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STABILITY AND HOPF BIFURCATION ON A TWO-NEURON SYSTEM WITH TIME DELAY IN THE FREQUENCY DOMAIN*

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In this paper, a general two-neuron model with time delay is considered, where the time delay is regarded as a parameter. It is found that Hopf bifurcation occurs when this delay passes through a sequence of critical value. By analyzing the characteristic equation and using the frequency domain approach, the existence of Hopf bifurcation is determined. The stability of bifurcating periodic solutions are determined by the harmonic balance approach, Nyquist criterion and the graphic Hopf bifurcation theorem. Numerical results are given to justify the theoretical analysis.

Keywords: Time delay; Hopf bifurcation; periodic solutions; harmonic balance; Nyquist criterion; graphic Hopf bifurcation theorem.

1. Introduction

In recent years, the dynamical characteristics (including stable, unstable, oscillatory, and chaotic behavior) of neural networks [Cao & Chen, 2004; Cao *et al.*, 2005; Cao & Li, 2005; Cao & Liang, 2004; Cao *et al.*, 2004; Cao & Wang, 2004, 2005; Liao *et al.*, 2001a, 2001b; Ruan & Wei, 2001, 2003; Yu & Cao, 2005] have attracted the attention of many researchers, and much efforts have been expended. It is well known that neural networks are complex and large-scale nonlinear systems, neural networks under study today have been dramatically simplified [Guo *et al.*, 2004; Liao *et al.*, 2001a, 2001b; Ruan & Wei, 2001, 2003; Song *et al.*, 2005; Song & Wei, 2005; Yu & Cao, 2005; Yu & Cao, in press]. These investigations of simplified models are still very useful, since the dynamical characteristics found in simple models can be carried over to large-scale networks in some way. So in order to know better the large-scale networks, we should study the simplified networks first.

In 1946, Tsypkin published his classical paper [Tsypkin, 1946] on feedback systems with delay. It is a major extension of the Nyquist criterion in which the problem of delay was solved in a single stroke simply and elegantly. An English translation of this paper was published in a volume in commemoration of Harry Nyquist edited by MacFarlane [Tsypkin, 1946]. It is fitting that this paper appeared immediately following that of Nyquist's original paper. The method of Tsypkin is of major significance considering that the analytical formulation of the problem of stability with delay is very complicated.

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There has been increasing interest in investigating the dynamics of neural networks since Hopfield [1984] constructed a simplified neural network model. Based on the Hopfield neural network model, Marcus and Westervelt [1989] argued that time delays always occur in the singal transmission and proposed a neural network model with delay. Afterward, a variety of artificial models have been estabished to describe neural networks with delays [Babcock & Westervelt, 1987; Baldi & Atiya, 1994; Hopfield, 1984; Kosko, 1988]. Many researchers [Gopalsamy & Leung, 1996, 1997; Liao *et al.*, 1999] focus their attention on the neural networks with time delays and study the dynamical characteristics of neural networks with time delays.

It is known to all that periodic solutions can cause a Hopf bifurcation. This occurs when an eigenvalue crosses the imaginary axis from left to right as a real parameter in the equation passing through a critical value. Recently, stability and Hopf bifurcation analysis have been studied in many neural network models [Guo *et al.*, 2004; Liao *et al.*, 2001a, 2001b; Ruan & Wei, 2001, 2003; Song *et al.*, 2005; Song & Wei, 2005; Yu & Cao, 2005; Yu & Cao, in press]. Since the general models are more complex and we cannot investigate the bifurcation analysis of them. Thus, networks of two neurons have been used as a prototype to understand the dynamics of large-scale neural networks. Hopf bifurcation and stability of bifurcating periodic solutions are often studied using the approach in [Hassard *et al.*, 1981] (see, for example [Guo *et al.*, 2004; Liao *et al.*, 2001a, 2001b; Ruan & Wei, 2001, 2003; Song *et al.*, 2005; Song & Wei, 2005; Yu & Cao, 2005, 2006]). In this paper, we will study a more general neural network model with time delay, using Nyquist criterion and the graphical Hopf bifurcation theorem stated in [Mees, 1981; Moiola & Chen, 1996] to determine the existence of Hopf bifurcation and stability of bifurcating periodic solutions.

Gopalsamy and Leung [1996] considered the following neural network of two neurons constituting an activator-inhibitor assembly by the delay differential system:

$$\begin{cases} \dfrac{dx(t)}{dt} = -x(t) + a\tanh[c_1 y(t-\tau)], \\ \dfrac{dy(t)}{dt} = -y(t) + a\tanh[-c_2 x(t-\tau)], \end{cases} \tag{1}$$

where a, c_1, c_2 and τ are positive constants, y denotes the activating potential of x, and x is the inhibiting potential. Gopalsamy and Leung showed that if the delay has a sufficiently large magnitude, the network is excited to exhibit a temporally periodic behavior, where the analytical mechanism for the onset of cyclic behavior is through a Hopf bifurcation. Approximate solutions to the periodic output of the netlet were calculated, and the stability of the temporally periodic cyclic was investigated.

Olien and Bèlair [1997], on the other hand, investigated the following system with two delays

$$\begin{cases} \dfrac{dx_1(t)}{dt} = -x_1(t) + a_{11} f(x_1(t-\tau_1)) \\ \qquad\qquad + a_{12} f(x_2(t-\tau_2)), \\ \dfrac{dx_2(t)}{dt} = -x_2(t) + a_{21} f(x_1(t-\tau_1)) \\ \qquad\qquad + a_{22} f(x_2(t-\tau_2)), \end{cases} \tag{2}$$

for which several cases, such as $\tau_1 = \tau_2$, $a_{11} = a_{22} = 0$, etc. were discussed. They obtained some sufficient conditions for the stability of the stationary point of model (2), and showed that (2) undergoes some bifurcations at certain values of the parameters. Wei and Ruan [1999] analyzed model (2) with two discrete delays. For the case without self-connections, they found that Hopf bifurcation occurs when the sum of the two delays passes through a sequence of critical values. The stability and direction of the Hopf bifurcation were also determined.

In this paper, we will consider a more general equation with a discrete delay, and study the existence of a Hopf bifurcation and the stability of bifurcating periodic solutions of equation.

The organization of this paper is as follows: In Sec. 2, we will discuss the stability of the trivial solutions and the existence of Hopf bifurcation. In Sec. 3, a formula for determining the stability of bifurcating periodic solutions will be given by using harmonic balance approach, Nyquist criterion and the graphic Hopf bifurcation theorem introduced at [Allwright, 1977; MacFarlane & Postlethwaite, 1977; Mees, 1981; Moiola & Chen, 1993a, 1993b, 1996]. In Sec. 4, numerical simulations aimed at justifying the theoretical analysis will be reported.

