Stability and Hopf bifurcation of a general delayed recurrent neural network

Wenwu Yu, *Student Member*, *IEEE*, Jinde Cao, *Senior Member*, *IEEE*, and Guanrong Chen, *Fellow*, *IEEE*

Abstract— In this paper, stability and bifurcation of a general recurrent neural network with multiple time delays is considered, where all the variables of the network can be regarded as bifurcation parameters. It is found that Hopf bifurcation occurs when these parameters pass through some critical values where the conditions for local asymptotical stability of the equilibrium are not satisfied. By analyzing the characteristic equation and using the frequency domain method, the existence of Hopf bifurcation is proved. The stability of bifurcating periodic solutions is determined by the harmonic balance approach, Nyquist criterion and graphic Hopf bifurcation theorem. Moreover, a critical condition is derived under which the stability is not guaranteed, thus a necessary and sufficient condition for ensuring the local asymptotical stability is well understood, and from which the essential dynamics of the delayed neural network are revealed. Finally, numerical results are given to verify the theoretical analysis, and some interesting phenomena are observed and reported.

Index Terms— Hopf bifurcation, frequency domain approach, harmonic balance, recurrent neural network, stability

I. INTRODUCTION

In the last three decades, neural networks, particularly the Hopfield Neural Network (HNN) [1] and Cohen-Grossberg Neural Network (CGNN) [2], have received increasing attention due to their wide and important applications in such areas as signal processing, image processing, pattern recognition and optimizations. Some applications of neural networks require the knowledge of dynamical behaviors of the neural networks, such as the uniqueness and asymptotical stability of an equilibrium point of a specific neural network. Therefore, the problem of stability analysis for neural networks has been a focal topic of research in this field.

In practice, due to the finite speeds of the switching and transmitting signals, time delays exist in various neural networks and therefore should be taken into consideration [3]-[6][37][38]. It is well known that time delays may result

W. Yu is with the Department of Mathematics, Southeast University, Nanjing 210096, China, and is with the Department of Electronic Engineering, City University of Hong Kong, Hong Kong. (e-mail: wenwuyu@gmail.com; wwyu@ee.columbia.edu).

J. Cao is with the Department of Mathematics, Southeast University, Nanjing 210096, China. (e-mail: jdcao@seu.edu.cn).

G. Chen is with the Department of Electronic Engineering, City University of Hong Kong, China (e-mail: eegchen@cityu.edu.hk).

in oscillatory behaviors or network instability (periodic oscillation and chaos) [10]-[18][30][31][34], hence the study of delayed neural networks is very important. In fact, stability problems of delayed neural networks have been intensively studied [7][8][28][38] where, however, most derived conditions are sufficient conditions for the asymptotical stability and are generally too conservative.

Stability of equilibrium point in neural networks is widely investigated in associative memory, pattern recognition and optimization, which can be considered as a single storage or memory pattern, or an optimum object. Owing to the limited information stored in equilibrium point, there have been great interests in using periodic solutions for associative memory and pattern recognition. It is known that periodic solutions can restore various complex patterns unlike most existing patterns based on the stable equilibrium point. When the dynamics of neural network pass the Hopf bifurcation, various local periodic solutions arise from the equilibrium point of neural networks. In order to realize a memory system, the authors examined the neural network dynamics phenomenon as bifurcations of attractors in [33]. A noisy self-organizing neural network with bifurcation dynamics for combinatorial optimization is investigated in [32]. Hopf bifurcation can not only provide a guide to design stable neural networks, but also paves the way for the application of periodic solutions. Therefore, in order to reveal the dynamics of artificial neural networks, it is very urgent and significant to study the bifurcation analysis of the neural network model.

In [17], Olien and Bélair investigated the following system with two delays:

$$\dot{x_1}(t) = -x_1(t) + a_{11}f(x_1(t-\tau_1)) + a_{12}f(x_2(t-\tau_2)),$$
$$\dot{x_2}(t) = -x_2(t) + a_{21}f(x_1(t-\tau_1)) + a_{22}f(x_2(t-\tau_2)),$$

for which several cases were discussed, such as $\tau_1 = \tau_2$, $a_{11} = a_{22} = 0$, etc.

In [11], Yu and Cao extended the above model and studied the following delayed network model:

$$\dot{x_1}(t) = -a_1 x_1 + b_{11}f_1(x_1(t-\tau)) + b_{12}f_2(x_2(t-\tau)),$$
$$\dot{x_2}(t) = -a_2 x_2 + b_{21}f_1(x_1(t-\tau)) + b_{22}f_2(x_2(t-\tau)),$$

where $a_i(i = 1, 2)$ are positive constants, $x_1(t)$ and $x_2(t)$ denote the activations of two neurons, τ denotes the synaptic transmission delay, $b_{ij}(1 \leq i, j \leq 2)$ are the synaptic weights, $f_i(i = 1, 2)$ are the activations function and $f_i : R \longrightarrow R$ are C^3 smooth functions with $f_i(0) = 0$.

In [18], Campbell *et al.* studied a neural network model

with multiple delays:

$$C_j \dot{u}_j(t) = -\frac{1}{R_j} + F_j(u_j(t-\sigma)) + G_j(u_{j-1}(t-\tau)),$$
$$j = 1, 2, \cdots, n,$$

where $C_j > 0$ and $R_j > 0$ represent the capacitance and resistance of each neuron, respectively, and F_j and G_j are nonlinear functions representing, respectively, the feedback from neuron j to itself and the connection from j to $j-1$, for which only the case of $n = 4$ was discussed.

In [13], Song *et al.* considered a simplified BAM neural network model as follows:

$$\begin{cases} \dot{x}_1(t) & = & -\mu_1 x_1(t) + c_{21} f_1(y_1(t-\tau_2)) \\ & & +c_{31} f_1(y_2(t-\tau_2)), \\ \dot{y}_1(t) & = & -\mu_2 y_1(t) + c_{12} f_2(x_1(t-\tau_1)), \\ \dot{y}_2(t) & = & -\mu_3 y_2(t) + c_{13} g_3(x_1(t-\tau_1)). \end{cases}$$

And, in [10], Yu and Cao studied a more complex BAM neural network model:

$$\begin{cases} \dot{x}_1(t) & = & -\mu_1 x_1(t) + c_{11} f_{11}(y_1(t-\tau_3)) \\ & & +c_{12} f_{12}(y_2(t-\tau_3)), \\ \dot{x}_2(t) & = & -\mu_2 x_2(t) + c_{21} f_{21}(y_1(t-\tau_4)) \\ & & +c_{22} f_{22}(y_2(t-\tau_4)), \\ \dot{y}_1(t) & = & -\mu_3 y_1(t) + d_{11} g_{11}(x_1(t-\tau_1)) \\ & & +d_{12} g_{12}(x_2(t-\tau_2)), \\ \dot{y}_2(t) & = & -\mu_4 y_2(t) + d_{21} g_{21}(x_1(t-\tau_1)) \\ & & +d_{22} g_{22}(x_2(t-\tau_2)). \end{cases}$$

From the above introduction, it is easy to see that bifurcation analysis about delayed systems under investigation today are mainly lower-dimensional systems with a few time delays. The dimensions of the delayed systems studied in [9]-[18][29] are no more than four, and actually with no more than two delays as bifurcation parameters. Though in [4], an n-dimensional delayed model was investigated, the model is very simple with only one delay. It is well known that neural networks are very complex and large-scale nonlinear systems, but neural network models under study today have been dramatically simplified [10]-[18][29]. In this paper, a general model with multiple delays is investigated, aiming at giving a clearer understanding of the transition from stability to bifurcation and explaining some phenomena that otherwise cannot be interpreted by the existing results.

