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Global Synchronization of Linearly Hybrid Coupled Networks with Time-Varying Delay*

Wenwu Yu†, Jinde Cao‡, and Jinhu Lü§

Abstract. Many real-world large-scale complex networks demonstrate a surprising degree of synchronization. To unravel the underlying mechanics of synchronization in these complex networks, a generally linearly hybrid coupled network with time-varying delay is proposed, and its global synchronization is then further investigated. Several effective sufficient conditions of global synchronization are attained based on the Lyapunov function and a linear matrix inequality (LMI). Both delay-independent and delay-dependent conditions are deduced. In particular, the coupling matrix may be nonsymmetric or nondiagonal. Moreover, the derivative of the time-varying delay is extended to any given value. Finally, a small-world network, a regular network, and scale-free networks with network size are constructed to show the effectiveness of the proposed synchronous criteria.

Key words. Lyapunov function, linear matrix inequality (LMI), global synchronization, time-varying delay, complex networks

AMS subject classifications. 94B50, 34C15, 34D20, 92B20

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1. Introduction.

Over the last few years, complex networks have received increasing attention from all fields of sciences and humanities [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40]. Networks are everywhere in the real world, such as food-webs, ecosystems, metabolic pathways, the Internet, the World Wide Web, social networks, and global economic markets [1, 2]. The ubiquity of networks in the biological, physical, engineering, and social sciences leads naturally to two important common problems: How does network structure affect network function? How do individual dynamics affect global dynamics?

Despite advances in understanding network structure and dynamical behaviors in idealized cases, relatively little is known about large-scale, real-world complex networks and their

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†Department of Mathematics, Southeast University, Nanjing 210096, China, Department of Electronic Engineering, City University of Hong Kong, Hong Kong, China, and Department of Electrical Engineering, Columbia University, New York, NY 10027 (wenwuyu@gmail.com, wy2137@columbia.edu).

‡Department of Mathematics, Southeast University, Nanjing 210096, China (jdcao@seu.edu.cn).

§Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China, and Department of Ecology and Evolutionary Biology, Princeton University, Princeton, NJ 08544 (jhlu@iss.ac.cn, jinhulu@princeton.edu).

dynamical characteristics, especially for the evolving networks [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 32, 33, 34]. Historically, many models were proposed to describe various complex networks, including regular graph, random graph, small-world network, scale-free network, evolving networks, etc. [1, 2]. Undoubtedly, these models well describe many real networks in nature, such as social, biological, and engineering networks.

On the other hand, one can also extend the existing network models by introducing dynamical elements into the network nodes [3, 4, 14, 32, 33, 34]. Over the last few years, nonlinear dynamics of complex networks have been intensively investigated. Synchronization is a kind of typical collective behavior and a basic motion in nature [14]. Our intuition is that loosely coupled dynamical systems tend to synchronize with respect to periodic behavior [18]. This synchronization is essentially a form of self-organization. Moreover, it has been demonstrated that many real-world problems have a close relationship with network synchronization. For example, theoretical and experimental results reveal that a mammalian brain not only displays its storage of associative memories but also modulates oscillatory neuronal synchronization by selective perceived attention [6].

Recently, network synchronization has been intensively investigated in various different fields [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 32, 33, 34]. For example, some researchers studied the synchronization of coupled connected neural networks [6, 8, 9, 10]; Yu, Cao, and their colleagues explored the synchronization of an array of linearly coupled networks with time-delay [7, 16]; Lü and his colleagues introduced several synchronization criteria for the time-varying complex dynamical networks [16, 32, 33, 34]; and Li and Chen studied the robust adaptive synchronization of some uncertain dynamical networks [15].

In this paper, we introduce a linearly hybrid coupled network with time-varying delay. Based on this network model, several simply sufficient conditions of global network synchronization are then deduced by using the Lyapunov function and a linear matrix inequality (LMI). Both delay-independent and delay-dependent sufficient conditions are also attained. It should be especially emphasized that we do not assume that the coupling matrix is symmetric or diagonal. However, most of the former works on network synchronization are based on this assumption. Furthermore, we extend the derivative of the time-varying delay to any given value. Last but not least, one constructs a small-world network, a regular network, and scale-free networks with network size to verify the effectiveness of the proposed synchronous criteria.

The remainder of this paper is organized as follows: In section 2, the main background of complex networks is briefly outlined, and a generally linearly hybrid coupled network with time-varying delay is proposed. The main theorems and corollaries for global network synchronization are then given in section 3. In section 4, a small-world network, a regular network, and scale-free networks with network size are constructed to show the effectiveness of the proposed global network synchronous criteria. The conclusions are finally drawn in section 5.

2. Preliminaries. Consider a complex dynamical network consisting of N identical nodes with linearly diffusive couplings [3, 4, 5, 14, 32, 33, 34], which is described by

$$\dot{x}_i(t) = f(x_i(t)) + c \sum_{j=1,\, j\neq i}^{N} G_{ij}\Gamma(x_j(t) - x_i(t)), \quad i = 1, 2, \ldots, N, \tag{1}$$

where $i = 1, 2, \ldots, N$, $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in R^n$ is the state vector of the ith node, $f : R^n \longrightarrow R^n$ is continuously differentiable, c is the coupling strength, $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) \in R^{n\times n}$ is a 0-1 diagonal matrix with specific $\gamma_i = 1$ and 0 for others, and $G = (G_{ij})_{N\times N}$ is a coupling configuration matrix representing the topological structure of the network, where G_{ij} is defined as follows: if there exists a connection from node i to another node j, then the coupling strength $G_{ij} = G_{ji} = 1$; otherwise, $G_{ij} = G_{ji} = 0$ $(j \neq i)$, and the diagonal elements of matrix G are defined by

$$G_{ii} = -\sum_{j=1,\, j\neq i}^{N} G_{ij} = -\sum_{j=1,\, j\neq i}^{N} G_{ji}. \tag{2}$$

Thus network (1) can be rewritten as follows:

$$\dot{x}_i(t) = f(x_i(t)) + c \sum_{j=1}^{N} G_{ij}\Gamma x_j(t), \quad i = 1, 2, \ldots, N. \tag{3}$$

Hereafter, suppose network (3) is connected in the sense that there are no isolate clusters. That is, the coupling configuration matrix G is an irreducible matrix.

However, most real-world complex networks are time-varying. To characterize the real-world evolving networks, Lü and Chen introduced a time-varying network [14, 32, 33, 34] which represents many real biological and engineering networks. Also, there inevitably exists time-delay in many practical complex networks because of the finite information exchanging speed. Considering the time-delay, we propose a simple complex network model as follows [13]. Recently, a linearly coupled complex network was presented and further studied [6, 7, 8]. Considering the time delay, Cao and his colleagues further introduced the following constant delayed complex dynamical network [16].

In this paper, we will consider the following linearly hybrid coupled network with time-varying delay:

$$\dot{x}_i(t) = -Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau)) + I(t) + \sum_{j=1}^{N} G_{ij}Dx_j(t) + \sum_{j=1}^{N} G_{ij}D_\tau x_j(t-\tau(t)), \tag{4}$$

where $i = 1, 2, \ldots, N$, $C = \mathrm{diag}(c_1, c_2, \ldots, c_n) \in R^{n\times n}$ is a diagonal matrix with positive diagonal entries $c_i > 0$, $i = 1, 2, \ldots, n$, $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are weight and delayed weight matrices, respectively, $I(t) = (I_1(t), I_2(t), \ldots, I_n(t))^T \in R^n$ is an external input vector, $D = (d_{ij}) \in R^{n\times n}$ and $D_\tau = (d_{\tau ij}) \in R^{n\times n}$ are constant and delayed inner coupling matrices of complex networks, respectively, $f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)), \ldots, f_n(x_{in}(t)))^T \in R^n$

corresponds to the activation functions of neurons, and $G = (G_{ij})_{N\times N}$ satisfies the diffusive condition (2). In particular, one does not assume that the constant and delayed inner coupling matrices $D = (d_{ij}) \in R^{n\times n}$ and $D_\tau = (d_{\tau ij}) \in R^{n\times n}$ are diagonal matrices.

Denote $x_i(t) = \phi_i(t) \in \mathcal{C}([-r, 0], R)$ $(i = 1, 2, \ldots, N)$, where $r = \sup_{t\in R}\{\tau(t)\}$ and $\mathcal{C}([-r, 0], R)$ is the set of continuous functions from $[-r, 0]$ to R. To simplify, one has the following fundamental assumptions.

A_1: $f_i(x_i)$ $(i = 1, 2, \ldots, n)$ are monotonically nondecreasing on R.

A_2: $f_i(x_i)$ $(i = 1, 2, \ldots, n)$ are Lipschitz continuous; i.e., there exist constants $F_i > 0$ such that

$$|f_i(\alpha_1) - f_i(\alpha_2)| \le F_i|\alpha_1 - \alpha_2| \quad \forall \alpha_1, \alpha_2 \in R. \tag{5}$$

A_3: $\tau(t)$ is a bounded differential function of time t satisfying

$$0 \le \dot{\tau}(t) \le h < 1,$$

where h is a positive real constant.

A_4: The coupling matrix G satisfies the conditions

$$G_{ij} \ge 0, \quad i \ne j, \qquad G_{ii} = -\sum_{j=1,\, j\ne i}^{N} G_{ij}, \quad i = 1, 2, \ldots, N. \tag{6}$$

Before stating the main results, some similar definitions and lemmas are given [6, 7, 8, 9, 10].

Definition 1. *Let $r = \max_{t\in R}\{\tau(t)\}$. Set $\mathbf{S} = \{x = (x_1(s), x_2(s), \ldots, x_N(s)) : x_i(s) \in \mathcal{C}([-r, 0], R),\ x_i(s) = x_j(s),\ i, j = 1, 2, \ldots, N\}$, which is called the synchronization manifold of network (4).*

Definition 2. *Let $\widehat{R}$ be a ring and $T(\widehat{R}, K) = \{$the set of matrices with entries $\widehat{R}$ such that the sum of the entries in each row is equal to K for some $K \in \widehat{R}\}$.*

Definition 3. *The set of $M_1^N(1)$: $M_1^N(1)$ is composed of matrices with N columns; each row (such as the ith row) of $\widetilde{M} \in M_1^N(1)$ has exactly one entry α_i and one entry $-\alpha_i$, where $\alpha_i \ne 0$, and all other entries are zeros.*

Definition 4. *The set of $M_1^N(n)$: $M_1^N(n) = \{\mathbf{M} = M \otimes I_n : M \in M_1^N(1),\ I_n$ is the n-dimensional identity matrix$\}$, where $\otimes$ is Kronecker product.*

Definition 5. *$M_2^N(n) \subset M_1^N(n)$: If $M \in M_2^N(n)$, for any pair of indices i and j, there exist indices $j_1, j_2, \ldots, j_l$ and $p_1, p_2, \ldots, p_{l-1}$ such that $M_{p_q,i_q} \ne 0$ and $M_{p_q,i_{q+1}} \ne 0$ for all $1 \le q < l$, where $j_1 = i$ and $j_l = j$.*

Definition 6. *Synchronization manifold $\mathbf{S}$ is said to be globally exponentially stable (or network (4) is globally exponentially synchronized) if there exist $\epsilon > 0$, $T > t_0$, and $M > 0$ such that*

$$\|x_i(t) - x_j(t)\| \le Me^{-\epsilon t}, \tag{7}$$

where $\phi_i \in \mathcal{C}([-r, 0], R)$, $t > T$, $i, j = 1, 2, \ldots, N$.

Definition 7. *Synchronization manifold* $\mathbf{S}$ *is said to be globally asymptotically stable (or network* (4) *is globally asymptotically synchronized) if for any* $\varepsilon > 0$, *there exists* $T > t_0$ *such that*

$$\|x_i(t) - x_j(t)\| \le \varepsilon, \tag{8}$$

where $\phi_i, \phi_j \in \mathcal{C}([-r, 0], R)$, $t > T$, $i, j = 1, 2, \ldots, N$.