2. Existence of Hopf Bifurcation

The neural networks with single delay considered in this paper are described by the following differential equations with delay:

$$\begin{cases} \dot{x}_1(t) = -a_1x_1 + b_{11}f_1(x_1(t-\tau)) \\ \qquad\qquad + b_{12}f_2(x_2(t-\tau)), \\ \dot{x}_2(t) = -a_2x_2 + b_{21}f_1(x_1(t-\tau)) \\ \qquad\qquad + b_{22}f_2(x_2(t-\tau)), \end{cases} \tag{3}$$

where $a_i(i = 1,2)$ are real and positive, $x_1(t)$ and $x_2(t)$ denote the activations of two neurons, τ denote the synaptic transmission delay, $b_{ij}(1 \le i,j \le 2)$ are the synaptic weights, $f_i(i = 1,2)$ is the activation function and $f_i : R \to R$ is a C^3 smooth function with $f_i(0) = 0$.

In a more simplified case, (3) can be written as

$$\dot{x}(t) = -Ax(t) + Bf(x(t-\tau)), \tag{4}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix},$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad f = \begin{pmatrix} f_1(x(t-\tau)) \\ f_2(x(t-\tau)) \end{pmatrix}.$$

By introducing a "state-feedback control" $u = g(y)$, one obtains a linear system with a nonlinear feedback, as follows

$$\begin{cases} \dot{x}(t) = -Ax(t) + Bu, \\ y(t) = -Cx(t), \\ u = g(y(t-\tau)), \end{cases} \tag{5}$$

where

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$g(y(t-\tau)) = \begin{pmatrix} g_1(y_1(t-\tau)) \\ g_2(y_2(t-\tau)) \end{pmatrix} = \begin{pmatrix} f_1(-y_1(t-\tau)) \\ f_2(-y_2(t-\tau)) \end{pmatrix}.$$

Next, taking a Laplace transform $\mathcal{L}(\bullet)$ on (5), yields

$$\mathcal{L}(x) = [sI + A]^{-1}B\mathcal{L}(g(y)),$$

and so

$$\begin{aligned} \mathcal{L}(y) &= -C\mathcal{L}(x) = -C[sI + A]^{-1}B\mathcal{L}(g(y)) \\ &= -G(s)\mathcal{L}(g(y)), \end{aligned} \tag{6}$$

where

$$G(s) = C[sI + A]^{-1}B \tag{7}$$

is the standard transfer matrix of the linear part of the system.

It follows from (6) that we may only deal with $y(t)$ in the frequency domain, without directly considering $x(t)$. In so doing, we first observe that if x^* is an equilibrium solution of the first equation of (5), then

$$y^*(t) = -G(0)g(y^*). \tag{8}$$

From (7), we have

$$\begin{aligned} G(s) &= C[sI + A]^{-1}B \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s + a_1 & 0 \\ 0 & s + a_2 \end{pmatrix}^{-1} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{1}{s + a_1} & 0 \\ 0 & \dfrac{1}{s + a_2} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{b_{11}}{s + a_1} & \dfrac{b_{12}}{s + a_1} \\ \dfrac{b_{21}}{s + a_2} & \dfrac{b_{22}}{s + a_2} \end{pmatrix}. \end{aligned} \tag{9}$$

Clearly, $y = 0$ is the equilibrium of the linearized feedback system, then the Jacobian is given by

$$J = \left(\frac{\partial g}{\partial y}\right)\bigg|_{y=0} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}, \tag{10}$$

where

$$g_{ij} = \frac{\partial g_i}{\partial y_j}\bigg|_{y=0} \quad (i,j = 1,2),$$

so one has

$$\begin{aligned} G(s)J &= \begin{pmatrix} \dfrac{b_{11}}{s + a_1} & \dfrac{b_{12}}{s + a_1} \\ \dfrac{b_{21}}{s + a_2} & \dfrac{b_{22}}{s + a_2} \end{pmatrix} \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{b_{11}g_{11}}{s + a_1} & \dfrac{b_{12}g_{22}}{s + a_1} \\ \dfrac{b_{21}g_{11}}{s + a_2} & \dfrac{b_{22}g_{22}}{s + a_2} \end{pmatrix}. \end{aligned} \tag{11}$$

Set

$$\begin{aligned} h(\lambda, s; \tau) &= \det|\lambda I - G(s)Je^{-s\tau}| \\ &= \begin{vmatrix} \lambda - \dfrac{b_{11}g_{11}}{s + a_1}e^{-s\tau} & -\dfrac{b_{12}g_{22}}{s + a_1}e^{-s\tau} \\ -\dfrac{b_{21}g_{11}}{s + a_2}e^{-s\tau} & \lambda - \dfrac{b_{22}g_{22}}{s + a_2}e^{-s\tau} \end{vmatrix} \end{aligned}$$

$$= \lambda^2 - \left(\frac{b_{11}g_{11}}{s+a_1} + \frac{b_{22}g_{22}}{s+a_2}\right)\lambda e^{-s\tau} + \frac{1}{(s+a_1)(s+a_2)} \times (b_{11}b_{22} - b_{12}b_{21})g_{11}g_{22}e^{-2s\tau}. \quad (12)$$

Then applying the generalized Nyquist stability criterion, the following results stated in [Mees, 1981; Moiola & Chen, 1993a, 1993b, 1996] can be established.

Lemma 2.1 [Moiola & Chen, 1996]. *If an eigenvalue of the corresponding Jacobian of the nonlinear system, in the time domain, assumes a purely imaginary value $i\omega_0$ at a particular value $\tau = \tau_0$, then the corresponding eigenvalue of the constant matrix $[G(i\omega_0)Je^{-i\omega_0\tau_0}]$ in the frequency domain must assume the value $-1 + i0$ at $\tau = \tau_0$.*

To apply Lemma 2.1, let $\hat{\lambda} = \hat{\lambda}(i\omega;\tau)$ be the eigenvalue of $G(i\omega)Je^{-i\omega\tau}$ that satisfies $\hat{\lambda}(i\omega_0;\tau_0) = -1 + i0$. Then

$$h(-1, i\omega_0; \tau_0) = 1 + \left(\frac{b_{11}g_{11}}{s+a_1} + \frac{b_{22}g_{22}}{s+a_2}\right)e^{-s\tau} + \frac{1}{(s+a_1)(s+a_2)}(b_{11}b_{22} - b_{12}b_{21}) \times g_{11}g_{22}e^{-2s\tau} = 0. \quad (13)$$

Thus, we obtained

$$s^2 + (a_1 + a_2)s + a_1a_2 + [(b_{11}g_{11} + b_{22}g_{22})s + a_2b_{11}g_{11} + a_1b_{22}g_{22}]e^{-s\tau} + [(b_{11}b_{22}-, b_{12}b_{21})g_{11}g_{22}]e^{-2s\tau} = 0, \quad (14)$$

and it can be written as

$$s^2 + d_1s + d_2 + (d_3s + d_4)e^{-s\tau} + d_5e^{-2s\tau} = 0, \quad (15)$$

where $d_1 = a_1 + a_2$, $d_2 = a_1a_2$, $d_3 = b_{11}g_{11} + b_{22}g_{22}$, $d_4 = a_2b_{11}g_{11} + a_1b_{22}g_{22}$, $d_5 = (b_{11}b_{22} - b_{12}b_{21})g_{11}g_{22}$.