Hopf bifurcation of dynamical systems is investigated in [9][10][22][35] by using the normal form method and center manifold theorem. Then a new method of multiple scales versus center manifold is used for order reduction of retarded nonlinear systems [36]. However, these approaches are still difficult to analyze Hopf bifurcation of a general system with multiple delays. To the best of our knowledge, the Hopf bifurcation of a general neural network model with multiple time delays has not been investigated elsewhere. In this paper, we try to find out some critical conditions under which the stability is not guaranteed and the Hopf bifurcation occurs when the parameters of the network model pass through some critical values. The results obtained in this paper give an explicit view of the dynamics of a general

delayed neural network based on Hopf bifurcation analysis from the harmonic balance approach, Nyquist criterion and graphic Hopf bifurcation theorem [20][21], rather than the normal form method and center manifold theorem [22].

The rest of the paper is organized as follows: in section 2, local asymptotical stability analysis is established by analyzing the characteristic equation and using Nyquist criterion. Then applying harmonic balance approach and graphic Hopf bifurcation theorem, existence and stability of bifurcating periodic solutions of a general neural network are investigated in section 3. In section 4, simulation examples are constructed to verify the theoretical analysis in this paper. Finally, the conclusions are drawn.

II. LOCAL ASYMPTOTICAL STABILITY ANALYSIS

Consider the following delayed recurrent neural network model:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-\tau)) + E, \quad (1)$$

namely,

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t))$$
$$+ \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j)) + E_i,$$
$$i = 1, 2, \cdots, n, \quad (2)$$

where n denotes the number of neurons in the network, τ_j $(j = 1, 2, \cdots, n)$ are the time delays, $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in R^n$ is the state vector associated with the neurons, $E = (E_1, E_2, \cdots, E_n)^T \in R^n$ is the external input vector, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \cdots, f_n(x_n(t)))^T \in R^n$ and $f(x(t-\tau)) = (f_1(x_1(t-\tau_1)), f_2(x_2(t-\tau_2)), \cdots, f_n(x_n(t-\tau_n)))^T \in R^n$ correspond to the activation functions and delayed activation functions of neurons, respectively, $C = diag(c_1, c_2, \cdots, c_n) > 0$, $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively, and the initial conditions are given by $\phi_i(t) \in \mathcal{C}([-r, 0], R)$, where $r = \max_{1 \leq i \leq n}\{\tau_i\}$, with $\mathcal{C}([-r, 0], R)$ denoting the set of all continuous functions from $[-r, 0]$ to R.

Assume that model (1) has an equilibrium $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ for a given E. Without loss of generality, assume the equilibrium point x^*. Using the transformation

$$y(t) = x(t) - x^*, \ y(t-\tau) = x(t-\tau) - x^*,$$

model (1) can be transformed into the following form:

$$\dot{y}(t) = -Cy(t) + Ag(y(t)) + Bg(y(t-\tau)), \quad (3)$$

namely,

$$\dot{y}_i(t) = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} g_j(y_j(t))$$

$$+ \sum_{j=1}^{n} b_{ij} g_j(y_j(t - \tau_j)), i = 1, 2, \cdots, n, \quad (4)$$

where $g(y(t)) = (g_1(y_1(t)), g_2(y_2(t)), \cdots, g_n(y_n(t)))^T \in R^n$ with $g_i(y_i(t)) = f_i(y_i(t) + x_i^*) - f_i(x_i^*)$ and $g(0) = 0$.

Next, the stability and Hopf bifurcation of delayed system (3) are discussed.

By introducing a 'state-feedback control' u, one obtains a linear system with a non-linear feedback, which is equivalent to (3), as follows:

$$\begin{cases} \dot{y}(t) &= -Cy(t) + u(z(t)), \\ z(t) &= -y(t), \\ u(z(t)) &= Ag(-z(t)) + Bg(-z(t - \tau)), \end{cases} \quad (5)$$

where $z(t) = (z_1(t), z_2(t), \cdots, z_n(t))^T \in R^n$ and $z(t-\tau) = (z_1(t - \tau_1), z_2(t - \tau_2), \cdots, z_n(t - \tau_n))^T \in R^n$.

In order to study the Hopf bifurcation of delayed neural networks, then one can parameterize the feedback system (5) as

$$\begin{cases} \dot{y}(t) &= -C(\mu)y(t) + u(z(t); \mu), \\ z(t) &= -y(t), \\ u(z(t); \mu) &= A(\mu)g(-z(t); \mu) + B(\mu) \\ & \quad \times g(-z(t - \tau(\mu)); \mu), \end{cases} \quad (6)$$

where μ is the bifurcation parameter.

Next, taking a Laplace transform $\mathcal{L}(\cdot)$ on (6) yields

$$\mathcal{L}(y) = [sI + C(\mu)]^{-1}\mathcal{L}(u(z; \mu)),$$

and

$$\begin{aligned} \mathcal{L}(z) &= -\mathcal{L}(y) = -[sI + C(\mu)]^{-1}\mathcal{L}(u(z; \mu)) \\ &= -G(s; \mu)\mathcal{L}(u(z; \mu)), \end{aligned} \quad (7)$$

where

$$G(s; \mu) = [sI + C(\mu)]^{-1} \quad (8)$$

is the standard transfer matrix of the linear part of the system. Throughout this paper, I is the n-dimensional identity matrix.

It follows from (7) that one may only deal with $z(t)$ in the frequency domain, without directly considering $y(t)$. In so doing, first observe that if y^* is an equilibrium solution of the first equation of (6), then

$$z^* = -G(0; \mu)u(z^*; \mu). \quad (9)$$

Let $u(z(t); \mu) = A(\mu)\widetilde{u}(z(t); \mu) + B\widetilde{u}(z(t-\tau(\mu)); \mu)$, where $\widetilde{u}(z(t); \mu) = g(-z(t); \mu)$.

Clearly, $y = 0$ is the equilibrium of the linearized feedback system. If one linearizes the feedback system about the equilibrium z^*, then the Jacobian of $\widetilde{u}$ is given by

$$J(\mu) = \left(\frac{\partial \widetilde{u}}{\partial z}\right)\Big|_{z=0}, \quad (10)$$

where $J(\mu) = (J_{ij})_{n \times n}, J_{ij} = \dfrac{\partial \widetilde{u}_i}{\partial z_j}\Big|_{y=0}$ $(i, j = 1, 2, \cdots, n)$. The Jacobian of the nonlinear feedback u is then given by $\widetilde{J}(s; \mu) = A(\mu)J(\mu) + B(\mu)J(\mu)e^{-s\tau(\mu)}$, where $e^{-s\tau(\mu)} = diag(e^{-s\tau_1(\mu)}, e^{-s\tau_2(\mu)}, \cdots, e^{-s\tau_n(\mu)})$. The closed-loop transfer matrix of the linearized feedback system (6) is

$$H(s; \mu) = -[I + G(s; \mu)\widetilde{J}(s; \mu)]^{-1}G(s; \mu). \quad (11)$$

To this end, stability analysis is established based on the Nyquist stability criterion [27]. Let $\Re\{\cdot\}$ be the real part of the complex constant.

Lemma 1 [21] Let $G \in R$ have $p(G)$ poles in $\Re(s) > 0$ and let $\mathcal{D}$ be a simple closed contour consisting of an interval $[-i\omega, i\omega]$ on the imaginary axis together with a semicircle in $\Re(s) > 0$, large enough to contain all the poles in $\Re(s) > 0$. Suppose $\mathcal{D}$ is indented if necessary to exclude any poles on the imaginary axis. Let Γ_G be the image of $\mathcal{D}$ under G as $\mathcal{D}$ is traversed clockwise. Then, H, defined in (11), has no poles in $\Re(s) \geq 0$ if Γ_G encircles $-1/\widetilde{J}(s)$ $p(G)$ times anticlockwise, and $|1 + G(s)\widetilde{J}(s)| \nrightarrow 0$ as $|s| \to \infty$.