Lemma 1 (see [9]). *Let* G *be an* $N \times N$ *matrix in the set* $T(\widehat{R}, K)$. *Then the* $(N-1)\times(N-1)$ *matrix* H *satisfies* $MG = HM$, *where* $H = MGJ$,

$$M = \begin{pmatrix} \mathbf{1} & -\mathbf{1} & & & \\ & \mathbf{1} & -\mathbf{1} & & \\ & & \ddots & & \\ & & & \mathbf{1} & -\mathbf{1} \end{pmatrix}_{((N-1)\times N)}, \quad J = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ & & \ddots & & \mathbf{1} \\ & & \cdots & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}_{(N\times(N-1))}, \tag{9}$$

in which $\mathbf{1}$ *is the multiplicative identity of* $\widehat{R}$. *Moreover, the matrix* H *can be rewritten explicitly as follows:* $H_{(i,j)} = \sum_{k=1}^{j} G_{(i,k)} - G_{(i+1,k)}$ *for* $i, j \in \{1, 2, \ldots, N-1\}$.

Lemma 2 (Schur complement [17]). *The LMI*

$$\begin{pmatrix} \widetilde{Q}(x) & \widetilde{S}(x) \\ \widetilde{S}(x)^T & \widetilde{R}(x) \end{pmatrix} > 0$$

is equivalent to one of the following conditions:

(i) $\widetilde{Q}(x) > 0$, $\widetilde{R}(x) - \widetilde{S}(x)^T\widetilde{Q}(x)^{-1}\widetilde{S}(x) > 0$,

(ii) $\widetilde{R}(x) > 0$, $\widetilde{Q}(x) - \widetilde{S}(x)\widetilde{R}(x)^{-1}\widetilde{S}(x)^T > 0$,

where $\widetilde{Q}(x) = \widetilde{Q}(x)^T$, $\widetilde{R}(x) = \widetilde{R}(x)^T$.

Lemma 3 (see [9]). *If matrix* G *is symmetric and also satisfies condition* A_4, *then* G *is irreducible iff there exists a* $p \times N$ *matrix* $M \in M_2^N(1)$, *such that* $G = -M^T M$.

Lemma 4 (see [9]). *Let* $x = (x_1, x_2, \ldots, x_N)^T$, *where* $x_i \in R^n$, $i = 1, 2, \ldots, N$. *Then* $x \in \mathbf{S}$ *iff there exists* $\mathbf{M} \in M_2^N(n)$ *satisfying* $\|\mathbf{M}x\| = 0$.

Denote

$$d(x) = \|\mathbf{M}x\|^2 = x^T\mathbf{M}^T\mathbf{M}x, \quad \mathbf{M} \in M_2^N(n). \tag{10}$$

Then $d(x)$ *is a nonnegative distance function. From the assumptions of* $\mathbf{M}$, *one has* $d(x) \longrightarrow 0$ *iff* $\|x_i(t) - x_j(t)\| \longrightarrow 0$ *for all* $i, j = 1, 2, \ldots, N$.

Lemma 5 (see [19]). *The Kronecker product has the following properties:*

(1) $(\alpha A) \otimes B = A \otimes (\alpha B)$;

(2) $(A + B) \otimes C = A \otimes C + B \otimes C$;

(3) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

Lemma 6 (Jensen inequality [29]). *Assume that the vector function* $\omega : [0, r] \in R^{m\times m}$ *is well defined for the following integrations. For any symmetric matrix* $W \in R^{m\times m}$ *and scalar*

$r > 0$, *one has*

$$r\int_0^r \omega(s)W\omega(s)\,ds \geq \left(\int_0^r \omega(s)\,ds\right)^T W\left(\int_0^r \omega(s)\,ds\right).$$

3. Main results. In this section, several novel criteria are proposed for the global synchronization of complex network (4) based on the Lyapunov function and an LMI.

To simplify the presentation, some notation is given in the following. Let T be a symmetric and irreducible matrix satisfying assumption A_4. From Lemma 3, there exists a $p\times N$ matrix $\widetilde{M} \in M_2^N(1)$ such that $T = -\widetilde{M}^T\widetilde{M}$. Denote $\mathbf{M} = \widetilde{M}\otimes I_n \in M_2^N(n)$, and let $\mathbf{M}_i = (M_{i1}, M_{i2}, \ldots, M_{iN})$ be the ith row of $\mathbf{M}$, where $M_{ii_1} = \alpha_i I_n$, $M_{ii_2} = -\alpha_i I_n$, and $M_{ij} = 0$ for $j \neq i_1, i_2$. Let $A\otimes B$ be the Kroneker product of matrices A and B.

$$\mathbf{C} = I_N\otimes C,\quad \mathbf{C^1} = I_p\otimes C,\quad \mathbf{A} = I_N\otimes A,\quad \mathbf{A^1} = I_p\otimes A,$$

$$\mathbf{B} = I_N\otimes B,\quad \mathbf{B^1} = I_p\otimes B,\quad \mathbf{G} = G\otimes D,\quad \mathbf{G}_\tau = G\otimes D_\tau,$$

$$x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T\ (\forall i = 1,2,\ldots,N),\quad x(t) = (x_1^T(t), x_2^T(t), \ldots, x_N^T(t))^T,$$

$$\mathbf{f}(x(t)) = (f^T(x_1(t)), f^T(x_2(t)), \ldots, f^T(x_N(t)))^T,\quad \mathbf{I}(t) = (I^T(t), I^T(t), \ldots, I^T(t))^T, \tag{11}$$

where I_N is the N dimensional identity matrix.

Then the complex network (4) can be recast as follows:

$$\dot{x}(t) = -\mathbf{C}x(t) + \mathbf{A}\mathbf{f}(x(t)) + \mathbf{B}\mathbf{f}(x(t-\tau(t))) + \mathbf{I}(t) + \mathbf{G}x(t) + \mathbf{G}_\tau x(t-\tau(t)),\quad i = 1,2,\ldots,N. \tag{12}$$

Theorem 1. *Suppose A_2–A_4 hold. Network (12) is globally exponentially synchronized if there exist positive definite matrices $P = (p_{ij})_{n\times n} \in R^{n\times n}$, $Q = (q_{ij})_{n\times n} \in R^{n\times n}$, and $\Omega = (\Omega_{ij})_{n\times n} \in R^{n\times n}$, a positive definite diagonal matrix $\Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_n) \in R^{n\times n}$, a symmetric matrix $\Delta = (\Delta_{ij})_{n\times n} \in R^{n\times n}$, and an irreducible symmetric matrix $T = (t_{ij}) \in R^{N\times N}$ satisfying A_4, such that*

$$\Lambda_0 = \begin{pmatrix} -2PC - \Delta + F\Sigma F & PA & PB \\ A^TP & -\Sigma + Q & 0 \\ B^TP & 0 & -(1-h)Q \end{pmatrix} < 0, \tag{13}$$

where $F = \mathrm{diag}(F_1, F_2, \ldots, F_n) \in R^{n\times n}$ and one of the following conditions holds:

$$\text{(i)}\ \Lambda_{ij} = \begin{pmatrix} \sum_{k=1}^N T_{ik}G_{kj}(PD + D^TP) + T_{ij}(\Omega+\Delta) & \sum_{k=1}^N T_{ik}G_{kj}PD_\tau \\ \sum_{k=1}^N T_{ik}G_{kj}D_\tau^TP & -(1-h)T_{ij}\Omega \end{pmatrix} < 0 \quad \forall 1 \leq i < j \leq N. \tag{14}$$

$$\text{(ii)}\ \widetilde{\Lambda}_{ij} = \begin{pmatrix} 2\sum_{k=1}^n p_{ik}d_{kj}TG + (\Omega_{ij}+\Delta_{ij})T & \sum_{k=1}^n p_{ik}d_{\tau kj}TG \\ \sum_{k=1}^n p_{ik}d_{\tau kj}G^TT & -(1-h)\Omega_{ij}T \end{pmatrix} > 0 \quad \forall 1 \leq i, j \leq n. \tag{15}$$

Proof. See Appendix A.

Instead of using inequality (A.4), from assumption A_1, one has

$$\begin{aligned}\mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{\Sigma}\mathbf{M}\mathbf{f}(x(t)) &= \textstyle\sum_{j=1}^{p}\alpha_j^2[f(x_{j_1}(t))-f(x_{j_2}(t))]^T\Sigma[f(x_{j_1}(t))-f(x_{j_2}(t))]\\ &\le \textstyle\sum_{j=1}^{p} y_j^T(t)F\Sigma[f(x_{j_1}(t))-f(x_{j_2}(t))].\end{aligned} \tag{16}$$

Similarly, following the same steps in part (i) of Theorem 1, one can easily attain the following corollary.

Corollary 1. *Suppose that assumptions A_1–A_4 hold. Then network (12) is globally exponentially synchronized if there exist positive definite matrices $P=(p_{ij})_{n\times n}\in R^{n\times n}$, $Q=(q_{ij})_{n\times n}\in R^{n\times n}$, and $\Omega=(\Omega_{ij})_{n\times n}\in R^{n\times n}$, a positive definite diagonal matrix $\Sigma=\mathrm{diag}(\Sigma_1,\Sigma_2,\dots,\Sigma_n)\in R^{n\times n}$, a symmetric matrix $\Delta=(\Delta_{ij})_{n\times n}\in R^{n\times n}$, and an irreducible symmetric matrix $T=(t_{ij})\in R^{N\times N}$ satisfying A_4, such that*

$$\begin{pmatrix} -2PC-\Delta & PA+F\Sigma & PB\\ A^TP+\Sigma F & -2\Sigma+Q & 0\\ B^TP & 0 & -(1-h)Q\end{pmatrix}<0, \tag{17}$$

where $F=\mathrm{diag}(F_1,F_2,\dots,F_n)\in R^{n\times n}$ and

$$\begin{pmatrix} \sum_{k=1}^{N}T_{ik}G_{kj}(PD+D^TP)+T_{ij}(\Omega+\Delta) & \sum_{k=1}^{N}T_{ik}G_{kj}PD_\tau\\ \sum_{k=1}^{N}T_{ik}G_{kj}D_\tau^TP & -(1-h)T_{ij}\Omega\end{pmatrix}<0 \quad \forall 1\le i<j\le N. \tag{18}$$

Denote $e=(1,1,\dots,1)^T\in R^N$, $J=ee^T$, $U=NI_N-J$. Let $T=-U=J-NI_N$; then $T_{ij}=1$ $(i\ne j)$, $T_{ij}=-(N-1)$ $(i=j)$, $i,j=1,2,\dots,N$. It is easy to verify that T satisfies assumption A_4. According to Lemma 3, there exists a $p\times N$ matrix $M\in M_2^N(1)$, such that $T=-M^TM$. Since G satisfies assumption A_4, then one has

$$\begin{aligned}\textstyle\sum_{k=1}^{N}T_{ik}G_{kj} &= (T_{ii}-1)G_{ij}+\textstyle\sum_{k=1,\,k\ne i}^{N}T_{ik}G_{kj}+G_{ij}\\ &= -NG_{ij}+\textstyle\sum_{k=1}^{N}G_{kj}=-NG_{ij}.\end{aligned} \tag{19}$$

Therefore, from Theorem 1, one gets the following corollary.

Corollary 2. *Suppose assumptions A_2–A_4 hold. Then network (12) is globally exponentially synchronized if there exist positive definite matrices $P=(p_{ij})_{n\times n}\in R^{n\times n}$, $Q=(q_{ij})_{n\times n}\in R^{n\times n}$, and $\Omega=(\Omega_{ij})_{n\times n}\in R^{n\times n}$, a positive definite diagonal matrix $\Sigma=\mathrm{diag}(\Sigma_1,\Sigma_2,\dots,\Sigma_n)\in R^{n\times n}$, and a symmetric matrix $\Delta=(\Delta_{ij})_{n\times n}\in R^{n\times n}$, such that*

$$\begin{pmatrix} -2PC-\Delta+F\Sigma F & PA & PB\\ A^TP & -\Sigma+Q & 0\\ B^TP & 0 & -(1-h)Q\end{pmatrix}<0, \tag{20}$$

where $F=\mathrm{diag}(F_1,F_2,\dots,F_n)\in R^{n\times n}$ and

$$\begin{pmatrix} -NG_{ij}(PD+D^TP)+(\Omega+\Delta) & -NG_{ij}PD_\tau\\ -NG_{ij}D_\tau^TP & -(1-h)\Omega\end{pmatrix}<0 \quad \forall 1\le i<j\le N. \tag{21}$$

Remark 1. In [16], Cao and his colleagues investigated the global synchronization of a coupled complex network with constant time-delay. The main theorem in [16] is Corollary 2 with $h = 0$, where the time-delay is a constant. Therefore, the main result in [16] is a special case of Theorem 1.