It is easy to see that (14) is equivalent to the characteristic equation of (3). Multiplying $e^{s\tau}$ on both sides of (15), we have

$$(s^2 + d_1s + d_2)e^{s\tau} + (d_3s + d_4) + d_5e^{-s\tau} = 0. \quad (16)$$

Let $s = i\omega_0, \tau = \tau_0$, and substituting these into (16), for the sake of simplicity, we denote ω_0 and τ_0 by ω, τ, respectively, then (16) becomes

$$(\cos(\omega\tau) + i\sin(\omega\tau))(-\omega^2 + d_1i\omega + d_2) + d_3i\omega + d_4 + d_5(\cos(\omega\tau) - i\sin(\omega\tau)) = 0. \quad (17)$$

Separating the real and imaginary parts, we have

$$\begin{cases} (\omega^2 - d_2 - d_5)\cos(\omega\tau) + d_1\omega\sin(\omega\tau) = d_4, \\ (\omega^2 - d_2 + d_5)\sin(\omega\tau) - d_1\omega\cos(\omega\tau) = d_3\omega. \end{cases} \quad (18)$$

By simple calculation, we obtained

$$\sin(\omega\tau) = \frac{\omega(d_3\omega^2 + d_1d_4 - d_2d_3 - d_3d_5)}{\omega^4 + (d_1^2 - 2d_2)\omega^2 + d_2^2 - d_5^2}, \quad (19)$$

and

$$\cos(\omega\tau) = \frac{(d_4 - d_1d_3)\omega^2 + (d_5d_4 - d_2d_4)}{\omega^4 + (d_1^2 - 2d_2)\omega^2 + d_2^2 - d_5^2}. \quad (20)$$

Let $e_1 = d_1^2 - 2d_2$, $e_2 = d_2^2 - d_5^2$, $e_3 = d_3$, $e_4 = d_1d_4 - d_2d_3 - d_3d_5$, $e_5 = d_4 - d_1d_3$, $e_6 = d_5d_4 - d_2d_4$, and $\sin(\omega\tau), \cos(\omega\tau)$ can be written as

$$\sin(\omega\tau) = \frac{\omega(e_3\omega^2 + e_4)}{\omega^4 + e_1\omega^2 + e_2}, \quad (21)$$

and

$$\cos(\omega\tau) = \frac{e_5\omega^2 + e_6}{\omega^4 + e_1\omega^2 + e_2}. \quad (22)$$

As is known to all that $\sin^2(\omega\tau) + \cos^2(\omega\tau) = 1$, we have

$$\omega^8 + f_3\omega^6 + f_2\omega^4 + f_1\omega^2 + f_0 = 0, \quad (23)$$

where $f_3 = 2e_1 - e_3^2$, $f_2 = e_1^2 + 2e_2 - 2e_3e_4 - e_5^2$, $f_1 = 2e_1e_2 - e_4^2 - 2e_5e_6$, $f_0 = e_2^2 - e_6^2$. Denote $z = \omega^2$, (23) becomes

$$z^4 + f_3z^3 + f_2z^2 + f_1z + f_0 = 0. \quad (24)$$

Let

$$l(z) = z^4 + f_3z^3 + f_2z^2 + f_1z + f_0.$$

Suppose (H1) (24) has at least one positive root.

If A, B, f of the system (4) are given, we can use the computer to calculate the roots of (24) easily. Since $\lim_{z\to\infty} l(z) = +\infty$, we conclude that if $f_0 < 0$, then (24) has at least one positive root.

Without loss of generality, we assume that it has four positive roots, defined by z_1, z_2, z_3, z_4, respectively. Then (23) will have four positive roots

$$\omega_1 = \sqrt{z_1}, \quad \omega_2 = \sqrt{z_2}, \quad \omega_3 = \sqrt{z_3}, \quad \omega_4 = \sqrt{z_4}.$$

By (22), we have

$$\cos(\omega_k\tau) = \frac{e_5\omega_k^2 + e_6}{\omega_k^4 + e_1\omega_k^2 + e_2}. \quad (25)$$

Thus, we denote

$$\tau_k^j = \frac{1}{\omega_k}\left\{\pm\arccos\left(\frac{e_5\omega_k^2 + e_6}{\omega_k^4 + e_1\omega_k^2 + e_2}\right) + 2j\pi\right\}, \quad (26)$$

where $k = 1, 2, 3, 4$; $j = 0, 1, \ldots$, then $\pm i\omega_k$ is a pair of purely imaginary roots of (14) with τ_k^j. Define

$$\tau_0 = \tau_{k_0}^0 = \min_{k\in\{1,2,3,4\}}\{\tau_k^0 : \tau_k^0 \geq 0\}, \ \omega_0 = \omega_{k_0}. \quad (27)$$

Note that when $\tau = 0$, (15) becomes

$$s^2 + ps + q = 0, \quad (28)$$

where $p = d_1 + d_3$, $q = d_2 + d_4 + d_5$. If

(H2): $p > 0$ and $q > 0$ holds, (28) has two roots with negative real parts and system (3) is stable near the equilibrium.

Till now, we can employ a result from [Ruan & Wei, 2001] to analyze (15), which is, for the convenience of the reader, stated as follows:

Lemma 2.2 [Ruan & Wei, 2001]. *Consider the exponential polynomial*

$$\begin{aligned} P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}) &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &\quad + [p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\tau_1} + \cdots \\ &\quad + [p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}]e^{-\lambda\tau_m}, \end{aligned}$$

where $\tau_i \geq 0 (i = 1, 2, \ldots, m)$ *and* $p_j^{(i)} (i = 0, 1, \ldots, m; j = 1, 2, \ldots, n)$ *are constants. As* $(\tau_1, \tau_2, \ldots, \tau_m)$ *vary, the sum of the order of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ *on the open right half plane can change only if a zero appears on or crosses the imaginary axis.*

From Lemmas 2.1 and 2.2, we have the following:

Theorem 2.3. *Suppose that* (H1) *and* (H2) *holds, then the following results hold*:

(I) *For Eq. (3), its zero solution is asymptotically stable for* $\tau \in [0, \tau_0)$,

(II) *Eq. (3) undergoes a Hopf bifurcation at the origin when* $\tau = \tau_0$. *That is, system (3) has a branch of periodic solutions bifurcating from the zero solution near* $\tau = \tau_0$.