Next, a theorem is given to ensure the local asymptotical stability of the nonlinear feedback system (6).

Theorem 1 [21] Let $G(s)\widetilde{J}(s) \in R^{n \times n}$ have characteristic functions $\lambda_1(s), \lambda_2(s), \cdots, \lambda_q(s)$, with a total of $p(G\widetilde{J})$ poles counted according to multiplicity. Let the jth characteristic locus Γ_{λ_j} encircle -1 for a total of n_j times anticlockwise. Then, the closed-loop system (5) is stable if $\sum_{j=1}^{q} n_j = p(G\widetilde{J})$. In this case, the recurrent neural network (3) is locally asymptotically stable.

Applying the generalized Nyquist stability criterion, the following result can be established.

Theorem 2 [20] If an eigenvalue of the corresponding Jacobian of the nonlinear system, in the time domain, assumes a purely imaginary value $i\omega_0$ at a particular value $\mu = \mu_0$, then the corresponding eigenvalue of the constant matrix $[G(i\omega_0; \mu_0)\widetilde{J}(i\omega_0; \mu_0)]$ in the frequency domain must assume the value $-1 + i0$ at $\mu = \mu_0$.

Set

$$h(\lambda, s; \mu) = |\lambda I - G(s; \mu)\widetilde{J}(s; \mu)|. \quad (12)$$

To apply Theorem 2, let $\widehat{\lambda} = \widehat{\lambda}(i\omega; \mu)$ be the eigenvalue of $G(i\omega; \mu)\widetilde{J}(i\omega; \mu)$ that satisfies $\widehat{\lambda}(i\omega_0; \mu_0) = -1 + i0$. Then

$$h(-1, s; \mu) = |-I - (sI + C)^{-1}(AJ + BJe^{-s\tau})|. \quad (13)$$

Let $h(-1, s; \mu) = 0$. Then

$$|(sI + C) + (AJ + BJe^{-s\tau})| = 0. \quad (14)$$

It is easy to see that (14) is equivalent to the characteristic equation of (3).

III. EXISTENCE AND STABILITY OF BIFURCATING PERIODIC SOLUTIONS

Based on Theorem 1, Theorem 2 and the results in [23]−[26], we now derive some formulas for the existence

and the stability of bifurcating periodic solutions.

Suppose that a second-order harmonic balance approximation for the solution has the following form:

$$z(t) = z^* + \Re\left\{\sum_{k=0}^{2} Z_k e^{ik\omega t}\right\}, \qquad (15)$$

where z^* is the equilibrium point.

Let $G(i\omega) = G(s;\mu)|_{s=i\omega}$. Then, equating the input and output of (6) gives

$$Z_k = -G(i\omega)F_k, k = 0, 1, 2, \qquad (16)$$

where F_k is the Fourier coefficients of $u(z(t);\mu)$. Formula (16) is known as the second-order harmonic balance equations [20][24][25][26]. By expanding the trial expression for z, which contains the time delays, one has

$$z(t - \tau(\mu)) = z^* + \Re\left\{\sum_{k=0}^{2} e^{-ik\omega\tau(\mu)} Z_k e^{ik\omega t}\right\}. \qquad (17)$$

In order to derive the main results, it is convenient to use the following notations:

$$
\begin{aligned}
D_2^1 &= A(\widetilde{\mu}) \left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} + B(\widetilde{\mu}) \left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} \\
&\quad \times (I \otimes e^{-i\widetilde{\omega}\tau(\widetilde{\mu})}),
\end{aligned}
\qquad (18)
$$

$$
\begin{aligned}
D_2^2 &= A(\widetilde{\mu}) \left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} + B(\widetilde{\mu}) \left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} \\
&\quad \times (e^{i\widetilde{\omega}\tau(\widetilde{\mu})} \otimes e^{-2i\widetilde{\omega}\tau(\widetilde{\mu})}),
\end{aligned}
\qquad (19)
$$

$$
\begin{aligned}
D_3 &= A(\widetilde{\mu}) \left.\frac{\partial^3 \widetilde{u}(z;\widetilde{\mu})}{\partial z^3}\right|_{z=0} + B(\widetilde{\mu}) \left.\frac{\partial^3 \widetilde{u}(z;\widetilde{\mu})}{\partial z^3}\right|_{z=0} \\
&\quad \times (e^{-i\widetilde{\omega}\tau(\widetilde{\mu})} \otimes e^{-i\widetilde{\omega}\tau(\widetilde{\mu})} \otimes e^{i\widetilde{\omega}\tau(\widetilde{\mu})}),
\end{aligned}
\qquad (20)
$$

$$
\begin{aligned}
D_2^3 &= A(\widetilde{\mu}) \left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} + B(\widetilde{\mu}) \left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} \\
&\quad \times (e^{-i\widetilde{\omega}\tau(\widetilde{\mu})} \otimes e^{i\widetilde{\omega}\tau(\widetilde{\mu})}),
\end{aligned}
\qquad (21)
$$

$$
\begin{aligned}
D_2^4 &= A(\widetilde{\mu}) \left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} + B(\widetilde{\mu}) \left.\frac{\partial^2 \widetilde{u}(z;\widetilde{\mu})}{\partial z^2}\right|_{z=0} \\
&\quad \times (e^{-i\widetilde{\omega}\tau(\widetilde{\mu})} \otimes e^{-i\widetilde{\omega}\tau(\widetilde{\mu})}),
\end{aligned}
\qquad (22)
$$

where $\otimes$ is the tensor product operator, $\widetilde{\mu}$ is the fixed value of the parameter μ, and $\widetilde{\omega}$ is the frequency of the intersection between the $\widehat{\lambda}$ locus and the negative real axis closest to the point $(-1 + i0)$.

Combining (15)-(22), one obtains

$$
\begin{aligned}
Z_0 &= -G(0;\widetilde{\mu})F_0. \\
&= -G(0;\widetilde{\mu})\left[J(0;\widetilde{\mu})Z_0 + \frac{D_2^3}{2!}\left(\frac{1}{2}Z_1 \otimes \bar{Z}_1\right) + \rho_0\right],
\end{aligned}
\qquad (23)
$$

$$
\begin{aligned}
Z_1 &= -G(i\widetilde{\omega};\widetilde{\mu})F_1. \\
&= -G(i\widetilde{\omega};\widetilde{\mu})\left[J(i\widetilde{\omega};\widetilde{\mu})Z_1 + \frac{D_2^1}{2!}(2Z_0 \otimes Z_1) + \frac{D_2^2}{2!}\right. \\
&\quad \times (\bar{Z}_1 \otimes Z_2) + \frac{D_3}{3!}\left.\left(\frac{3}{4}Z_1 \otimes Z_1 \otimes \bar{Z}_1\right) + \rho_1\right],
\end{aligned}
\qquad (24)
$$

$$
\begin{aligned}
Z_2 &= -G(2i\widetilde{\omega};\widetilde{\mu})F_2. \\
&= -G(2i\widetilde{\omega};\widetilde{\mu})\left[J(2i\widetilde{\omega};\widetilde{\mu})Z_2 + \frac{D_2^4}{2!}\left(\frac{1}{2}Z_1 \otimes Z_1\right)\right. \\
&\quad \left. + \frac{\rho_2}{2}\right],
\end{aligned}
\qquad (25)
$$

where $\bar{\cdot}$ denotes the complex conjugate, and ρ_0, ρ_1, ρ_2 are higher-order terms. From (23)-(25), one can calculate Z_0 and Z_2 as functions of Z_1, which yields

$$
\begin{aligned}
Z_0 &= -[I + G(0;\widetilde{\mu})J(0;\widetilde{\mu})]^{-1}G(0;\widetilde{\mu})\frac{D_2^3}{2!} \\
&\quad \times \left(\frac{1}{2}Z_1 \otimes \bar{Z}_1\right),
\end{aligned}
\qquad (26)
$$

$$
\begin{aligned}
Z_2 &= -[I + G(2i\widetilde{\omega};\widetilde{\mu})J(2i\widetilde{\omega};\widetilde{\mu})]^{-1}G(2i\widetilde{\omega};\widetilde{\mu}) \\
&\quad \times \frac{D_2^4}{2!}\left(\frac{1}{2}Z_1 \otimes Z_1\right).
\end{aligned}
\qquad (27)
$$