Let $G_\tau = 0$; i.e., there is no linearly delayed coupling in network (12) as that in [6, 7, 8]. Let $\Omega = \zeta I_n$, where ζ is a sufficient small positive number. Then one has the following corollary.

Corollary 3. *Suppose assumptions A_2–A_4 hold. Network (12) with $D_\tau = 0$ is globally exponentially synchronized if there exist positive definite matrices $P = (p_{ij})_{n\times n} \in R^{n\times n}$ and $Q = (q_{ij})_{n\times n} \in R^{n\times n}$, a positive definite diagonal matrix $\Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_n) \in R^{n\times n}$, a symmetric matrix $\Delta = (\Delta_{ij})_{n\times n} \in R^{n\times n}$, and an irreducible symmetric matrix $T = (t_{ij}) \in R^{N\times N}$ satisfying A_4, such that*

$$\begin{pmatrix} -2PC - \Delta + F\Sigma F & PA & PB \\ A^T P & -\Sigma + Q & 0 \\ B^T P & 0 & -(1-h)Q \end{pmatrix} < 0, \tag{22}$$

where $F = \mathrm{diag}(F_1, F_2, \ldots, F_n) \in R^{n\times n}$ and

$$\left(2\sum_{k=1}^{n} p_{ik} d_{kj} TG + \Delta_{ij} T \right) > 0 \quad \forall 1 \le i, j \le n. \tag{23}$$

It should be especially emphasized that the inner coupling matrix D is not necessarily a diagonal matrix in Theorem 1. If D is a diagonal matrix, then one gets the following corollary.

Corollary 4. *Suppose that assumptions A_2–A_4 hold. Assume also that $D_\tau = 0$ and D is diagonal. Network (12) is globally exponentially synchronized if there exist a positive definite matrix $Q = (q_{ij})_{n\times n} \in R^{n\times n}$, positive definite diagonal matrices $P = \mathrm{diag}(p_1, p_2, \ldots, p_n) \in R^{n\times n}$ and $\Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_n) \in R^{n\times n}$, a symmetric matrix $\Delta = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_n) \in R^{n\times n}$, and an irreducible symmetric matrix $T = (t_{ij}) \in R^{N\times N}$ satisfying A_4, such that*

$$\begin{pmatrix} -2PC - \Delta + F\Sigma F & PA & PB \\ A^T P & -\Sigma + Q & 0 \\ B^T P & 0 & -(1-h)Q \end{pmatrix} < 0, \tag{24}$$

where $F = \mathrm{diag}(F_1, F_2, \ldots, F_n) \in R^{n\times n}$ and

$$T\left(2p_i d_i G + \delta_i\right) > 0, \quad i = 1, 2, \ldots, n. \tag{25}$$

Let P, Ω, and Δ be diagonal matrices. According to part (ii) of Theorem 1, one has the following corollary.

Corollary 5. *Suppose assumptions A_2–A_4 hold. Suppose also that D and D_τ are diagonal matrices. Then network (12) is globally exponentially synchronized if there exist a positive definite matrix $Q = (q_{ij})_{n\times n} \in R^{n\times n}$, positive definite diagonal matrices $P = \mathrm{diag}(p_1, p_2, \ldots, p_n) \in R^{n\times n}$, $\Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_n) \in R^{n\times n}$, and $\Sigma = \mathrm{diag}(\Omega_1, \Omega_2, \ldots, \Omega_n) \in R^{n\times n}$, a* *symmetric matrix* $\Delta = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_n) \in R^{n\times n}$*, and an irreducible symmetric matrix* $T = (t_{ij}) \in R^{N\times N}$ *satisfying* A_4*, such that*

$$\begin{pmatrix} -2PC - \Delta + F\Sigma F & PA & PB \\ A^T P & -\Sigma + Q & 0 \\ B^T P & 0 & -(1-h)Q \end{pmatrix} < 0, \tag{26}$$

where $F = \mathrm{diag}(F_1, F_2, \ldots, F_n) \in R^{n\times n}$ *and*

$$\begin{pmatrix} 2p_i d_i TG + \Omega_i + \delta_i T & p_i d_{\tau i} TG \\ p_i d_{\tau i} G^T T & -(1-h)\Omega_i T \end{pmatrix} > 0, \quad i = 1, 2, \ldots, n. \tag{27}$$

Remark 2. In [6], Lu and Chen further investigated the synchronization of a coupled connected neural network with constant time-delay. Theorem 3 in [6] is Corollary 4 with $h = 0$. Therefore, the main result in [6] is a special case of Theorem 1. Moreover, in [6], the inner coupling matrix D and the inner delayed coupling matrix D_τ are both diagonal matrices. However, one removes these limit conditions in this paper.

Remark 3. To minimize the number of LMIs in the conditions of Theorem 1, one can apply the following rule: if $N < n$, one can use condition (i) of Theorem 1; otherwise, one can use condition (ii) of Theorem 1.

Since the conditions of Theorem 1 are relatively complex, one will simplify these LMIs by introducing some special $M \in M_2^N(1)$.

Theorem 2. *Suppose that assumptions* A_2–A_4 *hold. Network* (12) *is globally asymptotically synchronized if there exist positive definite matrices* $\mathbf{P} = (p_{ij})_{(N-1)n\times(N-1)n} \in R^{(N-1)n\times(N-1)n}$, $\mathbf{Q} = (q_{ij})_{(N-1)n\times(N-1)n} \in R^{(N-1)n\times(N-1)n}$*, and* $\mathbf{R} = (r_{ij})_{(N-1)n\times(N-1)n} \in R^{(N-1)n\times(N-1)n}$ *and a positive definite diagonal matrix* $\mathbf{\Sigma} = \mathrm{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_{(N-1)n}) \in R^{(N-1)n\times(N-1)n}$*, such that*

$$\mathbf{\Omega} = \begin{pmatrix} -2\mathbf{PC^1} + \mathbf{PH} + \mathbf{H}^T\mathbf{P} + \mathbf{F\Sigma F} + \mathbf{R} & \mathbf{PH}_\tau & \mathbf{PA^1} & \mathbf{PB^1} \\ \mathbf{H}_\tau^T\mathbf{P} & -(1-h)\mathbf{R} & 0 & 0 \\ \mathbf{A^1}^T\mathbf{P} & 0 & -\mathbf{\Sigma} + \mathbf{Q} & 0 \\ \mathbf{B^1}^T\mathbf{P} & 0 & 0 & -(1-h)\mathbf{Q} \end{pmatrix} < 0, \tag{28}$$

where $F = \mathrm{diag}(F_1, F_2, \ldots, F_n) \in R^{n\times n}$, $\mathbf{F} = I_{N-1} \otimes F$, $H = MGJ$, $\mathbf{H} = (MGJ) \otimes D$, $\mathbf{H}_\tau = (MGJ) \otimes D_\tau$*, and* M *and* J *are defined in* (9).

Proof. See Appendix B.

Instead of using inequality (B.4), from assumption A_1, one has

$$\begin{aligned} \mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{\Sigma}\mathbf{M}\mathbf{f}(x(t)) &= \textstyle\sum_{j=1}^{N-1} [f(x_j(t)) - f(x_{j+1}(t))]^T \mathbf{\Sigma}_j [f(x_j(t)) - f(x_{j+1}(t))] \\ &\le \textstyle\sum_{j=1}^{N-1} [x_j(t) - x_{j+1}(t)]^T F \mathbf{\Sigma}_j [x_j(t) - x_{j+1}(t)] \\ &= x^T(t)\mathbf{M}^T\mathbf{F\Sigma M}x(t), \end{aligned} \tag{29}$$

where $\mathbf{\Sigma}_j = \mathrm{diag}(\Sigma_{(j-1)n+1}, \ldots, \Sigma_{jn})$. Following the same steps in Theorem 2, other conditions can be similarly verified. Then the following corollary is obtained.

Corollary 6. *Suppose assumptions A_1–A_4 hold. Network (12) is globally asymptotically synchronized if there exist positive definite matrices* $\mathbf{P} = (p_{ij})_{(N-1)n\times(N-1)n} \in R^{(N-1)n\times(N-1)n}$, $\mathbf{Q} = (q_{ij})_{(N-1)n\times(N-1)n} \in R^{(N-1)n\times(N-1)n}$, *and* $\mathbf{R} = (r_{ij})_{(N-1)n\times(N-1)n} \in R^{(N-1)n\times(N-1)n}$ *and a positive definite diagonal matrix* $\mathbf{\Sigma} = \mathrm{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_{(N-1)n}) \in R^{(N-1)n\times(N-1)n}$, *such that*

$$
(30)\qquad \begin{pmatrix} -2\mathbf{PC^1} + \mathbf{PH} + \mathbf{H}^T\mathbf{P} + \mathbf{R} & \mathbf{PH}_\tau & \mathbf{PA^1} + \mathbf{F\Sigma} & \mathbf{PB^1} \\ \mathbf{H}_\tau^T\mathbf{P} & -(1-h)\mathbf{R} & 0 & 0 \\ \mathbf{A^1}^T\mathbf{P} + \mathbf{\Sigma F} & 0 & -2\mathbf{\Sigma} + \mathbf{Q} & 0 \\ \mathbf{B^1}^T\mathbf{P} & 0 & 0 & -(1-h)\mathbf{Q} \end{pmatrix} < 0,
$$

where $F = \mathrm{diag}(F_1, F_2, \ldots, F_n) \in R^{n\times n}$, $\mathbf{F} = I_{N-1} \otimes F$, $H = MGJ$, $\mathbf{H} = (MGJ) \otimes D$, $\mathbf{H}_\tau = (MGJ) \otimes D_\tau$, *and M and J are defined in* (9).

Corollary 7. *Suppose that assumptions A_2–A_4 hold. Network (12) is globally asymptotically synchronized if there exist positive definite matrices* $P = (p_{ij})_{n\times n} \in R^{n\times n}$, $Q = (q_{ij})_{n\times n} \in R^{n\times n}$, *and* $R = (r_{ij})_{n\times n} \in R^{n\times n}$, *a positive definite diagonal matrix* $\Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_n) \in R^{n\times n}$, *and a symmetric matrix* $\Delta = (\Delta_{ij})_{n\times n} \in R^{n\times n}$, *such that*

$$
(31)\qquad \Lambda_0 = \begin{pmatrix} -2PC - \Delta + F\Sigma F & PA & PB \\ A^TP & -\Sigma + Q & 0 \\ B^TP & 0 & -(1-h)Q \end{pmatrix} < 0
$$

and

$$
(32)\qquad \Xi = \begin{pmatrix} \mathbf{PH} + \mathbf{H}^T\mathbf{P} + \mathbf{R} + \mathbf{\Delta} & \mathbf{PH}_\tau \\ \mathbf{H}_\tau^T\mathbf{P} & -(1-h)\mathbf{R} \end{pmatrix} < 0,
$$

where $F = \mathrm{diag}(F_1, F_2, \ldots, F_n) \in R^{n\times n}$, $\mathbf{P} = I_{N-1} \otimes P$, $\mathbf{R} = I_{N-1} \otimes R$, $\mathbf{\Delta} = I_{N-1} \otimes \Delta$, $\mathbf{H} = H \otimes D$, $\mathbf{H}_\tau = H \otimes D_\tau$, $H = MGJ$, *and M and J are defined in* (9).

Proof. See Appendix C.

When $D_\tau = 0$, there is no linearly delayed coupling in network (12). Let $R = \zeta I_n$ in Corollary 7, where ζ is a sufficiently small positive number. Then Corollary 7 can be simplified as follows.