Remark 2.4. Yu and Cao [2005] studied a van der Pol equation. If we choose

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} -a & 1 \\ -1 & 0 \end{pmatrix}, J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the characteristic equation is

$$\lambda^2 + a\lambda e^{-\lambda\tau} + e^{-2\lambda\tau} = 0,$$

this is a special case in our characteristic Eq. (15).

Remark 2.5. In [Guo *et al.*, 2004], though Guo, Huang and Wang studied a two-neuron network model with three delays, the coefficients of the system must satisfy some conditions. We choose

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{11} \end{pmatrix},$$

$$J = \begin{pmatrix} -f'(0) & 0 \\ 0 & -f'(0) \end{pmatrix}, \quad \beta = -a_{11}g_{11},$$

$$\overline{a}_{12} = -a_{12}g_{22}, \quad \overline{a}_{21} = -a_{21}g_{11}$$

and the characteristic equation discussed in [Guo *et al.*, 2004] is

$$[\lambda + 1 - \beta e^{-\lambda\tau}]^2 - \overline{a_{12}a_{21}}e^{-2\lambda\tau} = 0.$$

It is also a special case in our characteristic Eq. (15).

Remark 2.6. Song *et al.* [2005] studied a simplified BAM neural network with three delays, but through a simple transformation the model can be changed into one time delay since the BAM neural network do not have self-connections. By the method studied in [Ruan & Wei, 2001], Song, Han and Wei studied the following characteristic equation

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_1\lambda + b_0)e^{-\lambda\tau} = 0,$$

and this characteristic equation is more simple than ours, also the approach used in that paper is more difficult than ours since it involves much mathematical analysis in that paper. We can also develop our model to a third degree exponential polynomial.

Remark 2.7. Song and Wei [2005] studied a delayed predator–prey system, the characteristic equation is

$$\lambda^2 + p\lambda + r + (s\lambda + q)e^{-\lambda\tau} = 0,$$

clearly, it is a special case in our characteristic Eq. (15).

For the coefficients given in the above characteristic equations, the reader may refer to the references. In this paper, we have a method to solve characteristic Eq. (15).

3. Stability of Bifurcating Periodic Solutions

Based on the Lemma 2.1 and the results in [Allwright, 1977; Mees, 1981; Moiola & Chen, 1993a, 1993b, 1996], we just give some conclusions for simplicity. By applying a second-order harmonic balance approximation in [Mees, 1981; Moiola & Chen, 1996] to the output, we have

$$y(t) = y^* + \Re\left\{\sum_{k=0}^{2} Y_k e^{ik\omega t}\right\}, \tag{29}$$

where y^* is the equilibrium point, $\Re\{\bullet\}$ is the real part of the complex constant, and the complex coefficients Y_k are determined by the approximation as shown below: we first define an auxiliary vector

$$\xi_1(\tilde{\omega}) = \frac{-w^T[G(i\tilde{\omega})]p_1 e^{-i\tilde{\omega}\tilde{\tau}}}{w^T v}, \tag{30}$$

where $\tilde{\tau}$ is the fixed value of the parameter τ, w^T and v are the left and right eigenvectors of $[G(i\tilde{\omega})]Je^{-i\tilde{\omega}\tau}$, respectively, associated with the value $\hat{\lambda}(i\tilde{\omega};\tilde{\tau})$, and

$$p_1 = \left[D_2\left(V_{02}\otimes v + \frac{1}{2}\overline{v}\otimes V_{22}\right) + \frac{1}{8}D_3 v\otimes v\otimes\overline{v}\right], \tag{31}$$

in which $\bar{\cdot}$ denotes the complex conjugate as usual, $\tilde{\omega}$ is the frequency of the intersection between the $\hat{\lambda}$ locus and the negative real axis closest to the point $(-1+i0)$, $\otimes$ is the tensor product operator, and

$$D_2 = \left.\frac{\partial^2 g(y;\tilde{\tau})}{\partial y^2}\right|_{y=0}, \tag{32}$$

$$D_3 = \left.\frac{\partial^3 g(y;\tilde{\tau})}{\partial y^3}\right|_{y=0}, \tag{33}$$

$$V_{02} = -\frac{1}{4}[I + G(0)J]^{-1}G(0)D_2 v\otimes\overline{v}, \tag{34}$$

$$V_{22} = -\frac{1}{4}[I + G(2i\tilde{\omega})Je^{-2i\tilde{\omega}\tilde{\tau}}]^{-1} \times G(2i\tilde{\omega})D_2 v\otimes v e^{-2i\tilde{\omega}\tilde{\tau}}. \tag{35}$$

Then, the following Hopf bifurcation theorem formulated in the frequency domain can be established [Moiola & Chen, 1996]:

Theorem 3.1. (*The Graphical Hopf Bifurcation Theorem*). *Suppose that when ω varies, the vector $\xi_1(\tilde{\omega}) \neq 0$, where $\xi_1(\tilde{\omega})$ is defined in (30), and that the half-line, starting from $-1+i0$ and pointing to the direction parallel to that of $\xi_1(\tilde{\omega})$, first intersects the locus of the eigenvalue $\hat{\lambda}(i\omega;\tilde{\tau})$ at the point*

$$\hat{P} = \hat{\lambda}(\hat{\omega};\tilde{\tau}) = -1 + \xi_1(\tilde{\omega})\theta^2, \tag{36}$$

at which $\omega = \hat{\omega}$ and the constant $\theta = \theta(\hat{\omega}) \geq 0$. Suppose furthermore, that the above intersection is transversal, namely,

$$\det\begin{vmatrix} \Re\{\xi_1(\hat{\omega})\} & \Im\{\xi_1(\hat{\omega})\} \\ \Re\left\{\left.\frac{d}{d\omega}\hat{\lambda}(\omega;\tilde{\tau})\right|_{\omega=\hat{\omega}}\right\} & \Im\left\{\left.\frac{d}{d\omega}\hat{\lambda}(\omega;\tilde{\tau})\right|_{\omega=\hat{\omega}}\right\}\end{vmatrix} \neq 0. \tag{37}$$

Then we have the following conclusions:

(1) *The nonlinear system (5) has a periodic solution $y(t) = y(t;\hat{y})$. Consequently, there exists a unique limit cycle for the nonlinear equation (3)*;
(2) *If the half-line L_1 first intersects the locus of $\hat{\lambda}(i\omega)$ when $\tilde{\tau} > \tau_0 (< \tau_0)$, then the bifurcating periodic solution exists and the Hopf bifurcation is supercritical (subcritical)*;
(3) *If the total number of anticlockwise encirclements of the point $P_1 = \hat{P} + \varepsilon\xi_1(\tilde{\omega})$, for a small enough $\varepsilon > 0$, is equal to the number of poles of $\lambda(s)$ that have positive real parts, then the limit cycle is stable*; *otherwise, it is unstable.*

In the above, as usual, $\Re\{\bullet\}$ and $\Im\{\bullet\}$ are the real and imaginary parts of the complex number, respectively.

