(t, \varepsilon)$ is $C^{L+1}$ jointly in $t$ and $\varepsilon$, the function $\beta(\varepsilon)$ is $C^{L+1}$ in $\varepsilon$. Next we will expand $\beta(\varepsilon)$.

If we write $\xi = x_1 + \mathrm{i}x_2$, then

$$\dot{x}_1 = \alpha x_1 - \omega x_2 + [(\mathrm{Re}\, c_1)x_1 - (\mathrm{Im}\, c_1)x_2]r^2 + \mathrm{O}(\varepsilon^4),$$

$$\dot{x}_2 = \omega x_1 + \alpha x_2 + [(\mathrm{Re}\, c_1)x_2 + (\mathrm{Im}\, c_1)x_1]r^2 + \mathrm{O}(\varepsilon^4)$$

and

$$tr\, \frac{\partial F}{\partial X}(P(t, \mu(\varepsilon))) = 2\alpha(\mu(\varepsilon)) + 4[\mathrm{Re}\, c_1(\mu(\varepsilon))]\varepsilon^2 + \mathrm{O}(\varepsilon^3),$$

where we have selectively used the facts that $\mu(\varepsilon) = \mathrm{O}(\varepsilon^2)$ and $r^2 = \varepsilon^2 + \mathrm{O}(\varepsilon^5)$, $(L \geqslant 2)$. Hence

$$\frac{1}{T(\varepsilon)} \int_0^{T(\varepsilon)} tr\, A(s, \varepsilon)\, \mathrm{d}s = 2\alpha(\mu(\varepsilon)) + 4[\mathrm{Re}\, c_1(\mu(\varepsilon))]\varepsilon^2 + \mathrm{O}(\varepsilon^3). \tag{A.19}$$

But

$$\alpha(\mu(\varepsilon)) = \alpha'(0)\mu_2\varepsilon^2 + \cdots = -\mathrm{Re}\, c_1(0)\varepsilon^2 + \cdots.$$

Therefore, following Floquet's theorem,

$$0 + \beta(\varepsilon) = 2\,\mathrm{Re}\, c_1(0)\varepsilon^2 + \mathrm{O}(\varepsilon^3).$$

Thus, $\beta(\varepsilon) < 0$ for $\varepsilon$ sufficiently small if $\mathrm{Re}\, c_1(0) < 0$, which is just the criterion derived earlier for asymptotic, orbital stability of $P(t, \mu(\varepsilon))$. However, the equality $\beta(\varepsilon) < 0$ implies that the bifurcating periodic solutions of the system (A.1) are asymptotically, orbitally stable with asymptotic phase.

The above computation can carried out to include terms of order $\varepsilon^4$, provided $L$ is at least 4. The result is

$$\beta(\varepsilon) = \beta_2\varepsilon^2 + \beta_4\varepsilon^4 + \mathrm{O}(\varepsilon^5),$$

where

$$\beta_2 = 2\,\mathrm{Re}\, c_1(0), \quad \beta_4 = 4\,\mathrm{Re}\, c_2(0) + 2\,\mathrm{Re}\, c_1'(0)\mu_2. \tag{A.20}$$

### *A.3. Reduction of two-dimensional systems to Poincaré normal form*

Before we can apply the bifurcation formulae, derived in the preceding sections, to various model systems, we must show how general autonomous systems, satisfying the hypothesis of Hopf Theorem, can be transformed into Poincaré normal form. We begin with the single complex equation

$$\dot{z} = \lambda z + g(z, \bar{z}, \mu), \tag{A.21}$$

where

$$g(z, \bar{z}, \mu) = \sum_{2 \leqslant i+j \leqslant L} g_{ij}(\mu) \frac{z^{\mathrm{i}} \bar{z}^j}{i!j!} + \mathrm{O}(|z|^{L+1}) \tag{A.22}$$

and

$$\lambda(\mu) = \alpha(\mu) + \mathrm{i}\omega(\mu).$$

We desire to transform (A.21) by means of a transformation:

$$z = \xi + \chi(\xi, \bar{\xi}, \mu) = \xi + \sum_{2 \leqslant i+j \leqslant L} \chi_{ij}(\mu) \frac{\xi^{\mathrm{i}} \bar{\xi}^j}{i!j!} \tag{A.23}$$

into the Poincaré normal form

$$\dot{\xi} = \lambda(\mu)\xi + \sum_{j=1}^{[L/2]} c_j(\mu)\xi|\xi|^{2j} + \mathrm{o}(|\xi||(\xi, \mu)|^{L+1}) = \lambda(\mu)\xi + \phi(\xi, \bar{\xi}, \mu). \tag{A.24}$$

By the chain rule

$$\dot{z} = \dot{\xi} + \chi_\xi \dot{\xi} + \chi_{\bar{\xi}} \dot{\bar{\xi}}$$

or

$$\lambda\xi\chi_\xi + \bar{\lambda}\bar{\xi}\chi_{\bar{\xi}} - \lambda\chi = g(\xi + \chi, \bar{\xi} + \bar{\chi}) - (\phi + \chi_\xi\phi + \chi_{\bar{\xi}}\bar{\phi}). \tag{A.25}$$

From this relation the coefficients $\chi_{ij}$ in (A.23) can be determined recursively. In powers of $\xi$ and $\bar{\xi}$ the left-hand side of (A.25) can be written as

$$\sum_{2 \leqslant i+j \leqslant L} (\chi_{ij})(\mathrm{i}\lambda + j\bar{\lambda} - \lambda) \frac{\xi^{\mathrm{i}} \bar{\xi}^j}{i!j!}. \tag{A.26}$$

Note that the expansion of the right-hand side of (A.25) to order $k = 2$ is independent of the $\chi_{ij}$ and to order $k(k = 3, \ldots, L)$ involves exactly the coefficients $\chi_{ij}$ for $2 \leqslant i + j \leqslant k$. Therefore the undetermined coefficients $\chi_{ij}$ with $i + j = k$ can be found by expanding (A.25) to order $k$. We begin with $k = 2$.