Corollary 8. *Suppose that assumptions A_2–A_4 hold. Network (12) with $D_\tau = 0$ is globally exponentially synchronized if there exist positive definite matrices* $P = (p_{ij})_{n\times n} \in R^{n\times n}$ *and* $Q = (q_{ij})_{n\times n} \in R^{n\times n}$, *a positive definite diagonal matrix* $\Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_n) \in R^{n\times n}$, *and a symmetric matrix* $\Delta = (\Delta_{ij})_{n\times n} \in R^{n\times n}$, *such that*

$$
\Lambda_0 = \begin{pmatrix} -2PC - \Delta + F\Sigma F & PA & PB \\ A^TP & -\Sigma + Q & 0 \\ B^TP & 0 & -(1-h)Q \end{pmatrix} < 0
$$

and

$$
\mathbf{PH} + \mathbf{H}^T\mathbf{P} + \mathbf{\Delta} < 0,
$$

where $F = \mathrm{diag}(F_1, F_2, \ldots, F_n) \in R^{n\times n}$, $\mathbf{P} = I_{N-1} \otimes P$, $\mathbf{\Delta} = I_{N-1} \otimes \Delta$, $\mathbf{H} = H \otimes D$, $H = MGJ$, *and M and J are defined in* (9).

Remark 4. In [7], Wang and Cao further studied the synchronization of an array of linearly coupled networks with time-varying delay. The main theorem, Theorem 2, in [7] is Corollary 8. Therefore, the main result in [7] is a special case of Theorem 2. Moreover, the inner coupling matrix D and inner delayed coupling matrix D_τ are both diagonal matrices in [7]. However, one removes these limit conditions. Furthermore, one also introduces the time-delay in the linear coupling in this paper.

If assumption A_3 is not satisfied, i.e., $\dot{\tau}(t) \geq 1$ for some t, one attains the following synchronous theorem.

Theorem 3. *Suppose assumptions A_2 and A_4 hold. Then network (12) is globally asymptotically synchronized if there exist positive definite matrices $\mathbf{P} = (p_{ij})_{(N-1)n\times(N-1)n} \in R^{(N-1)n\times(N-1)n}$, $\mathbf{Q} = (q_{ij})_{(N-1)n\times(N-1)n} \in R^{(N-1)n\times(N-1)n}$, $\mathbf{R} = (r_{ij})_{(N-1)n\times(N-1)n} \in R^{(N-1)n\times(N-1)n}$, and $\mathbf{T} = (t_{ij})_{(N-1)n\times(N-1)n} \in R^{(N-1)n\times(N-1)n}$, positive definite diagonal matrices $\mathbf{\Sigma} = \mathrm{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_{(N-1)n}) \in R^{(N-1)n\times(N-1)n}$ and $\mathbf{\Lambda} = \mathrm{diag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_{(N-1)n}) \in R^{(N-1)n\times(N-1)n}$, and a matrix $\mathbf{U} = (u_{ij})_{(N-1)n\times(N-1)n} \in R^{(N-1)n\times(N-1)n}$, such that*

(33)

$$\mathbf{\Omega_1} = \begin{pmatrix} \Psi & \mathbf{P}\mathbf{H}_\tau + \mathbf{U}^T & \mathbf{P}\mathbf{A^1} & \mathbf{P}\mathbf{B^1} & 0 & (-\mathbf{C^1}+\mathbf{H})^T\mathbf{T} \\ \mathbf{H}_\tau^T\mathbf{P} + \mathbf{U} & \Psi_1 & 0 & 0 & -\mathbf{U} & \mathbf{H}_\tau^T\mathbf{T} \\ \mathbf{A^1}^T\mathbf{P} & 0 & -\mathbf{\Sigma}+\mathbf{Q} & 0 & 0 & \mathbf{A^1}^T\mathbf{T} \\ \mathbf{B^1}^T\mathbf{P} & 0 & 0 & -(1-h)\mathbf{Q} - \mathbf{\Lambda} & 0 & \mathbf{B^1}^T\mathbf{T} \\ 0 & -\mathbf{U}^T & 0 & 0 & -\frac{1}{r}\mathbf{T} & 0 \\ \mathbf{T}(-\mathbf{C^1}+\mathbf{H}) & \mathbf{T}\mathbf{H}_\tau & \mathbf{T}\mathbf{A^1} & \mathbf{T}\mathbf{B^1} & 0 & -\frac{1}{r}\mathbf{T} \end{pmatrix} < 0,$$

where $\Psi = -2\mathbf{P}\mathbf{C^1} + \mathbf{P}\mathbf{H} + \mathbf{H}^T\mathbf{P} + \mathbf{R} + \mathbf{F}\mathbf{\Sigma}\mathbf{F}$, $\Psi_1 = -(1-h)\mathbf{R} - 2\mathbf{U} + \mathbf{F}\mathbf{\Lambda}\mathbf{F}$, $F = \mathrm{diag}(F_1, F_2, \ldots, F_n) \in R^{n\times n}$, $\mathbf{F} = I_{N-1} \otimes F$, $H = MGJ$, $\mathbf{H} = (MGJ) \otimes D$, $\mathbf{H}_\tau = (MGJ) \otimes D_\tau$, and M and J are defined in (9).

Proof. See Appendix D.

Remark 5. Although Theorems 1–3 and Corollaries 1–8 give some rigorously theoretical conditions for the synchronization of network (12), it is also difficult to fix the suitable parameters of matrixes in these conditions. However, in real-world control systems, one can easily use MATLAB LMI Toolbox to numerically solve these system parameters. For example, in Theorem 1, fixing matrix T as in Corollary 2, one can use MATLAB LMI Toolbox to solve (20) and (21); in Theorems 2–3, one can use MATLAB LMI Toolbox to solve (28) and (33), respectively.

In this paper, the delay-independent and delay-dependent conditions are both further investigated. It is well known that the delay-independent conditions tend to be conservative for small time-delay. In addition, for the coupled networks with time-varying delay, the state estimation criteria proposed in [7] are not applicable to the case in which the derivative of the time-varying delay is larger than 1. In this case, assumption A_3 is not satisfied. In this paper, one overcomes this difficulty in Theorem 3. Moreover, in [6, 7, 8], the coupling matrix G is a diagonal matrix. However, we do not need this assumption in all theorems.

4. Numerical simulations. To verify the effectiveness of the proposed theorems and corollaries, three simple examples are given in the following.

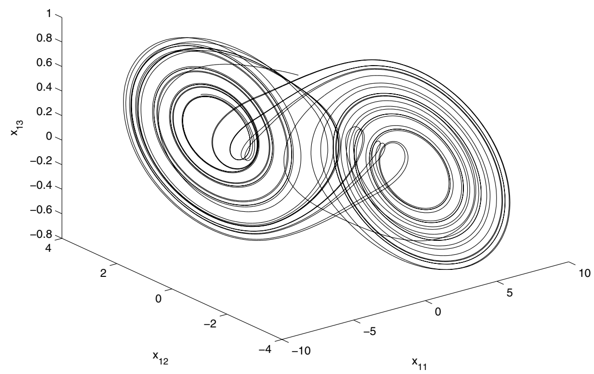

Figure 1. *Phase portrait of a single node.*

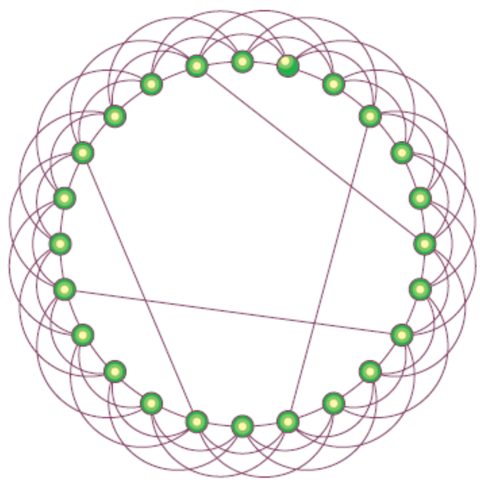

Figure 2. *Small-world network.*

4.1. Synchronization of small-world network. Consider the following Chua circuit described by [37]:

$$\begin{cases} \dot{x}_1 = \theta(-x_1 + x_2 - l(x_1)), \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -\beta x_2, \end{cases} \tag{34}$$

where $l(x_1) = bx_1 + 0.5(a-b)(|x_1+1| - |x_1-1|)$. The system (34) is chaotic as shown in Figure 1 when $\theta = 10$, $\beta = 18$, $a = -4/3$, and $b = -3/4$.

Now one considers network (4) with small-world connection as shown in Figure 2 [1], where the single node is given as above, and

$$D = \begin{pmatrix} 4 & 0.4 & 0.4 \\ 0.8 & 4 & 1.2 \\ -0.4 & -0.8 & 4 \end{pmatrix}.$$

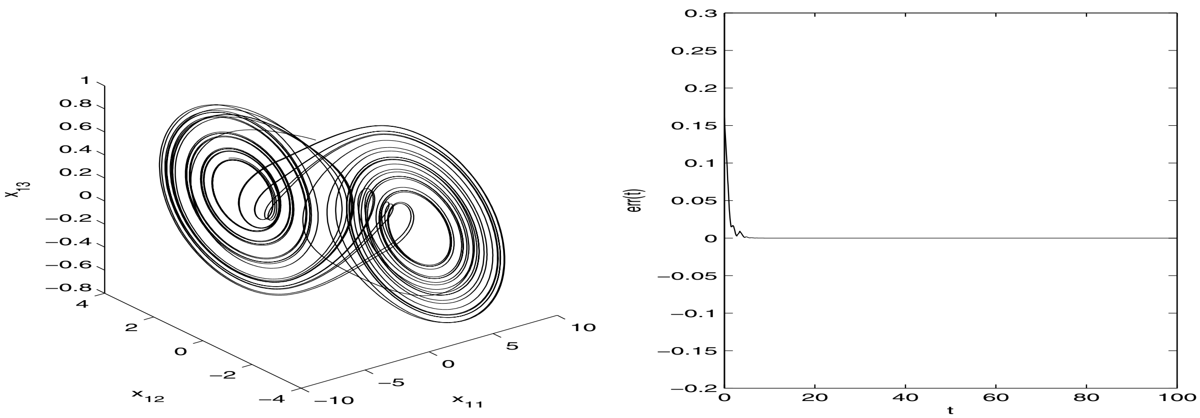

Figure 3. *Phase portrait of single node in network (4) and total synchronous error of the small-world network (4).*

According to Theorem 2 and MATLAB LMI Toolbox, one can easily attain the feasible solutions. Then network (4) is globally asymptotically synchronized. The total synchronous error of the small-world network is defined as follows:

$$err(t) = \frac{1}{25}\sum_{i=1}^{2}\sqrt{\sum_{j=1}^{25}[x_{1i}(t) - x_{ji}(t)]^2}.$$

Figure 3 shows the phase portrait of single node in network (4) and the total synchronous error of the small-world network (4). Similarly, one can verify the synchronization of network (4) with other topological structures, such as random graph and scale-free distribution.

4.2. Synchronization of a regular network. Consider the following 2-dimensional delayed system as a node, which is described by

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-\tau(t))) + I(t), \tag{35}$$

where $x(t) = (x_1(t), x_2(t))^T$, $f(x(t)) = (\tanh(x_1(t)), \tanh(x_2(t)))^T$, $I(t) = (0, 0)^T$,

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2.0 & -0.1 \\ -5.0 & 3.0 \end{pmatrix}, \quad B = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{pmatrix},$$

and $\tau(t) = 0.03[1 + \sin(40t)]$.

It is easy to verify that assumptions A_1 and A_2 hold for $F = I_2$ and assumption A_3 does not hold for $h = 1.2$, $r = 0.06$. The initial values are given as follows:

$$x_1(s) = 0.4, \quad x_2(s) = 0.6 \qquad \forall s \in [-1, 0].$$

Then system (35) has a chaotic attractor as shown in Figure 4.