From (32), one has

$$D_2 = \left.\frac{\partial^2 g(y;\tilde{\tau})}{\partial y^2}\right|_{y=0} = \begin{pmatrix} g_{111} & g_{112} & g_{121} & g_{122} \\ g_{211} & g_{212} & g_{221} & g_{222}\end{pmatrix} = \begin{pmatrix} g_{111} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{222}\end{pmatrix}, \tag{38}$$

where

$$g_{ijk} = \frac{\partial^2 g_i}{\partial y_j \partial y_k}(i,j,k = 1,2).$$

Also

$$D_3 = \frac{\partial^3 g(y;\tilde{\tau})}{\partial y^3}\Big|_{y=0} = \begin{pmatrix} g_{1111} & g_{1112} & g_{1121} & g_{1122} & g_{1211} & g_{1212} & g_{1221} & g_{1222} \\ g_{2111} & g_{2112} & g_{2121} & g_{2122} & g_{2211} & g_{2212} & g_{2221} & g_{2222} \end{pmatrix}$$

$$= \begin{pmatrix} g_{1111} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{2222} \end{pmatrix}, \tag{39}$$

where

$$g_{ijkl} = \frac{\partial^3 g_i}{\partial y_j \partial y_k \partial y_l}(i,j,k,l = 1,2).$$

As we know w^T and v are the left and right eigenvectors of $[G(i\tilde{\omega})]Je^{-i\tilde{\omega}\tau}$, respectively, associated with the value $\hat{\lambda}(i\tilde{\omega};\tilde{\tau}) = \tilde{\lambda}$, we have

$$v = \begin{pmatrix} 1 \\ \dfrac{i\tilde{\omega}+a_1}{b_{12}g_{22}}\tilde{\lambda}e^{i\tilde{\omega}\tilde{\tau}} - \dfrac{b_{11}g_{11}}{b_{12}g_{22}} \end{pmatrix} = \begin{pmatrix} 1 \\ v_2 \end{pmatrix}, \tag{40}$$

and

$$w = \begin{pmatrix} 1 \\ \dfrac{i\tilde{\omega}+a_2}{b_{21}g_{11}}\tilde{\lambda}e^{i\tilde{\omega}\tilde{\tau}} - \dfrac{i\tilde{\omega}+a_2}{b_{21}g_{11}}\dfrac{b_{11}g_{11}}{i\tilde{\omega}+a_1} \end{pmatrix} = \begin{pmatrix} 1 \\ w_2 \end{pmatrix}. \tag{41}$$

From (34) and (35), we obtained

$$\begin{aligned} V_{02} &= -\frac{1}{4}[I + G(0)J]^{-1}G(0)D_2 v\otimes\overline{v} \\ &= -\frac{1}{4}\begin{pmatrix} 1+\dfrac{b_{11}g_{11}}{a_1} & \dfrac{b_{12}g_{22}}{a_1} \\ \dfrac{b_{21}g_{11}}{a_2} & 1+\dfrac{b_{22}g_{22}}{a_2} \end{pmatrix}^{-1} \begin{pmatrix} \dfrac{b_{11}}{a_1} & \dfrac{b_{12}}{a_1} \\ \dfrac{b_{21}}{a_2} & \dfrac{b_{22}}{a_2} \end{pmatrix} \begin{pmatrix} g_{111} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{222} \end{pmatrix} \begin{pmatrix} 1 \\ \overline{v}_2 \\ v_2 \\ v_2\overline{v}_2 \end{pmatrix} \\ &= -\frac{1}{4\left[\left(1+\dfrac{b_{11}g_{11}}{a_1}\right)\left(1+\dfrac{b_{22}g_{22}}{a_2}\right) - \dfrac{b_{12}b_{21}g_{11}g_{22}}{a_1a_2}\right]} \\ &\quad\times \begin{pmatrix} \dfrac{b_{11}}{a_1} + \dfrac{(b_{11}b_{22}-b_{12}b_{21})g_{22}}{a_1a_2} & \dfrac{b_{12}}{a_1} \\ \dfrac{b_{21}}{a_2} & \dfrac{b_{22}}{a_2} + \dfrac{(b_{11}b_{22}-b_{12}b_{21})g_{11}}{a_1a_2} \end{pmatrix} \begin{pmatrix} g_{111} \\ g_{222}v_2\overline{v}_2 \end{pmatrix}, \end{aligned} \tag{42}$$

and

$$\begin{aligned} V_{22} &= -\frac{1}{4}[I + G(2i\tilde{\omega})Je^{-2i\tilde{\omega}\tilde{\tau}}]^{-1}G(2i\tilde{\omega})D_2 v\otimes v e^{-2i\tilde{\omega}\tilde{\tau}} \\ &= -\frac{1}{4}\begin{pmatrix} 1+\dfrac{b_{11}g_{11}}{2i\tilde{\omega}+a_1}e^{-2i\tilde{\omega}\tilde{\tau}} & \dfrac{b_{12}g_{22}}{2i\tilde{\omega}+a_1}e^{-2i\tilde{\omega}\tilde{\tau}} \\ \dfrac{b_{21}g_{11}}{2i\tilde{\omega}+a_2}e^{-2i\tilde{\omega}\tilde{\tau}} & 1+\dfrac{b_{22}g_{22}}{2i\tilde{\omega}+a_2}e^{-2i\tilde{\omega}\tilde{\tau}} \end{pmatrix}^{-1} \begin{pmatrix} \dfrac{b_{11}}{2i\tilde{\omega}+a_1} & \dfrac{b_{12}}{2i\tilde{\omega}+a_1} \\ \dfrac{b_{21}}{2i\tilde{\omega}+a_2} & \dfrac{b_{22}}{2i\tilde{\omega}+a_2} \end{pmatrix} \end{aligned}$$