The complex number $\xi_1(\widetilde{\omega})$, which is used to calculate the amplitude of the emerging periodic solution, is given by

$$\xi_1(\widetilde{\omega}) = \frac{-w^T[G(i\widetilde{\omega};\widetilde{\mu})]p_1}{w^T v}, \qquad (28)$$

where

$$
\begin{aligned}
p_1 &= \left[D_2^1(V_{02} \otimes v) + \frac{1}{2}D_2^2(\bar{v} \otimes V_{22})\right. \\
&\quad \left. + \frac{1}{8}D_3(v \otimes v \otimes \bar{v})\right],
\end{aligned}
\qquad (29)
$$

$$V_{02} = -\frac{1}{4}[I + G(0;\widetilde{\mu})J(0;\widetilde{\mu})]^{-1}G(0;\widetilde{\mu})D_2^3(v \otimes \bar{v}), \qquad (30)$$

$$
\begin{aligned}
V_{22} &= -\frac{1}{4}[I + G(2i\widetilde{\omega};\widetilde{\mu})J(2i\widetilde{\omega};\widetilde{\mu})]^{-1}G(2i\widetilde{\omega};\widetilde{\mu}) \\
&\quad \times D_2^4(v \otimes v),
\end{aligned}
\qquad (31)
$$

and w^T and v are the left and right eigenvectors of $[G(i\widetilde{\omega};\widetilde{\mu})]J(i\widetilde{\omega};\widetilde{\mu})$, respectively, associated with the value $\lambda(i\widetilde{\omega};\widetilde{\mu})$ that is closest eigenvalue to the critical point $(-1 + i0)$. Clearly, V_{02} and V_{22} are given by (30) and (31) after the replacements $Y_1 = v\theta$, $Y_0 = V_{02}\theta^2$ and $Y_2 = V_{02}\theta^2$, where θ is a measure of the amplitude of the periodic solution [24][25][26]. For more details about the harmonic balance approach, the reader is referred to the book [20].

Now, the following Hopf bifurcation theorem formulated in the frequency domain can be stated [20]:

Theorem 3 (The Graphical Hopf Bifurcation Theorem)

Suppose that when ω varies, the vector $\xi_1(\widetilde{\omega}) \neq 0$, where $\xi_1(\widetilde{\omega})$ is defined in (28), and that the half-line L_1, starting from $-1+i0$ and pointing to the direction parallel to that of $\xi_1(\widetilde{\omega})$, first intersects the locus of the eigenvalue $\widehat{\lambda}(i\omega; \widetilde{\mu})$ at the point

$$\widehat{P} = \widehat{\lambda}(\widehat{\omega}; \widetilde{\mu}) = -1 + \xi_1(\widetilde{\omega})\theta^2, \tag{32}$$

at which $\omega = \widehat{\omega}$ and the constant $\theta = \theta(\widehat{\omega}) \geq 0$. Suppose, furthermore, that the above intersection is transversal, namely,

$$\left| \begin{array}{cc} \Re\{\xi_1(\widehat{\omega})\} & \Im\{\xi_1(\widehat{\omega})\} \\ \Re\{\frac{d}{d\omega}\widehat{\lambda}(\omega; \widetilde{\mu})|_{\omega=\widehat{\omega}}\} & \Im\{\frac{d}{d\omega}\widehat{\lambda}(\omega; \widetilde{\mu})|_{\omega=\widehat{\omega}}\} \end{array} \right| \neq 0. \tag{33}$$

Then, the following conclusions hold:
(1) The nonlinear system (6) has a periodic solution $y(t) = y(t; \widehat{y})$. Consequently, there exists a unique limit cycle in the nonlinear equation (3).
(2) If the half-line L_1 first intersects the locus of $\widehat{\lambda}(i\omega)$ when $\widetilde{\mu} > \mu_0 (< \mu_0)$, then the bifurcating periodic solution exists and the Hopf bifurcation is supercritical (subcritical).
(3) If the total number of anticlockwise encirclements of the point $P_1 = \widehat{P} + \varepsilon\xi_1(\widetilde{\omega})$, for a small enough $\varepsilon > 0$, is equal to the number of poles of $\lambda(s)$ that have positive real parts, then the limit cycle is stable; otherwise, it is unstable.

It is easy to see that $s = -c_i$ $(i = 1, 2, \cdots, n)$ are the poles of $\lambda(s)$, and the number of poles of $\lambda(s)$ that have positive real parts is zero. Hence, the following Corollary is established:

Corollary 1 Let k be the total number of anticlockwise encirclements of the point $P_1 = \widehat{P} + \varepsilon\xi_1(\widetilde{\omega})$ for a small enough $\varepsilon > 0$, where $\widehat{P}$ is the intersection of the half-line L_1 and the locus $\widehat{\lambda}(i\omega)$. Then, the following conclusions hold:
(1) If $k = 0$, then the bifurcating periodic solutions of system (3) are stable.
(2) If $k \neq 0$, then the bifurcating periodic solutions of system (3) are unstable.

IV. NUMERICAL EXAMPLES

In this section, some numerical examples are given to verify the theoretical analysis. Stability of system (3) can be justified by Theorem 1, and thus the stability of the delayed system (1) can be discussed by a transformation. The half-line L_1 and the locus $\widehat{\lambda}(i\omega)$ are shown by the corresponding frequency graphs. If they intersect, a limit cycle exists, or else, no limit cycle exists. Corollary 1 implies that the stability of the bifurcating periodic solution is determined by the total number k of the anticlockwise encirclements of the point $P_1 = \widehat{P} + \varepsilon\xi_1(\widetilde{\omega})$ for a small enough $\varepsilon > 0$. Suppose that the half-line L_1 and the locus $\widehat{\lambda}(i\omega)$ intersect. If $k = 0$, the bifurcating periodic solutions of system (3) is stable; if $k \neq 0$, the bifurcating periodic solutions of system (3) is unstable.

Example 1. Consider the following delayed recurrent neural network:

$$\dot{y}(t) = -Cy(t) + Ag(y(t)) + Bg(y(t - \tau)), \tag{34}$$

where $C = \left(\begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right)$, $A = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$, $B = \left(\begin{array}{cc} 1 & 2 \\ 2 & 3 \end{array} \right)$, $g(y) = \left(\begin{array}{c} -\tanh(y) \\ -\tanh(y) \end{array} \right)$ are the same as those discussed in [11]. From [11], $\tau_0 = 0.5183$ is a bifurcation parameter if $\tau_1 = \tau_2$. If $\tau_1 = \tau_2 < \tau_0$, system (34) is locally asymptotically stable; otherwise, if $\tau_1 = \tau_2 > \tau_0$, a periodic solution emerges. A lot of published papers [10]-[18] have considered the case of only one delay as the bifurcation parameter.

In this paper, the two delays τ_1 and τ_2 are both considered as bifurcation parameters. First, we choose $\tau_1 = 0.48$ and $\tau_2 = 0.52$, respectively. The corresponding waveform, phase and frequency graph are shown in Fig. 1. The half-line L_1 and locus $\widehat{\lambda}(i\omega)$ do not intersect, so no limit cycle exists. By Theorem 1 and Theorem 3, we know that in Fig. 1 its zero solution is asymptotically stable.