Consider a regular network (4), where $A, B, C, I(t), f, \tau(t)$ are given above, and

$$G = \begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0.4 \\ 0.8 & 4 \end{pmatrix}, \quad D_\tau = \begin{pmatrix} 1 & 0.2 \\ 0.1 & 1 \end{pmatrix}.$$

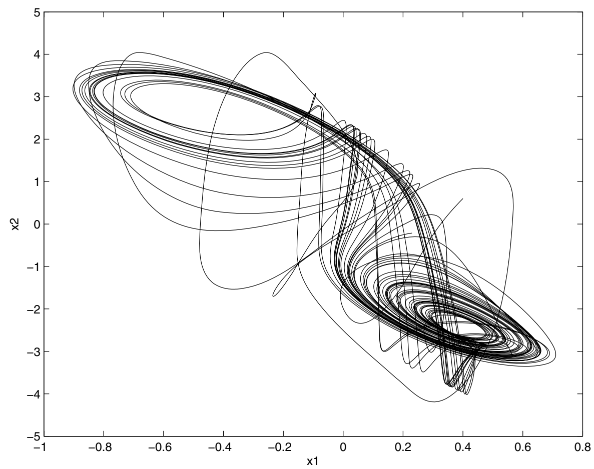

Figure 4. *Phase portrait of a single node.*

From Theorem 3, one gets the feasible solutions as follows:

$$\mathbf{P} = \begin{pmatrix} 9.4243 & -0.9377 & -1.6670 & 0.2372 \\ -0.9377 & 7.7092 & 0.2372 & -1.2747 \\ -1.6670 & 0.2372 & 9.4243 & -0.9377 \\ 0.2372 & -1.2747 & -0.9377 & 7.7092 \end{pmatrix},$$

$$\mathbf{Q} = \begin{pmatrix} 31.5252 & 8.0596 & 2.1015 & -0.8115 \\ 8.0596 & 21.9258 & -0.8115 & 0.8043 \\ 2.1015 & -0.8115 & 31.5252 & 8.0596 \\ -0.8115 & 0.8043 & 8.0596 & 21.9258 \end{pmatrix},$$

$$\mathbf{R} = \begin{pmatrix} 38.3723 & 2.7562 & -3.4028 & -3.1834 \\ 2.7562 & 20.8588 & -3.1834 & 1.0167 \\ -3.4028 & -3.1834 & 38.3723 & 2.7562 \\ -3.1834 & 1.0167 & 2.7562 & 20.8588 \end{pmatrix},$$

$$\mathbf{T} = \begin{pmatrix} 4.5225 & 0.1097 & -0.4335 & -0.1062 \\ 0.1097 & 3.1546 & -0.1062 & -0.0711 \\ -0.4335 & -0.1062 & 4.5225 & 0.1097 \\ -0.1062 & -0.0711 & 0.1097 & 3.1546 \end{pmatrix},$$

$$\mathbf{U} = \begin{pmatrix} 52.2980 & 3.7695 & 4.8829 & 1.0437 \\ 3.7695 & 44.0932 & 1.0437 & 5.5130 \\ 4.8829 & 1.0437 & 52.2980 & 3.7695 \\ 1.0437 & 5.5130 & 3.7695 & 44.0932 \end{pmatrix},$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} 118.7802 & 0 & 0 & 0 \\ 0 & 66.5769 & 0 & 0 \\ 0 & 0 & 118.7802 & 0 \\ 0 & 0 & 0 & 66.5769 \end{pmatrix},$$

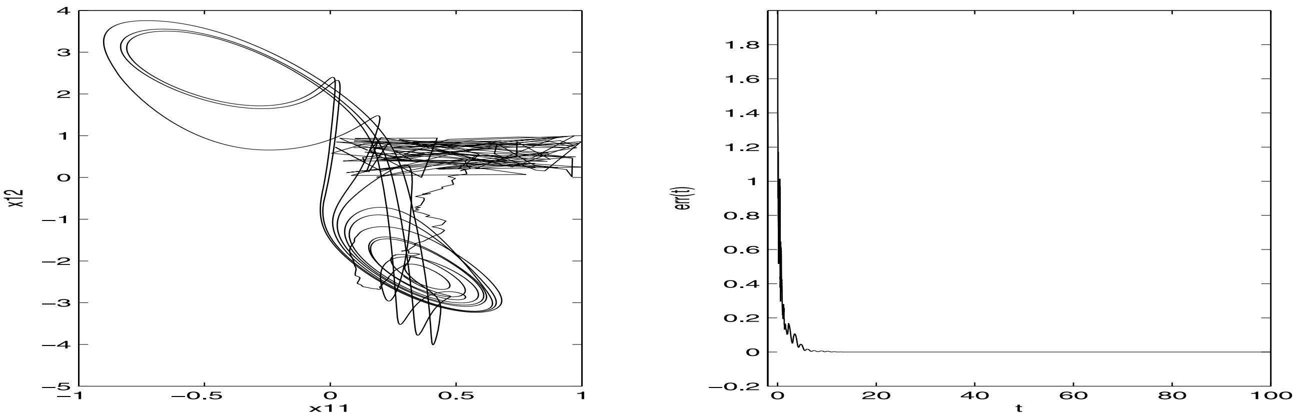

Figure 5. *Total synchronous error of network (4) with a regular structure.*

$$\mathbf{\Lambda} = \begin{pmatrix} 35.4724 & 0 & 0 & 0 \\ 0 & 30.6460 & 0 & 0 \\ 0 & 0 & 35.4724 & 0 \\ 0 & 0 & 0 & 30.6460 \end{pmatrix}.$$

According to Theorem 3, network (4) is globally asymptotically synchronized. The total error of network (4) is defined by

$$err(t) = \frac{1}{3}\sum_{i=1}^{2}\sqrt{[x_{1i}(t) - x_{2i}(t)]^2 + [x_{1i}(t) - x_{3i}(t)]^2}.$$

Figure 5 shows the total synchronous error of network (5), where the initial values are given by

$$x_1(s) = \begin{pmatrix} 0.1 \\ -0.3 \end{pmatrix}, \quad x_2(s) = \begin{pmatrix} 0.5 \\ -1 \end{pmatrix}, \quad x_3(s) = \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}.$$

4.3. Synchronization of scale-free networks with network size. The scale-free network model was proposed by Barabási and Albert [40]; they generated the network as follows:

(1) *Growth*: Starting with a small number (m_0) of nodes, at every time step a new node is introduced and connected to m ($\leq m_0$) existing nodes by undirected links.

(2) *Preferential attachment*: The probability that the new node is connected to node i is based on the degree k_i of node i:

$$p_i = \frac{k_i}{\sum_{j=1}^{N} k_j}.$$

After t time steps, this complex network has $N = t+m_0$ nodes and mt links. In the simulation, we take $m_0 = m = 5$, with each node being the same system as (35) except $\tau(t) = \frac{e^t}{1+e^t}$. It is obvious that $0 < \tau(t) < 1$, $\dot{\tau}(t) = \frac{e^t}{(1+e^t)^2} \leq \frac{1}{2} < 1$. G is connected in the scale-free network sense: if there exists a connection from node i to another node j in the scale-free

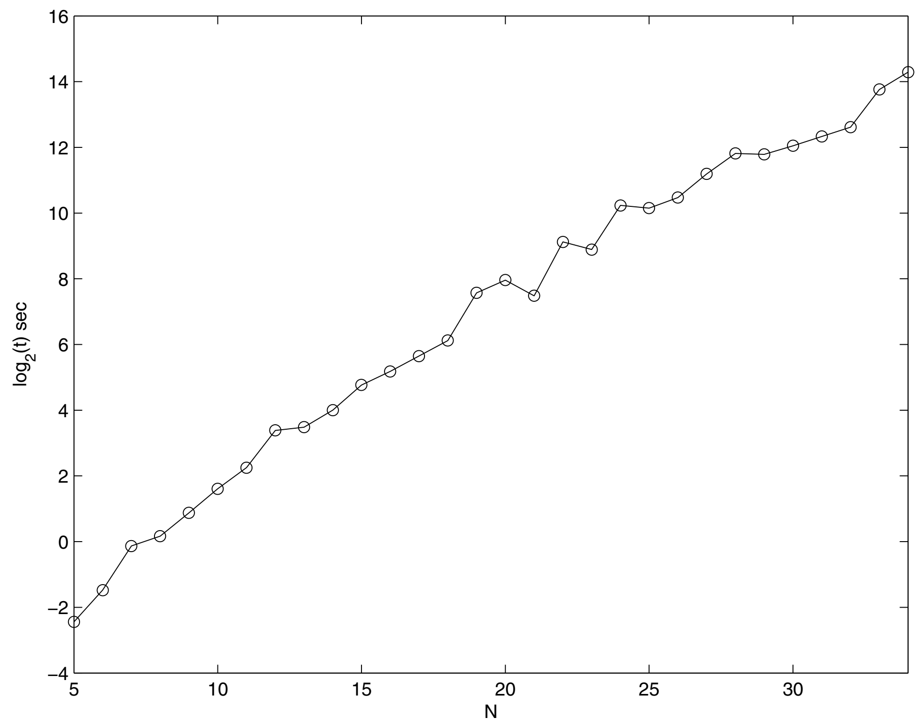

Figure 6. *The time for solving the LMI with network size in scale-free networks.*

network, then the coupling strength $G_{ij} = G_{ji} = 1$; otherwise, $G_{ij} = G_{ji} = 0$ $(j \neq i)$, and $G_{ii} = -\sum_{j=1,\, j\neq i}^{N} G_{ij}$.

$$D = \begin{pmatrix} 10 & 1 \\ 2 & 10 \end{pmatrix}, \quad D_\tau = \begin{pmatrix} 1 & 0.2 \\ 0.1 & 1 \end{pmatrix}.$$

Theorem 1 is available only when the scale-free network size $N = 5$ or $N = 6$. Therefore, the previous works [6, 7, 16] cannot be used to solve the network with size $N > 6$. However, Theorem 2 in this paper can be used to solve this problem. The sufficient condition in Theorem 2 is satisfied with more larger network size N. The time for solving the LMI in MATLAB Toolbox by using a normal computer and the corresponding network size are given in Figure 6.

Though the size of matrices $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{\Sigma}$ in Theorem 2 are of order $N \times N$, it can provide general and good results which is applicable for ensuring the synchronization of coupled networks with large size N. If the network size N is very large, the computation of LMI conditions is very difficult as in Figure 6 which should be further considered in the near future.

5. Conclusions. We have developed a generally linearly hybrid coupled network with time-varying delay and also further investigated its global synchronization. Based on this model, several effective sufficient conditions of global network synchronization are then deduced by using the Lyapunov function and an LMI. Both delay-independent and delay-dependent sufficient conditions are attained. It should be especially pointed out that we do not assume that the coupling matrix is symmetric or diagonal. However, most of the former works on network synchronization are based on this assumption. Moreover, we also generalize the derivative of the time-varying delay to any given value in this paper, although most of the former results are based on assumption A_3. To verify the effectiveness of the proposed synchronous criteria, a small-world network, a regular network, and scale-free networks with increasing network size are finally constructed to show the global network synchronization.

The proposed network model builds a platform for the study of network synchronization and other network dynamical behaviors. These network synchronous criteria also provide some new insight for the underlying mechanics of network synchronization. Furthermore, there are some abundant dynamical behaviors in the network which deserve to be also further investigated in the near future, such as the relation between network structure and function, the individual and global dynamics, etc. We will also explore the possible applications for these criteria in the real-world biological and engineering networks.

Appendix A. Proof of Theorem 1.

Let

$$\mathbf{\Sigma} = I_N \otimes \Sigma, \quad \mathbf{\Sigma^1} = I_p \otimes \Sigma, \quad \mathbf{\Delta} = I_p \otimes \Delta, \quad \mathbf{P} = I_p \otimes P,$$

$$\mathbf{Q} = I_p \otimes Q, \quad \mathbf{\Omega} = I_p \otimes \Omega, \quad \mathbf{I_n} = I_N \otimes I_n, \quad \mathbf{I_n^1} = I_p \otimes I_n.$$

Let $y(t) = \mathbf{M}x(t) = (y_1^T(t), y_2^T(t), \ldots, y_p^T(t))^T$, $y_i(t) = (y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t))^T$, $i = 1, 2, \ldots, p$.