$$\times \begin{pmatrix} g_{111} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{222} \end{pmatrix} \begin{pmatrix} 1 \\ v_2 \\ v_2 \\ v_2 v_2 \end{pmatrix}$$

$$= -\frac{1}{4\left[\left(1+\frac{b_{11}g_{11}}{2i\tilde{\omega}+a_1}e^{-2i\tilde{\omega}\tilde{\tau}}\right)\left(1+\frac{b_{22}g_{22}}{2i\tilde{\omega}+a_2}e^{-2i\tilde{\omega}\tilde{\tau}}\right)-\frac{b_{12}b_{21}g_{11}g_{22}e^{-4i\tilde{\omega}\tilde{\tau}}}{(2i\tilde{\omega}+a_1)(2i\tilde{\omega}+a_2)}\right]}$$

$$\times \begin{pmatrix} \frac{b_{11}}{2i\tilde{\omega}+a_1}+\frac{(b_{11}b_{22}-b_{12}b_{21})g_{22}e^{-2i\tilde{\omega}\tilde{\tau}}}{(2i\tilde{\omega}+a_1)(2i\tilde{\omega}+a_2)} & \frac{b_{12}}{2i\tilde{\omega}+a_1} \\ \frac{b_{21}}{2i\tilde{\omega}+a_2} & \frac{b_{22}}{2i\tilde{\omega}+a_2}+\frac{(b_{11}b_{22}-b_{12}b_{21})g_{11}e^{-2i\tilde{\omega}\tilde{\tau}}}{(2i\tilde{\omega}+a_1)(2i\tilde{\omega}+a_2)} \end{pmatrix}$$

$$\times \begin{pmatrix} g_{111} \\ g_{222}v_2v_2 \end{pmatrix} e^{-2i\tilde{\omega}\tilde{\tau}}. \quad (43)$$

Let

$$V_{02} = \begin{pmatrix} V_{02}(1) \\ V_{02}(2) \end{pmatrix} \quad \text{and} \quad V_{22} = \begin{pmatrix} V_{22}(1) \\ V_{22}(2) \end{pmatrix},$$

substituting (42) and (43) into (31), we obtained

$$p_1 = \left[D_2\left(V_{02}\otimes v + \frac{1}{2}\overline{v}\otimes V_{22}\right) + \frac{1}{8}D_3 v\otimes v\otimes\overline{v}\right]$$

$$= \begin{pmatrix} g_{111}V_{02}(1) + \frac{1}{2}g_{111}V_{22}(1) + \frac{1}{8}g_{1111} \\ g_{222}V_{02}(2)v_2 + \frac{1}{2}g_{222}V_{22}(2)\overline{v}_2 + \frac{1}{8}g_{2222}v_2^2\overline{v}_2 \end{pmatrix}. \quad (44)$$

Substituting (39)–(44) into (30), we can obtain $\xi_1(\tilde{\omega})$.

Corollary 3.2. *Let k be the total number of anti-clockwise encirclements of the point $P_1 = \hat{P}+\varepsilon\xi_1(\tilde{\omega})$ for a small enough $\varepsilon > 0$, where $\hat{P}$ is the intersection of the half-line L_1 and the locus $\hat{\lambda}(i\omega)$. Then*

(1) *If $k = 0$, the bifurcating periodic solutions of system (3) are stable*;
(2) *If $k \neq 0$, the bifurcating periodic solutions of system (3) are unstable.*

Remark 3.3. In this paper we study the stability of bifurcating periodic solutions using the harmonic balance approach, Nyquist criterion and the graphic Hopf bifurcation theorem. It is an algebraic and graphical approach and more simple than the normal form method and center manifold theorem introduced by Hassard *et al.* [1981]. It does not involve much mathematical analysis. The stability of bifurcating periodic orbits have been analyzed drawing the amplitude locus, L_1, and the locus $\hat{\lambda}(i\omega)$ in a neighborhood of the Hopf bifurcation point.

4. Numerical Examples

In this section, some numerical results of simulating system (3) are presented. The half-line and locus $\hat{\lambda}(i\omega)$ are shown in the corresponding frequency graphs. If they intersect, a limit cycle exists, or else, no limit cycle exists. Corollary 3.2 implies that the stabilities of the bifurcating periodic solutions are determined by the total number k of the anticlockwise encirclements of the point $P_1 = \hat{P} + \varepsilon\xi_1(\tilde{\omega})$ for a small enough $\varepsilon > 0$. Suppose that the half-line L_1 and the locus $\hat{\lambda}(i\omega)$ intersect. If $k = 0$, the bifurcating periodic solutions of system (3) are stable; if $k \neq 0$, the bifurcating periodic solutions of system (3) are unstable.

In order to verify the theoretical analysis results derived above, system (3) is simulated in different cases.

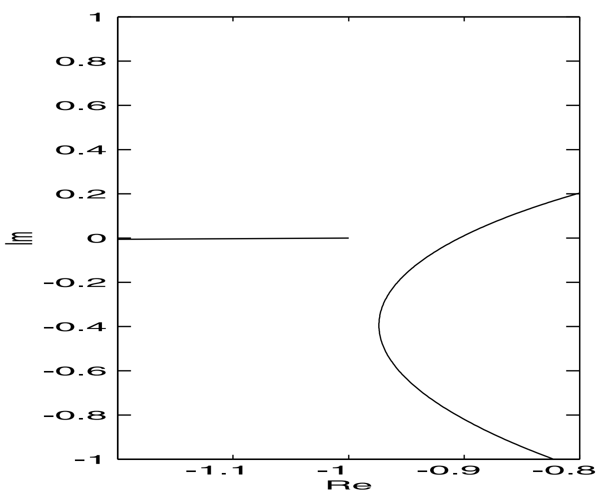
$$\text{(i)} \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix},$$

$$f(x) = \begin{pmatrix} -\tanh(x) \\ -\tanh(x) \end{pmatrix}.$$

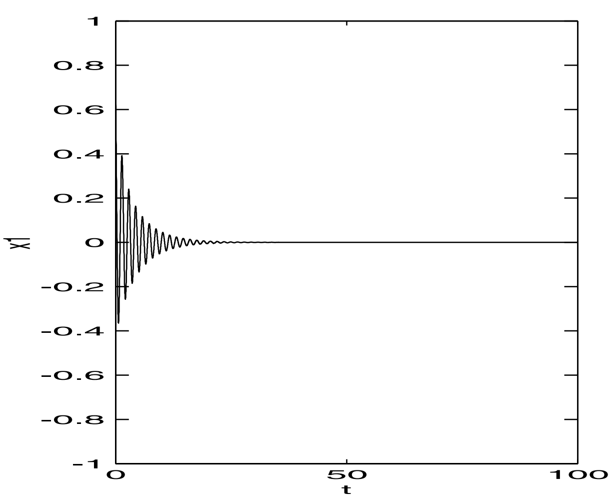
Equation (28) have two negative roots -1 and -6, Eq. (24) has one positive root 14.6834, from Eq. (26), we have