Next, we choose $\tau_1 = 0.50$ and $\tau_2 = 0.53$, respectively. The corresponding waveform, phase and frequency graph are shown in Fig. 2. By Theorem 3, we know that the half-line L_1 intersects the locus $\widehat{\lambda}(i\omega)$, so a limit cycle exists. The total number k of the anticlockwise encirclements of the point $P_1 = \widehat{P} + \varepsilon\xi_1(\widetilde{\omega})$ for a small enough $\varepsilon > 0$ is 0, i.e. $k = 0$, so by Corollary 1 a stable periodic solution exists.

Finally, we show a bifurcation diagram in a local region to verify the theoretical analysis, which is shown in Fig. 3. Here, τ_1 and τ_2 are considered as parameters. In some region of Fig. 3, system (34) is locally asymptotically stable, while in some other region it is unstable. Hopf bifurcation occurs when τ_1 and τ_2 pass through some critical values where the stability condition of the equilibrium is not satisfied.

Example 2. Consider the following delayed recurrent neural network:

$$\dot{y}(t) = -Cy(t) + Ag(y(t)) + Bg(y(t - \tau)), \tag{35}$$

where $C = \left(\begin{array}{cc} c_1 & 0 \\ 0 & 2 \end{array} \right)$, $A = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$, $B = \left(\begin{array}{cc} 1 & 2 \\ 2 & 3 \end{array} \right)$, $g(y) = \left(\begin{array}{c} -\tanh(y) \\ -\tanh(y) \end{array} \right)$, and $\tau_1 = 0.51$ is fixed. Here, τ_2 and c_1 are both considered as bifurcation parameters. The results can be hardly obtained by the previous analysis. First, we choose $\tau_2 = 0.515$ and $c_1 = 1.1$, respectively. The corresponding waveform, phase and frequency graph are shown in Fig. 4. The half-line L_1 and locus $\widehat{\lambda}(i\omega)$ do not intersect, so no limit cycle exists. By Theorem 1 and Theorem 3, we know that in Fig. 4 its zero solution is asymptotically stable.

Next, we choose $\tau_2 = 0.53$ and $c_1 = 0.9$, respectively. The corresponding waveform, phase and frequency graph are shown in Fig. 5. By Theorem 3, we know that the half-line L_1 intersects the locus $\widehat{\lambda}(i\omega)$, so a limit cycle exists. The total number k of the anticlockwise encirclements of the point $P_1 = \widehat{P} + \varepsilon\xi_1(\widetilde{\omega})$ for a small enough $\varepsilon > 0$ is 0, i.e. $k = 0$, so by Corollary 1 a stable periodic solution exists.