According to (13), there exists a sufficiently small $\varepsilon > 0$, such that

$$\widetilde{\Lambda_0} = \begin{pmatrix} -2P(C - \varepsilon I_n) - \Delta + F\Sigma F & PA & PB \\ A^T P & -\Sigma + Q & 0 \\ B^T P & 0 & -(1-h)Q \end{pmatrix} < 0.$$

(i) Consider the Lyapunov candidate

$$V(t) = e^{2\varepsilon t} x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{M}x(t) + \int_{t-\tau(t)}^{t} e^{2\varepsilon s}\mathbf{f}^T(x(s))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(s))\, ds + \int_{t-\tau(t)}^{t} e^{2\varepsilon s} x^T(s)\mathbf{M}^T\mathbf{\Omega}\mathbf{M}x(s)\, ds. \tag{A.1}$$

Differentiating $V(t)$ along the trajectories of (12) yields

$$
\begin{aligned}
\dot V(t)|_{(12)} &= 2\varepsilon e^{2\varepsilon t}x^T(t)\mathbf{M}^T\mathbf{PM}x(t) + 2e^{2\varepsilon t}x^T(t)\mathbf{M}^T\mathbf{PM}\dot x(t) + e^{2\varepsilon t}\mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{QMf}(x(t)) \\
&\quad - (1-\dot\tau(t))e^{2\varepsilon(t-\tau(t))}\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\mathbf{QMf}(x(t-\tau(t))) \\
&\quad + e^{2\varepsilon t}x^T(t)\mathbf{M}^T\mathbf{\Omega M}x(t) \\
&\quad - (1-\dot\tau(t))e^{2\varepsilon(t-\tau(t))}x^T(t-\tau(t))\mathbf{M}^T\mathbf{\Omega M}x(t-\tau(t)) \\
&\le 2e^{2\varepsilon t}x^T(t)\mathbf{M}^T\mathbf{PM}[-(\mathbf{C}-\varepsilon\mathbf{I_n})x(t) + \mathbf{Af}(x(t)) + \mathbf{Bf}(x(t-\tau(t))) + \mathbf{I}(t) \\
&\quad + \mathbf{G}x(t) + \mathbf{G}_\tau x(t-\tau(t))] + e^{2\varepsilon t}\mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{QMf}(x(t)) \\
&\quad - (1-h)e^{2\varepsilon t}\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\mathbf{QMf}(x(t-\tau(t))) \\
&\quad + e^{2\varepsilon t}x^T(t)\mathbf{M}^T(\mathbf{\Omega}+\mathbf{\Delta}-\mathbf{\Delta})\mathbf{M}x(t) \\
&\quad - (1-h)e^{2\varepsilon t}x^T(t-\tau(t))\mathbf{M}^T\mathbf{\Omega M}x(t-\tau(t)).
\end{aligned} \tag{A.2}
$$

From the definition of $\mathbf{M}$, one gets

$$
\mathbf{MC} = \mathbf{C^1M}, \quad \mathbf{MA} = \mathbf{A^1M}, \quad \mathbf{MB} = \mathbf{B^1M}, \quad \mathbf{MI_n} = \mathbf{I_n^1M}, \quad \mathbf{MI}(t) = 0.
$$

Therefore, one has

$$
\begin{aligned}
\dot V(t)|_{(12)} &\le 2e^{2\varepsilon t}x^T(t)\mathbf{M}^T\mathbf{P}[-(\mathbf{C^1}-\varepsilon\mathbf{I_n^1})\mathbf{M}x(t) + \mathbf{A^1Mf}(x(t)) + \mathbf{B^1Mf}(x(t-\tau(t))) \\
&\quad + \mathbf{MG}x(t) + \mathbf{MG}_\tau x(t-\tau(t))] + e^{2\varepsilon t}\mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{QMf}(x(t)) \\
&\quad - (1-h)e^{2\varepsilon t}\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\mathbf{QMf}(x(t-\tau(t))) \\
&\quad + e^{2\varepsilon t}x^T(t)\mathbf{M}^T(\mathbf{\Omega}+\mathbf{\Delta}-\mathbf{\Delta})\mathbf{M}x(t) \\
&\quad - (1-h)e^{2\varepsilon t}x^T(t-\tau(t))\mathbf{M}^T\mathbf{\Omega M}x(t-\tau(t)).
\end{aligned} \tag{A.3}
$$

According to assumption A_2, one gets

$$
\begin{aligned}
\mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{\Sigma Mf}(x(t)) &= \textstyle\sum_{j=1}^p \alpha_j^2[f(x_{j_1}(t)) - f(x_{j_2}(t))]^T\Sigma[f(x_{j_1}(t)) - f(x_{j_2}(t))] \\
&\le \textstyle\sum_{j=1}^p y_j^T(t)F\Sigma F y_j(t).
\end{aligned} \tag{A.4}
$$

Let

$$
\xi_j = \begin{pmatrix} y_j^T(t) & \alpha_j(f(x_{j_1}(t)) - f(x_{j_2}(t)))^T & \alpha_j(f(x_{j_1}(t-\tau(t))) - f(x_{j_2}(t-\tau(t))))^T \end{pmatrix}^T.
$$

It follows from (A.3)–(A.4) that

$$
\begin{aligned}
\dot V(t)|_{(12)} &\le e^{2\varepsilon t}\textstyle\sum_{j=1}^{p}\xi_j^T\widetilde{\Lambda_0}\xi_j + 2e^{2\varepsilon t}x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{M}[\mathbf{G}x(t)+\mathbf{G}_\tau x(t-\tau(t))] + e^{2\varepsilon t}x^T(t)\mathbf{M}^T\\
&\quad\times(\mathbf{\Omega}+\mathbf{\Delta})\mathbf{M}x(t) - (1-h)e^{2\varepsilon t}x^T(t-\tau(t))\mathbf{M}^T\mathbf{\Omega}\mathbf{M}x(t-\tau(t))\\
&= e^{2\varepsilon t}\textstyle\sum_{j=1}^{p}\xi_j^T\widetilde{\Lambda_0}\xi_j + 2e^{2\varepsilon t}x^T(t)(\widetilde{M}^T\otimes I_n)(I_p\otimes P)(\widetilde{M}\otimes I_n)(G\otimes D)x(t)\\
&\quad+2e^{2\varepsilon t}x^T(t)(\widetilde{M}^T\otimes I_n)(I_p\otimes P)(\widetilde{M}\otimes I_n)(G\otimes D_\tau)x(t-\tau(t))\\
&\quad+e^{2\varepsilon t}x^T(t)(\widetilde{M}^T\otimes I_n)(I_p\otimes(\Omega+\Delta))(\widetilde{M}\otimes I_n)x(t)\\
&\quad-(1-h)e^{2\varepsilon t}x^T(t-\tau(t))(\widetilde{M}^T\otimes I_n)(I_p\otimes\Omega)(\widetilde{M}\otimes I_n)x(t-\tau(t))\\
&= e^{2\varepsilon t}\textstyle\sum_{j=1}^{p}\xi_j^T\widetilde{\Lambda_0}\xi_j + 2e^{2\varepsilon t}x^T(t)(\widetilde{M}^T\widetilde{M}G\otimes PD)x(t) + 2e^{2\varepsilon t}x^T(t)(\widetilde{M}^T\widetilde{M}G\otimes PD_\tau)\\
&\quad\times x(t-\tau(t)) + e^{2\varepsilon t}x^T(t)(\widetilde{M}^T\widetilde{M}\otimes(\Omega+\Delta))x(t) - (1-h)e^{2\varepsilon t}x^T(t-\tau(t))\\
&\quad\times(\widetilde{M}^T\widetilde{M}\otimes\Omega)x(t-\tau(t))\\
&= e^{2\varepsilon t}\textstyle\sum_{j=1}^{p}\xi_j^T\widetilde{\Lambda_0}\xi_j - 2e^{2\varepsilon t}x^T(t)(TG\otimes PD)x(t) - 2e^{2\varepsilon t}x^T(t)(TG\otimes PD_\tau)x(t-\tau(t))\\
&\quad-e^{2\varepsilon t}x^T(t)(T\otimes(\Omega+\Delta))x(t) + (1-h)e^{2\varepsilon t}x^T(t-\tau(t))(T\otimes\Omega)x(t-\tau(t))\\
&= e^{2\varepsilon t}\textstyle\sum_{j=1}^{p}\xi_j^T\widetilde{\Lambda_0}\xi_j - 2e^{2\varepsilon t}\sum_{i=1}^{N}\sum_{j=1}^{N}x_i^T(t)(\sum_{k=1}^{N}T_{ik}G_{kj}PD)x_j(t)\\
&\quad-2e^{2\varepsilon t}\textstyle\sum_{i=1}^{N}\sum_{j=1}^{N}x_i^T(t)(\sum_{k=1}^{N}T_{ik}G_{kj}PD_\tau)x_j(t-\tau(t)) - e^{2\varepsilon t}\sum_{i=1}^{N}\sum_{j=1}^{N}x_i^T(t)T_{ij}\\
&\quad\times(\Omega+\Delta)x_j(t) + (1-h)e^{2\varepsilon t}\textstyle\sum_{i=1}^{N}\sum_{j=1}^{N}x_i^T(t-\tau(t))T_{ij}\Omega x_j(t-\tau(t)).
\end{aligned}
\tag{A.5}
$$

Denote $L_{ij}=\sum_{k=1}^{N}T_{ik}G_{kj}$. Then one obtains

$$
\begin{aligned}
\textstyle\sum_{j=1}^{N}L_{ij} &= \textstyle\sum_{j=1,\, j\ne i}^{N}L_{ij}+L_{ii}\\
&= \textstyle\sum_{j=1,\, j\ne i}^{N}\sum_{k=1}^{N}T_{ik}G_{kj}+\sum_{k=1}^{N}T_{ik}G_{ki}\\
&= \textstyle\sum_{k=1}^{N}(\sum_{j=1,\, j\ne i}^{N}T_{ik}G_{kj}+T_{ik}G_{ki})\\
&= \textstyle\sum_{k=1}^{N}T_{ik}(\sum_{j=1}G_{kj})=0.
\end{aligned}
\tag{A.6}
$$

Thus one has

$$
L_{ii}=-\sum_{j=1,\, j\ne i}^{N}L_{ij},\quad i=1,2,\dots,N. \tag{A.7}
$$

Let $\eta_{ij} = (x_i^T(t) - x_j^T(t),\, x_i^T(t-\tau(t)) - x_j^T(t-\tau(t)))^T$. From (A.5), one gets