$$\tau_j = 0.5183 + 1.6397j \ (j = 0, 1, \ldots,),$$

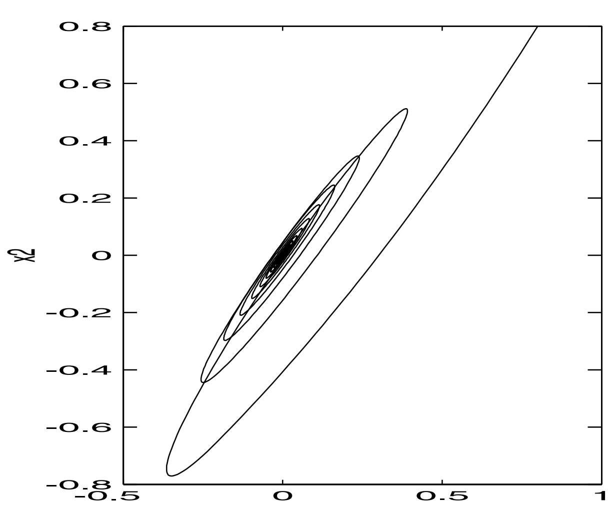
and $\tau_0 = 0.5183$.



Frequency graph

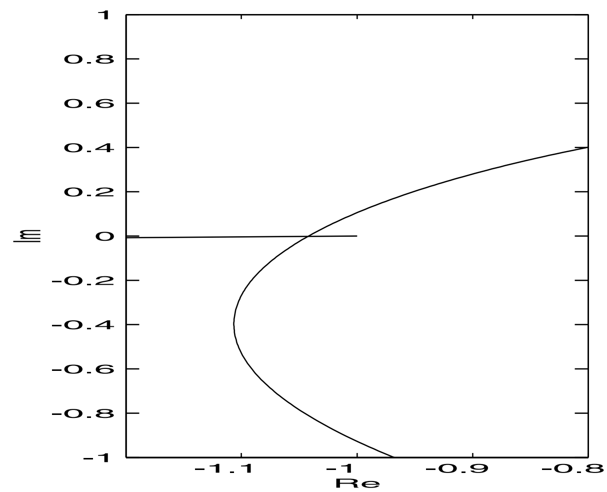


Waveform graph

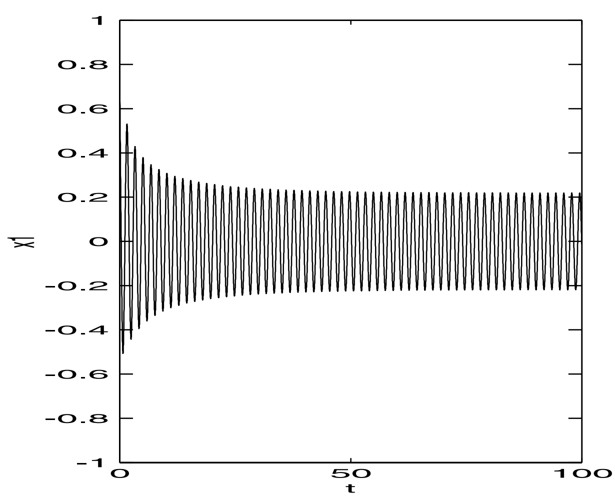


Phase graph

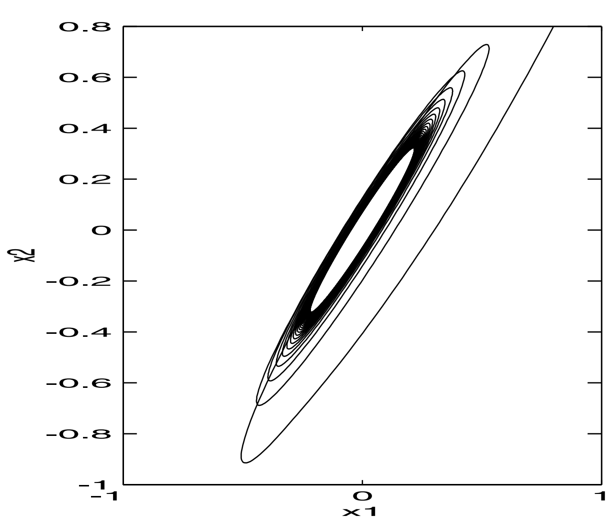
Fig. 1. $\tau = 0.45$. The half-line L_1 does not intersect the locus $\hat{\lambda}(i\omega)$, so no periodic solution exists.



Frequency graph



Waveform graph



Phase graph

Fig. 2. $\tau = 0.55$. The half-line L_1 intersects the locus $\hat{\lambda}(i\omega)$, and $k = 0$, so a stable periodic solution exists.