Finally, we show a bifurcation diagram in a local region to verify the theoretical analysis, which is shown in Fig. 6. Here, τ_2 and c_1 are considered as parameters. In some region
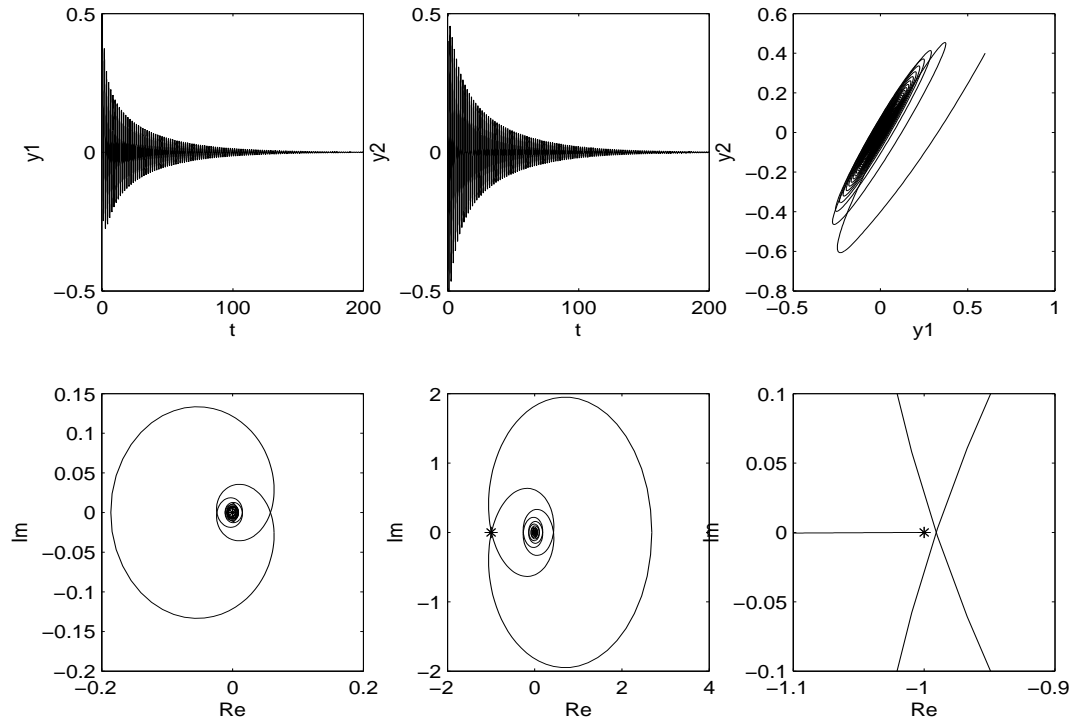
Fig. 1 Waveform, phase, frequency and magnification of the frequency graph ($\tau_1 = 0.48$, $\tau_2 = 0.52$).
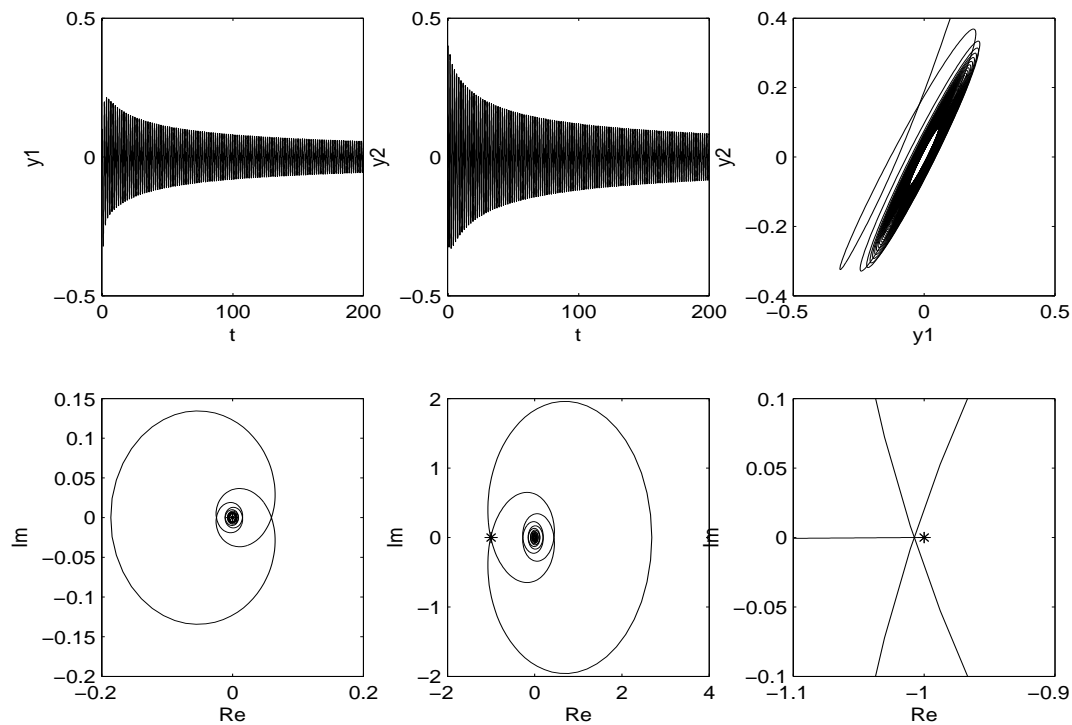
Fig. 2 Waveform, phase, frequency and magnification of the frequency graph ($\tau_1 = 0.50$, $\tau_2 = 0.53$).
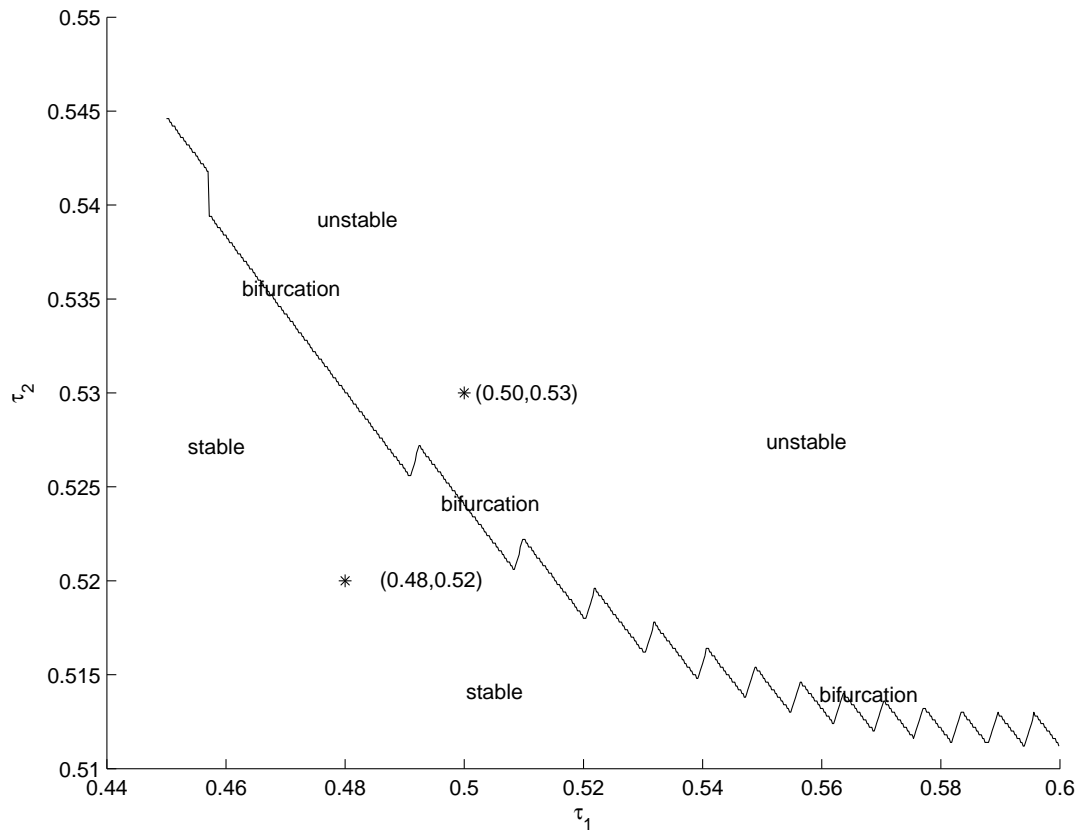
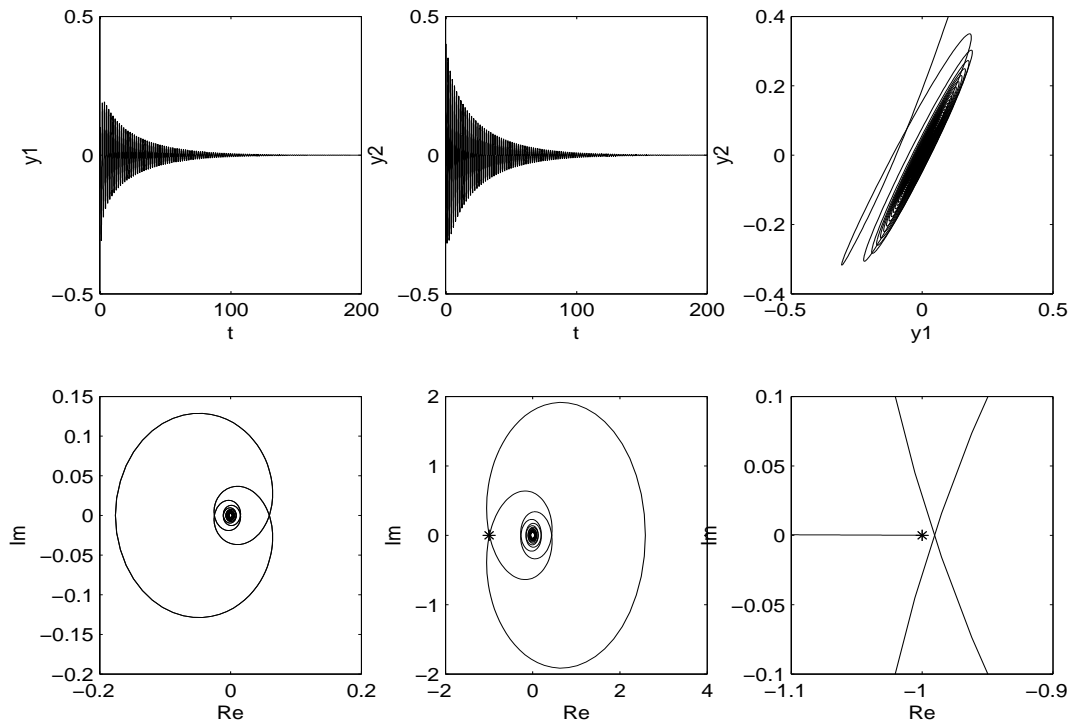Fig. 3 Two-parameter (τ_1 and τ_2) bifurcation diagram.