(A.8)
$$
\begin{aligned}
\dot{V}(t)|_{(12)} \le{}& e^{2\varepsilon t}\textstyle\sum_{j=1}^{p} \xi_j^T\widetilde{\Lambda_0}\xi_j - 2e^{2\varepsilon t}\sum_{i=1}^{N}\sum_{j=1}^{N} x_i^T(t)(L_{ij}PD)x_j(t)\\
&- 2e^{2\varepsilon t}\textstyle\sum_{i=1}^{N}\sum_{j=1}^{N} x_i^T(t)(L_{ij}PD_\tau)x_j(t-\tau(t))\\
&- e^{2\varepsilon t}\textstyle\sum_{i=1}^{N}\sum_{j=1}^{N} x_i^T(t)T_{ij}(\Omega+\Delta)x_j(t)\\
&+ (1-h)e^{2\varepsilon t}\textstyle\sum_{i=1}^{N}\sum_{j=1}^{N} x_i^T(t-\tau(t))T_{ij}\Omega x_j(t-\tau(t))\\
={}& e^{2\varepsilon t}\textstyle\sum_{j=1}^{p} \xi_j^T\widetilde{\Lambda_0}\xi_j - 2e^{2\varepsilon t}\sum_{i=1}^{N}\Big(\sum_{j=1,\, j\neq i}^{N} x_i^T(t)(L_{ij}PD)x_j(t) + x_i^T(t)L_{ii}PDx_i(t)\Big)\\
&- 2e^{2\varepsilon t}\textstyle\sum_{i=1}^{N}\Big(\sum_{j=1,\, j\neq i}^{N} x_i^T(t)(L_{ij}PD_\tau)x_j(t-\tau(t)) + x_i^T(t)L_{ii}PD_\tau x_i(t-\tau(t))\Big)\\
&- e^{2\varepsilon t}\textstyle\sum_{i=1}^{N}\Big(\sum_{j=1,\, j\neq i}^{N} x_i^T(t)T_{ij}(\Omega+\Delta)x_j(t) + x_i^T(t)T_{ii}(\Omega+\Delta)x_i(t)\Big) + (1-h)\\
&\times e^{2\varepsilon t}\textstyle\sum_{i=1}^{N}\Big(\sum_{j=1,\, j\neq i}^{N} x_i^T(t-\tau(t))T_{ij}\Omega x_j(t-\tau(t)) + x_i^T(t-\tau(t))T_{ii}\Omega x_i(t-\tau(t))\Big)\\
={}& e^{2\varepsilon t}\textstyle\sum_{j=1}^{p} \xi_j^T\widetilde{\Lambda_0}\xi_j + 2e^{2\varepsilon t}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} (x_i(t)-x_j(t))^T (L_{ij}PD)(x_i(t)-x_j(t))\\
&+ 2e^{2\varepsilon t}\textstyle\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} (x_i(t)-x_j(t))^T (L_{ij}PD_\tau)(x_i(t-\tau(t))-x_j(t-\tau(t)))\\
&+ e^{2\varepsilon t}\textstyle\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} (x_i(t)-x_j(t))^T T_{ij}(\Omega+\Delta)(x_i(t)-x_j(t)) - (1-h)\\
&\times e^{2\varepsilon t}\textstyle\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} (x_i(t-\tau(t))-x_j(t-\tau(t)))^T T_{ij}\Omega(x_i(t-\tau(t))-x_j(t-\tau(t)))\\
={}& e^{2\varepsilon t}\textstyle\sum_{j=1}^{p} \xi_j^T\widetilde{\Lambda_0}\xi_j + e^{2\varepsilon t}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \eta_{ij}^T\Lambda_{ij}\eta_{ij}.
\end{aligned}
$$

According to Lemma 4 and (A.8), under conditions (13)–(14), $\dot{V}(y(t)) \le 0$ and $V(t) \le V(0)$. That is, $V(t)$ is a bounded function and $\|y(t)\| = O(e^{-\varepsilon t})$. This completes the proof of part (i).

(ii) Let $y(t) = \mathbf{M}x(t) = (y_1^T(t), y_2^T(t), \dots, y_p^T(t))^T$, $y_i(t) = (y_{i1}(t), y_{i2}(t), \dots, y_{in}(t))^T$, $i = 1, 2, \dots, p$, $\widetilde{x}_j(t) = (x_{1j}(t), x_{2j}(t), \dots, x_{Nj}(t))^T$, and $\widetilde{y}_j(t) = (y_{1j}(t), y_{2j}(t), \dots, y_{pj}(t))^T$. Then $\widetilde{y}_j(t) = \widetilde{M}\widetilde{x}_j(t)$ for $j = 1, 2, \dots, n$.

Following the same method in part (i), from (A.5), one has

(A.9)
$$
\begin{aligned}
\dot{V}(t)|_{(12)} \le{}& e^{2\varepsilon t}\textstyle\sum_{j=1}^{p} \xi_j^T\widetilde{\Lambda_0}\xi_j - 2e^{2\varepsilon t}x^T(t)(TG\otimes PD)x(t) - 2e^{2\varepsilon t}x^T(t)(TG\otimes PD_\tau)x(t-\tau(t))\\
&- e^{2\varepsilon t}x^T(t)(T\otimes(\Omega+\Delta))x(t) + (1-h)e^{2\varepsilon t}x^T(t-\tau(t))(T\otimes\Omega)x(t-\tau(t))\\
={}& e^{2\varepsilon t}\textstyle\sum_{j=1}^{p} \xi_j^T\widetilde{\Lambda_0}\xi_j - 2e^{2\varepsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n} \widetilde{x}_i^T(t)(\sum_{k=1}^{n} p_{ik}d_{kj}TG)\widetilde{x}_j(t)\\
&- 2e^{2\varepsilon t}\textstyle\sum_{i=1}^{n}\sum_{j=1}^{n} \widetilde{x}_i^T(t)(\sum_{k=1}^{n} p_{ik}d_{\tau kj}TG)\widetilde{x}_j(t-\tau(t)) - e^{2\varepsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n} \widetilde{x}_i^T(t)\\
&\times(\Omega_{ij}+\Delta_{ij})T\widetilde{x}_j(t) + (1-h)e^{2\varepsilon t}\textstyle\sum_{i=1}^{n}\sum_{j=1}^{n} \widetilde{x}_i^T(t-\tau(t))\Omega_{ij}T\widetilde{x}_j(t-\tau(t)).
\end{aligned}
$$

Let $\widetilde{\eta}_{ij} = (\widetilde{x}_i^T(t) - \widetilde{x}_j^T(t),\, \widetilde{x}_i^T(t-\tau(t)) - \widetilde{x}_j^T(t-\tau(t)))^T$. According to (A.9), one obtains

$$
\dot{V}(t)|_{(12)} \le e^{2\varepsilon t}\sum_{j=1}^{p} \xi_j^T\widetilde{\Lambda_0}\xi_j + e^{2\varepsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n} \widetilde{\eta}_{ij}^T\widetilde{\Lambda}_{ij}\widetilde{\eta}_{ij}. \tag{A.10}
$$

From Lemma 4 and (A.10), under conditions (13) and (15), $\dot{V}(y(t)) \leq 0$ and $V(t) \leq V(0)$. That is, $V(t)$ is a bounded function and $\|y(t)\| = O(e^{-\varepsilon t})$. This completes the proof of part (ii).

Appendix B. Proof of Theorem 2. Consider the following Lyapunov candidate:

(B.1)
$$V(t) = x^T(t)\mathbf{M}^T\mathbf{PM}x(t) + \int_{t-\tau(t)}^{t} \mathbf{f}^T(x(s))\mathbf{M}^T\mathbf{QMf}(x(s))\,ds + \int_{t-\tau(t)}^{t} x^T(s)\mathbf{M}^T\mathbf{RM}x(s)\,ds.$$

Differentiating $V(t)$ along the trajectories of (12) yields

(B.2)
$$\begin{aligned}
\dot{V}(t)|_{(12)} &= 2x^T(t)\mathbf{M}^T\mathbf{PM}\dot{x}(t) + \mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{QMf}(x(t)) - (1-\dot{\tau}(t))\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T \\
&\quad \times \mathbf{QMf}(x(t-\tau(t))) + x^T(t)\mathbf{M}^T\mathbf{RM}x(t) \\
&\quad - (1-\dot{\tau}(t))x^T(t-\tau(t))\mathbf{M}^T\mathbf{RM}x(t-\tau(t)) \\
&\leq 2x^T(t)\mathbf{M}^T\mathbf{PM}[-\mathbf{C}x(t) + \mathbf{Af}(x(t)) + \mathbf{Bf}(x(t-\tau(t))) + \mathbf{I}(t) + \mathbf{G}x(t) \\
&\qquad\qquad + \mathbf{G}_\tau x(t-\tau(t))] \\
&\quad + \mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{QMf}(x(t)) - (1-h)\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\mathbf{QMf}(x(t-\tau(t))) \\
&\quad + x^T(t)\mathbf{M}^T\mathbf{RM}x(t) - (1-\dot{\tau}(t))x^T(t-\tau(t))\mathbf{M}^T\mathbf{RM}x(t-\tau(t)).
\end{aligned}$$

From the definition of $\mathbf{M}$, one has

$$\mathbf{MC} = \mathbf{C^1M}, \quad \mathbf{MA} = \mathbf{A^1M}, \quad \mathbf{MB} = \mathbf{B^1M}, \quad \mathbf{MI}(t) = 0.$$

Therefore, one obtains

(B.3)
$$\begin{aligned}
\dot{V}(t)|_{(12)} &\leq 2x^T(t)\mathbf{M}^T\mathbf{P}[-\mathbf{C^1M}x(t) + \mathbf{A^1Mf}(x(t)) + \mathbf{B^1Mf}(x(t-\tau(t))) + \mathbf{MG}x(t) \\
&\qquad\qquad + \mathbf{G}_\tau x(t-\tau(t))] \\
&\quad + \mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{QMf}(x(t)) - (1-h)\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\mathbf{QMf}(x(t-\tau(t))) \\
&\quad + x^T(t)\mathbf{M}^T\mathbf{RM}x(t) - (1-h)x^T(t-\tau(t))\mathbf{M}^T\mathbf{RM}x(t-\tau(t)).
\end{aligned}$$

According to assumption A_2, one gets

(B.4)
$$\begin{aligned}
\mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{\Sigma Mf}(x(t)) &= \textstyle\sum_{j=1}^{N-1} [f(x_j(t)) - f(x_{j+1}(t))]^T\mathbf{\Sigma}_j[f(x_j(t)) - f(x_{j+1}(t))] \\
&\leq \textstyle\sum_{j=1}^{N-1} [x_j(t) - x_{j+1}(t)]^T F\mathbf{\Sigma}_j F[x_j(t) - x_{j+1}(t)] \\
&= x^T(t)\mathbf{M}^T\mathbf{F\Sigma FM}x(t),
\end{aligned}$$

where $\mathbf{\Sigma}_j = \mathrm{diag}(\Sigma_{(j-1)n+1}, \ldots, \Sigma_{jn})$. From Lemmas 1 and 5, one has

(B.5)
$$\begin{aligned}
2x^T(t)\mathbf{M}^T\mathbf{PMG}x(t) &= 2x^T(t)\mathbf{M}^T\mathbf{P}[(M \otimes I_n)(G \otimes D)]x(t) \\
&= 2x^T(t)\mathbf{M}^T\mathbf{P}[MG \otimes D]x(t) \\
&= 2x^T(t)\mathbf{M}^T\mathbf{P}[HM \otimes D]x(t) \\
&= 2x^T(t)\mathbf{M}^T\mathbf{P}[(H \otimes D)(M \otimes I_n)]x(t) \\
&= 2x^T(t)\mathbf{M}^T\mathbf{PHM}x(t)
\end{aligned}$$

and

$$
\begin{aligned}
2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{M}\mathbf{G}_\tau x(t-\tau(t)) &= 2x^T(t)\mathbf{M}^T\mathbf{P}[(M\otimes I_n)(G\otimes D_\tau)]x(t-\tau(t))\\
&= 2x^T(t)\mathbf{M}^T\mathbf{P}[MG\otimes D_\tau]x(t-\tau(t))\\
&= 2x^T(t)\mathbf{M}^T\mathbf{P}[HM\otimes D_\tau]x(t-\tau(t))\\
&= 2x^T(t)\mathbf{M}^T\mathbf{P}[(H\otimes D_\tau)(M\otimes I_n)]x(t-\tau(t))\\
&= 2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{H}_\tau\mathbf{M}x(t-\tau(t)),
\end{aligned}\tag{B.6}
$$

where $H = MGJ$, $\mathbf{H} = (MGJ)\otimes D$, $\mathbf{H}_\tau = (MGJ)\otimes D_\tau$, and M and J are defined in (9).

According to (B.3)–(B.6), one obtains

$$
\begin{aligned}
\dot{V}(t)|_{(12)} \le{}& -2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{C}^{\mathbf{1}}\mathbf{M}x(t) + 2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{A}^{\mathbf{1}}\mathbf{M}\mathbf{f}(x(t))\\
&+ 2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{B}^{\mathbf{1}}\mathbf{M}\mathbf{f}(x(t-\tau(t)))\\
&+ 2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{H}\mathbf{M}x(t) + 2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{H}_\tau\mathbf{M}x(t-\tau(t)) + x^T(t)\mathbf{M}^T\mathbf{F}\mathbf{\Sigma}\mathbf{F}\mathbf{M}x(t)\\
&- \mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{\Sigma}\mathbf{M}\mathbf{f}(x(t)) + \mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(t)) - (1-h)\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\\
&\times \mathbf{Q}\mathbf{M}\mathbf{f}(x(t-\tau(t))) + x^T(t)\mathbf{M}^T\mathbf{R}\mathbf{M}x(t) - (1-h)x^T(t-\tau(t))\mathbf{M}^T\mathbf{R}\mathbf{M}x(t-\tau(t))\\
={}& \eta^T(t)\mathbf{\Omega}\eta(t),
\end{aligned}\tag{B.7}
$$

where

$$
\eta(t) = \begin{pmatrix} x^T(t)\mathbf{M}^T & x^T(t-\tau(t))\mathbf{M}^T & \mathbf{f}^T(x(t))\mathbf{M}^T & \mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T \end{pmatrix}^T.
$$

From Lemma 4 and (B.7), under the condition (28), $\dot{V}(t)\le 0$ and $V(t)\le V(0)$. That is, $V(t)$ is a bounded function and $\|\mathbf{M}x(t)\| \longrightarrow 0$. This proof is thus completed.