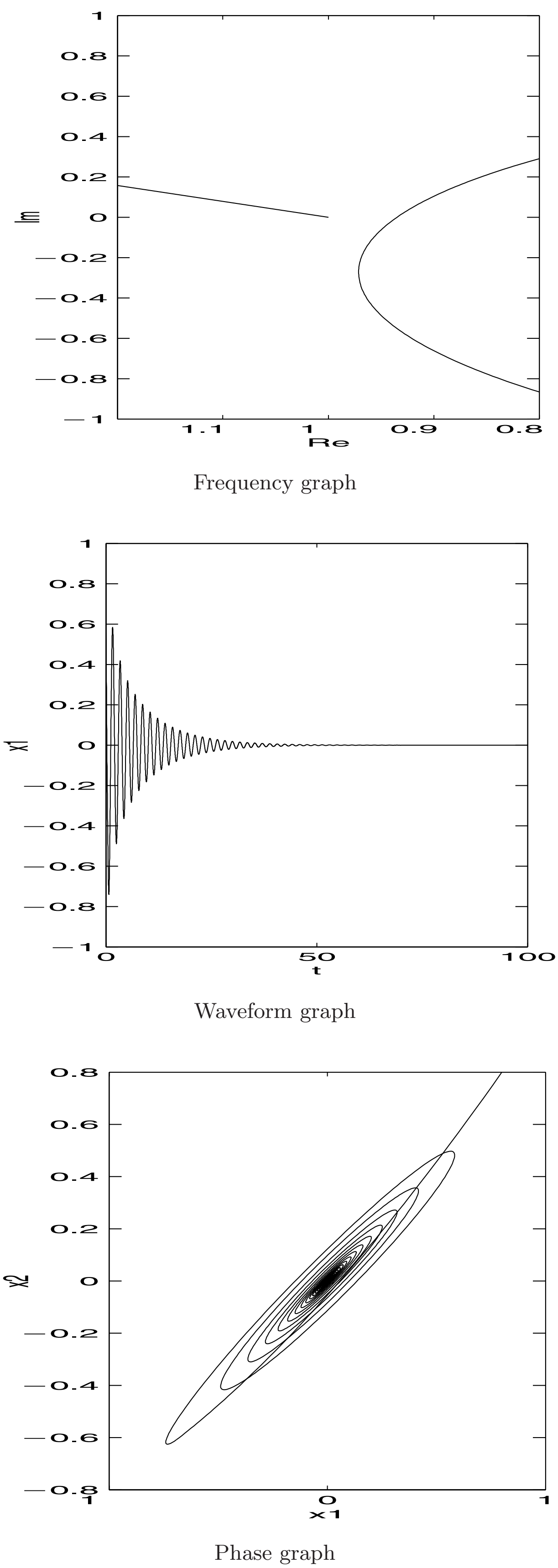


Fig. 3. $\tau = 0.60$. The half-line L_1 does not intersect the locus $\hat{\lambda}(i\omega)$, so no periodic solution exists.

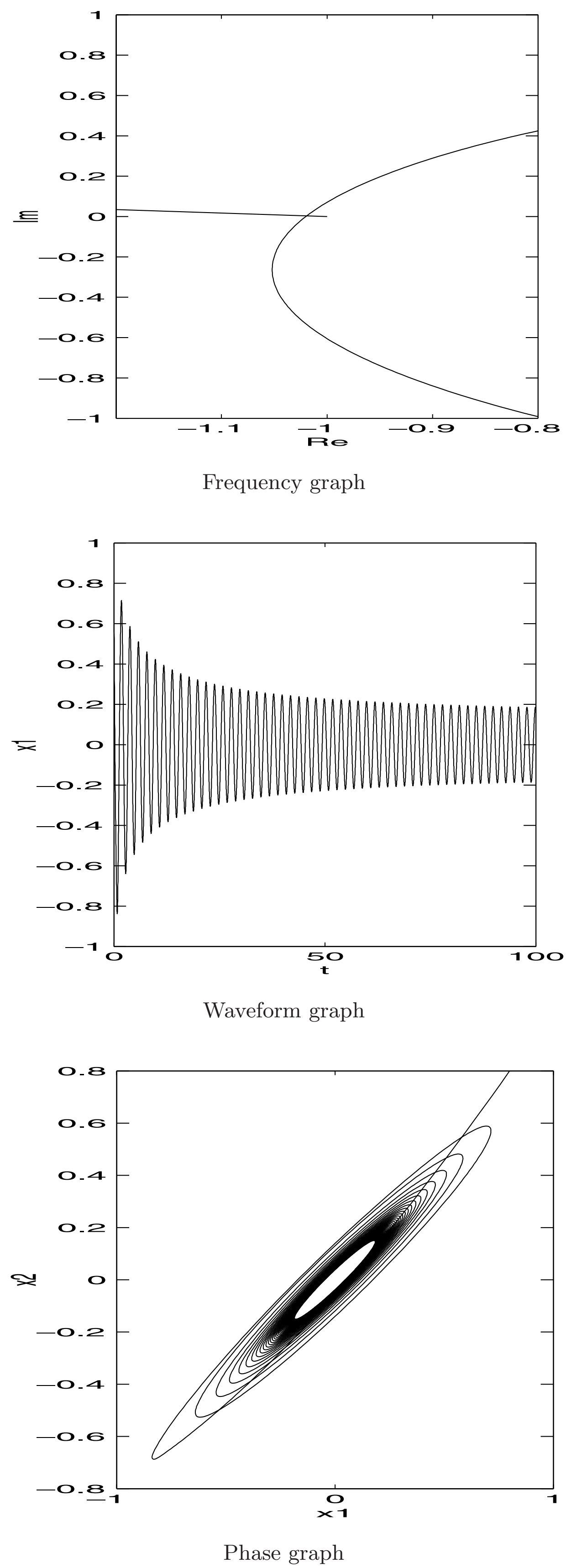


Fig. 4. $\tau = 0.70$. The half-line L_1 intersects the locus $\hat{\lambda}(i\omega)$, and $k = 0$, so a stable periodic solution exists.

We choose $\tau = 0.45 < \tau_0$ and $\tau = 0.55 > \tau_0$, respectively. The corresponding frequency, waveform and phase graph are shown in Figs. 1 and 2. By Lemma 2.1, Theorems 2.3 and 3.1 we know in Fig. 1 its zero solution is asymptotically stable, in Fig. 2 the bifurcating periodic solution is stable and the system undergoes a Hopf bifurcation at the origin.

$$\text{(ii)} \qquad A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix},$$

$$f(x) = \begin{pmatrix} -\tanh(x) \\ -\tanh(x) \end{pmatrix}.$$

Equation (28) have two negative roots -6.4142 and -3.5858, Eq. (24) has one positive root -1.2259, from Eq. (26), we have

$$\tau_j = 0.6751 + 1.9460j \ (j = 0, 1, \ldots,),$$

and $\tau_0 = 0.6751$.

We choose $\tau = 0.60 < \tau_0$ and $\tau = 0.70 > \tau_0$, respectively. The corresponding frequency, waveform and phase graph are shown in Figs. 3 and 4. By Lemma 2.1, Theorems 2.3 and 3.1 we know in Fig. 3 its zero solution is asymptotically stable, Fig. 4 undergoes a Hopf bifurcation at the origin.

5. Conclusions

A more general two-neuron model with time delay studied in this paper from the frequency domain approach turns out to be not so mathematically involved and so difficult as analyzing in the time domain [Guo *et al.*, 2004; Liao *et al.*, 2001a, 2001b; Ruan & Wei, 2001, 2003; Song *et al.*, 2005; Song & Wei, 2005; Yu & Cao, 2005, 2006]. By using the time delay as the bifurcation parameter, it has been shown that a Hopf bifurcation occurs when this parameter passes through a critical value. The stability of bifurcating periodic orbits have been analyzed drawing the amplitude locus, L_1, and the locus $\hat{\lambda}(i\omega)$ in a neighborhood of the Hopf bifurcation point. It is very difficult to solve large-scale neural networks with time delays, since the characteristic equation in large-scale neural networks is a more complex transcendental equation. In studying the stability and Hopf bifurcation analysis, there are still much work to be done, we should focus on large-scale neural networks with more time delays.

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