Fig. 4 Waveform, phase, frequency and magnification of the frequency graph ($c_1 = 1.1$, $\tau_2 = 0.515$).
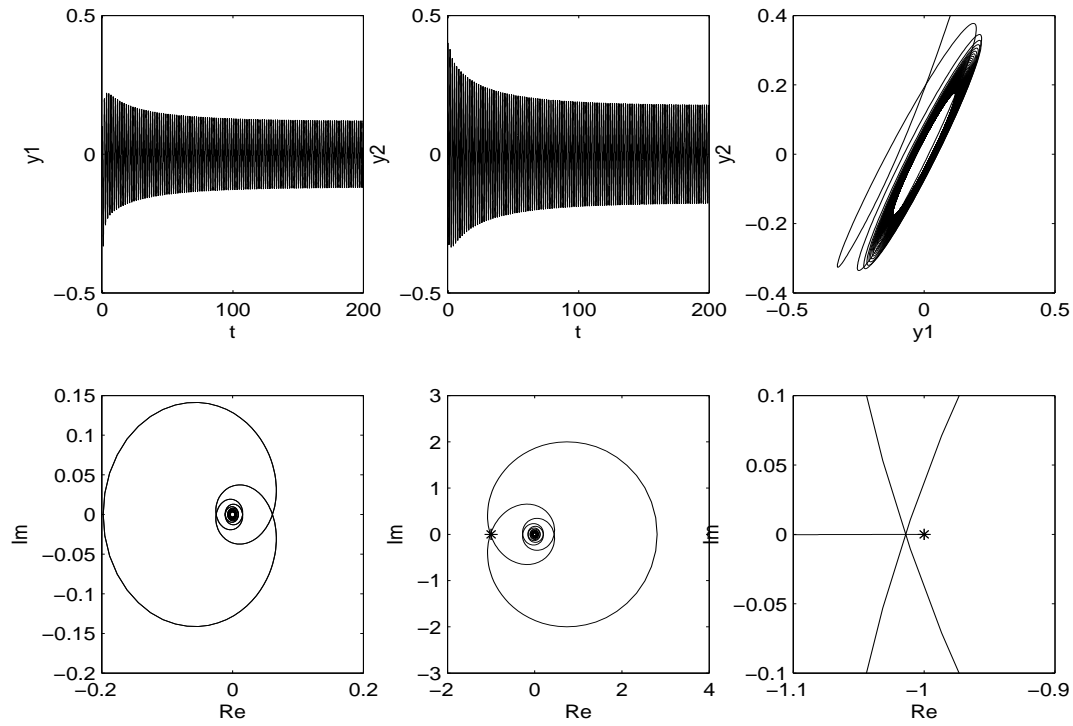
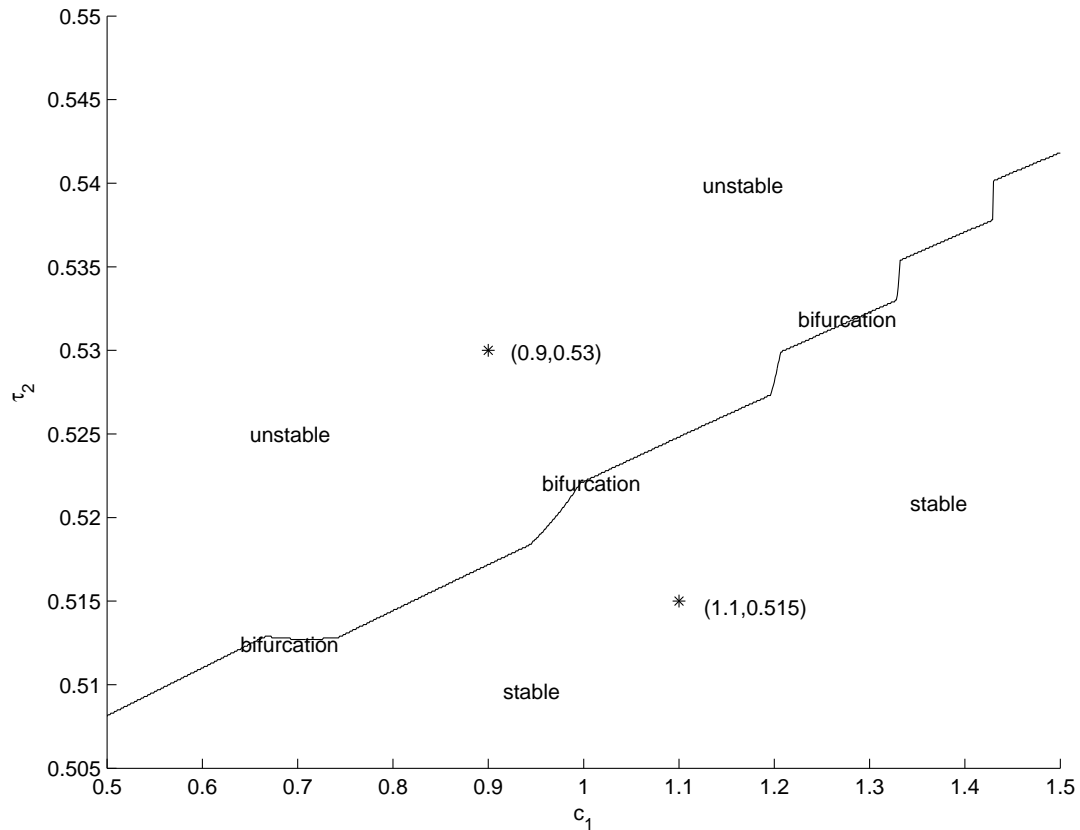Fig. 5 Waveform, phase, frequency and magnification of the frequency graph ($c_1 = 0.9$, $\tau_2 = 0.53$).

Fig. 6 Two-parameter (c_1 and τ_2) bifurcation diagram.

of Fig. 6, system (35) is locally asymptotically stable, while in some other region it is unstable. Hopf bifurcation occurs when τ_2 and c_1 pass through some critical values where the stability condition of the equilibrium is not satisfied.

In [28], some delay-dependent conditions are given to ensure the stability of the equilibrium point. However, these delay-dependent conditions are too conservative and depend on the maximum bound of all the time delays. The method used in many papers [10]-[17] can not be used to solve the above problem, where more than one time delay (which are system bifurcation parameters) are greater than τ_0. The method developed in this paper, however, can be applied. In addition, a general high-dimensional Hopf bifurcation is analyzed.

V. CONCLUSIONS

In this paper, we have discussed the local asymptotical stability and Hopf bifurcation of a general model of delayed recurrent neural networks, which gives a better understand of the situation when the stability of delayed recurrent neural networks is not guaranteed and a Hopf bifurcation occurs. To the best of our knowledge, there are very few results available about the bifurcation analysis of higher-dimensional system with multiple delays, which may lead the system to instability. The analytical results obtained in this paper may therefore give new insights on the dynamics of multi-delayed recurrent neural networks.

REFERENCES

[1] J.J. Hopfield, Neurons with graded response have collective computational properties like those of two-state neurons, Proc. Nat. Acad. Sci. USA, vol. 81, pp. 3088-3092, 1984.

[2] M.A. Cohen, S. Grossberg, Absolute stability of global pattern formation and parallel memory storage by competitive neural networks, IEEE Trans. Syst. Man Cyber., vol. 13, pp. 815-825, 1983.

[3] H. Ye, A.N. Michel, K. Wang, Qualitive analysis of Cohen-Grossberg neural networks with multiple delays, Phys. Rev. E, vol. 51, pp. 2611-2618, 1995.

[4] C. M. Marcus, R. M. Westervelt, Stability of analog neural networks with delay, Phys. Rev. A, vol. 39, pp. 347-359, 1989.

[5] K. Gopalsamy, I. Leung, Convergence under dynamical thresholds with delays, IEEE Trans. Neural Networks, vol. 8, pp. 341-348, 1997.

[6] P. Baldi, A. Atiya, How delays affect neural dynamics and learning, IEEE Trans. Neural Networks, vol. 5, pp. 612-621, 1994.

[7] J. Cao, L. Wang, Exponential stability and periodic oscillatory solution in BAM networks with delays, IEEE Trans. Neural Networks, vol. 13, no. 2, pp. 457-463, 2002.

[8] J. Cao, J. Wang, Global asymptotic and robust stability of recurrent neural networks with time delays, IEEE Trans. Circuits Systems-I, vol. 52, no. 2, pp. 417-426, 2005.

[9] W. Yu, J. Cao, Hopf bifurcation and stability of periodic solutions for van der Pol equation with time delay, Nonlinear Analysis, vol. 62, pp. 141-165, 2005.

[10] W. Yu, J. Cao, Stability and Hopf bifurcation analysis on a four-neuron BAM neural network with time delays, Phys. Lett. A, vol. 351, pp. 64-78, 2006.

[11] W. Yu, J. Cao, Stability and Hopf bifurcation on a two-neuron system with time delay in the frequency domain, Int. J. Bifurcation and Chaos, vol. 17, no. 4, pp. 1355-1366, 2007.

[12] S. Guo, L. Huang, Linear stability and Hopf bifurcation in a two-neuron network with three delays, Int. J. Bifurcation and Chaos, vol. 14, pp. 2790-2810, 2004.

[13] Y. Song, M. Han, J. Wei, Stability and Hopf bifurcation analysis on a simplified BAM neural network with delays, Physica D, vol. 200, pp. 185-204, 2005.

[14] Y. Song, J. Wei, Local Hopf bifurcation and global periodic solutions in a delayed predator-prey system, J. Math. Anal. Appl., vol. 301, pp. 1-21, 2005.

[15] S. Ruan, J. Wei, On the zeros of transcendental functions with applications to stability of delay differential equations with two delays, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., vol. 10, pp. 863-874, 2003.

[16] S. Ruan, J. Wei, On the zeros of a third degree exponential polynomial with applications to a delayed model for the control of testosterone secretion, IMA J. Math. Appl. Med. Biol., vol. 18, pp. 41-52, 2001.