Appendix C. Proof of Corollary 7. Select the Lyapunov candidate (B.1), where M and J are also defined in (9), $\mathbf{P} = I_{N-1}\otimes P$, $\mathbf{Q} = I_{N-1}\otimes Q$, $\mathbf{R} = I_{N-1}\otimes R$, and $\mathbf{\Sigma} = I_{N-1}\otimes \Sigma$.

From (B.7), one obtains

$$
\begin{aligned}
\dot{V}(t)|_{(12)} &\le \eta^T(t)\mathbf{\Omega}\eta(t)\\
&= \textstyle\sum_{j=1}^{N-1}\xi_j^T\Lambda_0\xi_j + \left(x^T(t)\mathbf{M}^T, x^T(t-\tau(t))\mathbf{M}^T\right)\Xi\left(\mathbf{M}x^T(t), \mathbf{M}x^T(t-\tau(t))\right)^T,
\end{aligned}\tag{C.1}
$$

where

$$
\xi_j = \begin{pmatrix} (x_j(t)-x_{j+1}(t))^T & (f(x_j(t))-f(x_{j+1}(t)))^T & (f(x_j(t-\tau(t)))-f(x_{j+1}(t-\tau(t))))^T \end{pmatrix}^T
$$

and $\mathbf{\Omega}$ is defined in (28). According to Lemma 4 and (C.1), under the conditions (31)–(32), $\dot{V}(t)\le 0$ and $V(t)\le V(0)$. That is, $V(t)$ is a bounded function, and $\|\mathbf{M}x(t)\| \longrightarrow 0$. This completes the proof.

Appendix D. Proof of Theorem 3. Construct the Lyapunov function as follows:

(D.1)
$$V(t) = x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{M}x(t) + \int_{t-\tau(t)}^t \mathbf{f}^T(x(s))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(s))\,ds + \int_{t-\tau(t)}^t x^T(s)\mathbf{M}^T\mathbf{R}\mathbf{M}x(s)\,ds + \int_{-r}^0 d\theta \int_{t+\theta}^t \dot{x}^T(s)\mathbf{M}^T\mathbf{T}\mathbf{M}\dot{x}(s)\,ds.$$

From Lemma 6, differentiating $V(t)$ along the trajectories of (12) results in

(D.2)
$$\begin{aligned}
\dot{V}(t)|_{(12)} &= 2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{M}\dot{x}(t) + \mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(t)) - (1-\dot{\tau}(t))\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T \\
&\quad \times \mathbf{Q}\mathbf{M}\mathbf{f}(x(t-\tau(t))) + x^T(t)\mathbf{M}^T\mathbf{R}\mathbf{M}x(t) \\
&\quad - (1-\dot{\tau}(t))x^T(t-\tau(t))\mathbf{M}^T\mathbf{R}\mathbf{M}x(t-\tau(t)) \\
&\quad + r\dot{x}^T(t)\mathbf{M}^T\mathbf{T}\mathbf{M}\dot{x}(t) - \int_{t-r}^t \dot{x}^T(\theta)\mathbf{M}^T\mathbf{T}\mathbf{M}\dot{x}(\theta)\,d\theta \\
&\le 2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{M}[-\mathbf{C}x(t) + \mathbf{A}\mathbf{f}(x(t)) + \mathbf{B}\mathbf{f}(x(t-\tau(t))) + \mathbf{I}(t) + \mathbf{G}x(t) \\
&\qquad + \mathbf{G}_\tau x(t-\tau(t))] \\
&\quad + \mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(t)) - (1-h)\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(t-\tau(t))) \\
&\quad + x^T(t)\mathbf{M}^T\mathbf{R}\mathbf{M}x(t) - (1-h)x^T(t-\tau(t))\mathbf{M}^T\mathbf{R}\mathbf{M}x(t-\tau(t)) + r\dot{x}^T(t)\mathbf{M}^T\mathbf{T}\mathbf{M}\dot{x}(t) \\
&\quad - \frac{1}{r}\left(\int_{t-\tau(t)}^t \mathbf{M}\dot{x}(\theta)\,d\theta\right)^T \mathbf{T}\left(\int_{t-\tau(t)}^t \mathbf{M}\dot{x}(\theta)\,d\theta\right).
\end{aligned}$$

From the definition of $\mathbf{M}$, one has

$$\mathbf{MC} = \mathbf{C^1M}, \quad \mathbf{MA} = \mathbf{A^1M}, \quad \mathbf{MB} = \mathbf{B^1M}, \quad \mathbf{MI}(t) = 0.$$

According to (D.2), one obtains

(D.3)
$$\begin{aligned}
\dot{V}(t)|_{(12)} &\le 2x^T(t)\mathbf{M}^T\mathbf{P}[-\mathbf{C^1M}x(t) + \mathbf{A^1M}\mathbf{f}(x(t)) + \mathbf{B^1M}\mathbf{f}(x(t-\tau(t))) + \mathbf{HM}x(t) \\
&\qquad + \mathbf{H}_\tau\mathbf{M}x(t-\tau(t))] \\
&\quad + \mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(t)) - (1-h)\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(t-\tau(t))) \\
&\quad + x^T(t)\mathbf{M}^T\mathbf{R}\mathbf{M}x(t) - (1-h)x^T(t-\tau(t))\mathbf{M}^T\mathbf{R}\mathbf{M}x(t-\tau(t)) + r\dot{x}^T(t)\mathbf{M}^T\mathbf{T}\mathbf{M}\dot{x}(t) \\
&\quad - \frac{1}{r}\left(\int_{t-\tau(t)}^t \mathbf{M}\dot{x}(\theta)\,d\theta\right)^T \mathbf{T}\left(\int_{t-\tau(t)}^t \mathbf{M}\dot{x}(\theta)\,d\theta\right).
\end{aligned}$$

Similar to (B.4), one has

$$\mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{\Sigma}\mathbf{M}\mathbf{f}(x(t)) \le x^T(t)\mathbf{M}^T\mathbf{F\Sigma F M}x(t) \tag{D.4}$$

and

$$\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\mathbf{\Lambda}\mathbf{M}\mathbf{f}(x(t-\tau(t))) \le x^T(t-\tau(t))\mathbf{M}^T\mathbf{F\Lambda F M}x(t-\tau(t)). \tag{D.5}$$

From the Leibniz–Newton formula, for any matrix $\mathbf{U}$ with appropriate dimensions, one gets

$$x^T(t-\tau(t))\mathbf{M}^T\mathbf{U}\mathbf{M}\left(x(t)-x(t-\tau(t))-\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds\right)=0. \tag{D.6}$$

Let

$$\xi(t)=\begin{pmatrix} x^T(t)\mathbf{M}^T & x^T(t-\tau(t))\mathbf{M}^T & \mathbf{f}^T(x(t))\mathbf{M}^T & \mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T & \int_{t-\tau(t)}^{t}\dot{x}^T(s)\,ds\,\mathbf{M}^T \end{pmatrix}^T,$$
$$\Pi=\begin{pmatrix} -\mathbf{C^1}+\mathbf{H} & \mathbf{H}_\tau & \mathbf{A^1} & \mathbf{B^1} & 0 \end{pmatrix};$$

then one has

$$\dot{x}^T(t)\mathbf{M}^T\mathbf{T}\mathbf{M}\dot{x}(t)=\xi^T(t)\Pi^T\mathbf{T}\Pi\xi(t). \tag{D.7}$$

Combining this with (D.3)–(D.7), one obtains

(D.8)
$$\begin{aligned}
\dot{V}(t)|_{(12)} \le\ & -2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{C^1}\mathbf{M}x(t)+2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{A^1}\mathbf{M}\mathbf{f}(x(t))+2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{B^1}\mathbf{M}\mathbf{f}(x(t-\tau(t)))\\
& +2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{H}\mathbf{M}x(t)+2x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{H}_\tau\mathbf{M}x(t-\tau(t))+x^T(t)\mathbf{M}^T\mathbf{F}\boldsymbol{\Sigma}\mathbf{F}\mathbf{M}x(t)\\
& -\mathbf{f}^T(x(t))\mathbf{M}^T\boldsymbol{\Sigma}\mathbf{M}\mathbf{f}(x(t))+x^T(t-\tau(t))\mathbf{M}^T\mathbf{F}\boldsymbol{\Lambda}\mathbf{F}\mathbf{M}x(t-\tau(t))-\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\\
& \times\boldsymbol{\Lambda}\mathbf{M}\mathbf{f}(x(t-\tau(t)))+\mathbf{f}^T(x(t))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(t))-(1-h)\mathbf{f}^T(x(t-\tau(t)))\mathbf{M}^T\mathbf{Q}\mathbf{M}\\
& \times\mathbf{f}(x(t-\tau(t)))+x^T(t)\mathbf{M}^T\mathbf{R}\mathbf{M}x(t)-(1-h)x^T(t-\tau(t))\mathbf{M}^T\mathbf{R}\mathbf{M}x(t-\tau(t))\\
& +r\xi^T(t)\Pi^T\mathbf{T}\Pi\xi(t)-\frac{1}{r}\left(\int_{t-\tau(t)}^{t}\mathbf{M}\dot{x}(\theta)\,d\theta\right)^T\mathbf{T}\left(\int_{t-\tau(t)}^{t}\mathbf{M}\dot{x}(\theta)\,d\theta\right)\\
& +2x^T(t-\tau(t))\mathbf{M}^T\mathbf{U}\mathbf{M}\left(x(t)-x(t-\tau(t))-\textstyle\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds\right)\\
=\ & \xi^T(t)(\boldsymbol{\Xi}+r\Pi^T\mathbf{T}\Pi)\xi(t),
\end{aligned}$$

where

$$\boldsymbol{\Xi}=\begin{pmatrix}
\Psi & \mathbf{P}\mathbf{H}_\tau+\mathbf{U}^T & \mathbf{P}\mathbf{A^1} & \mathbf{P}\mathbf{B^1} & 0\\
\mathbf{H}_\tau^T\mathbf{P}+\mathbf{U} & -(1-h)\mathbf{R}-2\mathbf{U}+\mathbf{F}\boldsymbol{\Lambda}\mathbf{F} & 0 & 0 & -\mathbf{U}\\
\mathbf{A^1}^T\mathbf{P} & 0 & -\boldsymbol{\Sigma}+\mathbf{Q} & 0 & 0\\
\mathbf{B^1}^T\mathbf{P} & 0 & 0 & -(1-h)\mathbf{Q}-\boldsymbol{\Lambda} & 0\\
0 & -\mathbf{U}^T & 0 & 0 & -\dfrac{1}{r}\mathbf{T}
\end{pmatrix}$$

and $\Psi=-2\mathbf{P}\mathbf{C^1}+\mathbf{P}\mathbf{H}+\mathbf{H}^T\mathbf{P}+\mathbf{R}+\mathbf{F}\boldsymbol{\Sigma}\mathbf{F}$.

According to Schur complement Lemma 2, $\boldsymbol{\Xi}+r\Pi^T\mathbf{T}\Pi<0$ is equivalent to $\boldsymbol{\Omega_1}<0$. From Lemma 4 and (D.8), under the condition (33), $\dot{V}(t)\le 0$ and $V(t)\le V(0)$. That is, $V(t)$ is a bounded function, and $\|\mathbf{M}x(t)\|\longrightarrow 0$. Thus the proof is completed.

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