To order $|\xi|^2$, (A.25) is

$$\lambda\chi_{20}\frac{\xi^2}{2} + \bar{\lambda}\chi_{11}\xi\bar{\xi} + (2\bar{\lambda} - \lambda)\chi_{02}\frac{\bar{\xi}^2}{2} = g_{20}\frac{\xi^2}{2} + g_{11}\xi\bar{\xi} + g_{02}\frac{\bar{\xi}^2}{2}.$$

Thus

$$\chi_{20} = \frac{g_{20}}{\lambda}, \quad \chi_{11} = \frac{g_{11}}{\bar{\lambda}}, \quad \chi_{02} = \frac{g_{02}}{2\bar{\lambda} - \lambda}. \tag{A.27}$$

Taking this into account, we next find that equating coefficients of $\xi^2\bar{\xi}$ on both sides of (A.25) implies that

$$c_1(\mu) = \frac{g_{20}g_{11}(2\lambda + \bar{\lambda})}{2|\lambda|^2} + \frac{|g_{11}|^2}{\lambda} + \frac{|g_{02}|^2}{2(2\lambda - \bar{\lambda})} + \frac{g_{21}}{2}. \tag{A.28}$$

Thus, for $\mu = 0$, we obtain

$$c_1(0) = \frac{\mathrm{i}}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \tag{A.29}$$

where we have used the hypothesis that $\lambda(0) = \mathrm{i}\omega_0$.

### *A.4. Restriction to the center manifold*

Writing (A.1) as

$$\dot{X} = A(\mu)X + f(X, \mu) = F(X, \mu). \tag{A.30}$$

Let $q(\mu)$ and $q^*(\mu)$ be eigenvectors for

$$A(\mu) = F_X(0, \mu)$$

and $A^{\mathrm{T}}$, respectively, corresponding to the simple eigenvalues

$$\lambda(\mu) = \alpha(\mu) + \mathrm{i}\omega(\mu)$$

and $\bar{\lambda}(\mu)$ of $A(\mu)$; that is

$$Aq = \lambda q, \quad A^{\mathrm{T}}q^* = \bar{\lambda}q^*. \tag{A.31}$$

We normalize $q^*$ relative to $q$ so that

$$\langle q^*, q\rangle = 1, \tag{A.32}$$

where $\langle\cdot,\cdot\rangle$ denotes the Hermitian product $\langle u, v\rangle = \sum_{i=1}^{n} \bar{u}_i v_i$.

For any solution $x$ of (A.30), we define

$$z(t) = \langle q^*, x(t)\rangle. \tag{A.33}$$

We shall use $z$ and $\bar{z}$ (in the directions $q$ and $\bar{q}$) as local coordinates. We also define

$$w(t) = x(t) - z(t)q(\mu) - \bar{z}\bar{q}(\mu) = x(t) - 2\,\mathrm{Re}[z(t)q(\mu)]. \tag{A.34}$$

Because $zq$ and $\bar{z}\bar{q}$ will always occur together, our choice of complex coordinates will not introduce complex-valued solutions of (A.30).

Note that $\langle q^*, \bar{q}\rangle = 0$. A slightly different way of looking at the decomposition of $x(t)$ into $z(t)$ and $w(t)$ is by means of the projection matrices

$$P_{\|} = q(\bar{q}^*)^{\mathrm{T}} + \bar{q}(q^*)^{\mathrm{T}} = 2\,\mathrm{Re}[q(\bar{q}^*)^{\mathrm{T}}],$$

and

$$P_{\perp} = I - P_{\|} = I - 2\,\mathrm{Re}[q(\bar{q}^*)^{\mathrm{T}}].$$

These obey

$$P_{\|}^2 = P_{\|}, \quad P_{\perp}^2 = P_{\perp}, \quad P_{\|}P_{\perp} = 0, \quad P_{\perp}P_{\|} = 0.$$

In terms of $P_{\|}$ and $P_{\perp}$

$$z(t)q + \bar{z}(t)\bar{q} = P_{\|}x(t)$$

and

$$w(t) = P_{\perp}x(t).$$

In the variables $z$ and $w$, (A.30) becomes

$$\begin{aligned}\dot{z} &= \lambda(\mu)z + G(z, \bar{z}, w, \mu),\\ \dot{w} &= A(\mu)w + H(z, \bar{z}, w, \mu),\end{aligned} \tag{A.35}$$

where

$$\begin{aligned}G(z, \bar{z}, w, \mu) &= \langle q^*, f(w + 2\,\mathrm{Re}[zq], \mu)\rangle\\ H(z, \bar{z}, w, \mu) &= f(w + 2\,\mathrm{Re}[zq], \mu) - 2\,\mathrm{Re}[qG].\end{aligned} \tag{A.36}$$

Since

$$\langle \mathrm{Re}\, q^*, w\rangle = 0 \quad \text{and} \quad \langle \mathrm{Im}\, q^*, w\rangle = 0, \quad \langle q^*, w\rangle = 0. \tag{A.37}$$

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