[17] L. Olien, J. Bélair, Bifurcations, stability and monotonicity properties of a delayed neural network model, Physica D, vol. 102, pp. 349-363, 1997.

[18] S.A. Campbell, S. Ruan and J. Wei, Qualitative analysis of a neural network model with multiple time delays. Int. J. Bifurcation and Chaos, vol. 9, no. 8, pp. 1585-1595, 1999.

[19] Y. Tsypkin, Stability of systems with delayed feedback, Automat. Telemekh, vol. 7, pp. 107-129, 1946. Also reprinted in A.G.J. MacFarlane (Editor), Frequency Response Methods in Control Systems IEEE Press, pp. 45-56, 1979.

[20] J.L. Moiola, G. Chen, Hopf Bifurcation Analysis: A Frequency Domain Approach, World Scientific, Singapore, 1996.

[21] A.I. Mees, Dynamics of Feedback Systems, John Wiley & Sons, Chichester, UK, 1981.

[22] B.D. Hassard, N.D. Kazarinoff, and Y.H. Wan, Theory and Application of Hopf Bifurcation, Cambridge University Press, Cambridge, 1981.

[23] D.J. Allwright, Harmonic balance and the Hopf bifurcation theorem, Mathematical Proceedings of Cambrige Philosphical Society, vol. 82, pp. 453-467, 1977.

[24] A.I. Mees, L.O. Chua, The Hopf bifurcation theorem and its applications to nonlinear oscillations in circuits and systems, IEEE Trans. Circuits Systems, vol. 26, pp. 235-254, 1979.

[25] J.L. Moiola, G. Chen, Computation of limit cycles via higher-order harmonic balance approximation. IEEE Trans. Automatic Control, vol. 38, pp. 782-790, 1993.

[26] J.L. Moiola, G. Chen, Frequency domain approach to computation and analysis of bifurcations and limit cycles: A tutorial, Int. J. Bifurcation and Chaos, vol. 3, pp. 843-867, 1993.

[27] A.G.J. MacFarlane, I. Postlethwaite, The generalized Nyquist stability criterion and multivariable root loci, Int. J. Control, vol. 25, pp. 81-127, 1977.

[28] W. Chen, Z. Guan, X. Lu, Delay-dependent exponential stability of neural networks with variable delays, Phys. Lett. A, vol. 326, pp. 355-363, 2004.

[29] J. Cao, M. Xiao, Stability and Hopf bifurcation in a simplified BAM neural network with two time delays, IEEE Trans. Neural Networks, vol. 18, no. 2, pp. 416-430, 2007.

[30] W. Yu, J. Cao, Adaptive Q-S (lag, anticipated, and complete) time-varying synchronization and parameters identification of uncertain delayed neural networks, Chaos, vol. 16, art.no. 023119, 2006.

[31] Y. Liu, Z. You, L. Cao, On the almost periodic solution of cellular neural networks with distributed delays, IEEE Trans. Neural Networks, vol. 18, no. 1, pp. 295-300, Jan. 2007.

[32] T. Kwok, K.A. Smith, A noisy self-organizing neural network with bifurcation dynamics for combinatorial optimization, IEEE Trans. Neural Networks, vol. 15, no. 1, pp. 84-98, Jan. 2004.

[33] H. Kadone, Y. Nakamura, Symbolic memory for humanoid robots using hierarchical bifurcations of attractors in nonmonotonic neural networks, IEEE International Conference on Intelligent Robots and Systems, pp. 3548-3553, Aug. 2005.

[34] E. Enikov, G. Stepan, Microchaotic motion of digital controlled machines, Journal of Vibration and Control, vol. 4, no. 4, pp. 427-443, 1998.

[35] G. Stepan, Retarded Dynamical Systems: Stability and Characteristic Functions, Longman Scientific and Technical Publishers, 1989.

[36] A.H. Nayfeh, Order reduction of retarded nonlinear systems-the method of multiple scales versus center-manifold reduction, Nonlinear dynamics, DOI:10.1007/s11071-007-9237-y.

[37] W. Yu, J. Cao, G. Chen, Robust adaptive control of unknown modified Cohen -Grossberg neural networks with delay, IEEE Trans. Circuits Syst. II, vol. 54, no. 6, pp. 502-506, 2007.

[38] W. Yu, J. Cao, J. Wang, An LMI approach to global asymptotic stability of the delayed Cohen-Grossberg neural network via nonsmooth analysis, Neural Networks, vol. 20, no. 7, pp. 810-818, 2007.

Guanrong Chen (M'89,SM'92,F'97) received the M.Sc. degree in Computer Science from Sun Yat-sen (Zhongshan) University, China and the Ph.D. degree in Applied Mathematics from Texas A&M University, USA. Currently he is a Chair Professor and the Director of the "Centre for Chaos and Complex Networks" at the City University of Hong Kong. He is a Fellow of the IEEE for his fundamental contributions to the theory and applications of chaos control and bifurcation analysis.

Prof. Chen has numerous publications since 1981 in the fields of nonlinear systems, in both dynamics and controls, which include 17 monographs and textbooks, 400 some journal papers and about 250 conference abstracts. Prof. Chen served and is serving as Editors in various ranks for several international journals, including the IEEE TRANSACTION and MAGAZINE ON CIRCUITS AND SYSTEMS, and the INTERNATIONAL JOURNAL OF BIFURCATION AND CHAOS. He received the 1998 Harden-Simons Annual Prize for Outstanding Journal Paper from the American Society of Engineering Education, the 2001 IEEE M. Barry Carlton Best Annual Paper Award from the IEEE Aerospace and Electronic Systems Society, the 2002 Best Paper Award from the Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, and the 2005 IEEE Guillemin-Cauer Best Annual Paper Award from the IEEE Circuits and Systems Society. He is Honorary Professor of more than 10 universities in Argentina, Australia, and China.

Wenwu Yu (S'07) received both the B.S. degree and the M.S. degree from the Department of Mathematics, Southeast University, Nanjing, China in 2004 and 2007, respectively. From November 2006 to January 2007, he was a Research Assistant in the Department of Electronic Engineering, City University of Hong Kong, Hong Kong. He is currently pursuing his PhD degree in Department of Electrical Engineering, Columbia University, New York, USA.

He is the author or coauthor of about 20 journal papers, and a reviewer of several journals. His research interests include stability theory, bifurcation analysis, chaos synchronization and control, complex networks and systems, stochastic systems, neural networks, cryptography, and communications.

Jinde CAO (M'07,SM'07) received the B.S. degree from Anhui Normal University, Wuhu, China, the M.S. degree from Yunnan University, Kunming, China, and the Ph.D. degree from Sichuan University, Chengdu, China, all in mathematics/applied mathematics, in 1986, 1989, and 1998, respectively. From March 1989 to May 2000, he was with the Yunnan University. In May 2000, he joined the Department of Mathematics, Southeast University, Nanjing, China. From July 2001 to June 2002, he was a Post Doctoral Research Fellow in the Department of Automation and Computer-aided Engineering, Chinese University of Hong Kong, Hong Kong. In 2006 and 2007, he was a Visiting Research Fellow and a Visiting Professor in the School of Information Systems, Computing and Mathematics, Brunel University, UK.

He is currently a TePin Professor and Doctoral Advisor at the Southeast University, prior to which he was a Professor at Yunnan University from 1996 to 2000. He is the author or coauthor of more than 150 journal papers and five edited books and a reviewer of MATHEMATICAL REVIEWS and ZENTRALBLATT-MATH.

His research interests include nonlinear systems, neural networks, complex systems and complex networks, stability theory, and applied mathematics.

Professor Cao is an Associate Editor of the IEEE TRANSACTION ON NEURAL NETWORKS, JOURNAL OF THE FRANKLIN INSTITUTE, MATHEMATICS AND COMPUTERS IN SIMULATION, and NEURO-COMPUTING.