Parameter identification of dynamical systems from time series

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In this paper, synchronization based parameter identification of dynamical systems from time series is carefully revisited. It is shown, based on rigorous theoretical analysis and concrete counterexamples, that some recent research reports on this issue are incomplete or even incorrect. A linear independence condition is pointed out, which is sufficient for such parameter identification of general dynamical systems.

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Since the pioneering work of Ott, Gregogi, and Yorke [1], and Pecora and Carroll [2], chaos control and synchronization have received increasing attention [3–7] due to their potential applications in secure communication, laser systems, chemical reactions, biological systems, and so on. An important application of synchronization and control are in adaptive parameter estimation methods where parameters in a model are adjusted dynamically in order to minimize the synchronization error [8–15]. Such parameter identification methods are useful for instance in chaos communications [16] where the states of two identical chaotic systems are required to asymptotically synchronize and the (unknown) parameters of the transmitter system will be estimated. In another field, stable equilibrium points in neural networks have been widely investigated in associative memories, where they are used to store and recover a (single) pattern. However, due to the limited information stored in such a stable equilibrium point, there is also great interest in using periodic solutions and chaotic orbits for associative memory and pattern recognition [17]. In these neural networks, the learning algorithm is based on the adjustment of the weight matrix, which can be considered as the estimation of network parameters.

It is well known in the field of control systems that system parameters identification may have "parameter drift," i.e., the estimated value departs from the true value of a parameter by a constant that cannot be eliminated unless the so-called persistent excitation condition is satisfied which requires rich enough training information (input signals) [18].

Recently, some researchers began to question about whether the parameters can really be identified from a time series in deterministic dynamical systems based on the synchronization method, particularly from chaotic systems, due to the lack of complete and rigorous mathematical theory about this topic in the field. In [19], for example, a counterexample to the model investigated in [13] was given, declaring that parameters cannot be estimated from stable systems by using the adaptive control method. Later, in [20], the same model was revisited with a detailed proof to show that parameters in periodic and chaotic systems can be estimated. But the systems discussed therein are too special and cannot be extended to the general cases. More recently, in [14], adaptive synchronization was studied with an application to estimating system parameters, which provides a detailed analysis, with a claim that "the chaotic behavior is necessary to realize such techniques of parameter identification." However, in the present paper, based on rigorous theoretical analysis and concrete counterexamples, we show that all reports [8–14] referred to the above issue are either incomplete or even incorrect.

In the following we shall demonstrate that parameter identification is almost impossible for systems converging to a fixed point and may also fail for chaotic systems. A linear independence condition will be pointed out, which is essential for ensuring parameters identification of general dynamical systems. Furthermore, it will be shown that under this sufficient condition, parameters can be well estimated from a time series of dynamical systems based on synchronization.

In the master-slave framework, consider the following master system:

$$\dot{x}(t) = F(x,p) + G(x(t-\tau),r), \tag{1}$$

where $x(t)=(x_1(t),x_2(t),\ldots,x_n(t))^T \in R^n$ is the state vector, $\tau>0$ is a time delay, $F(x,p)=(F_1(x,p), F_2(x,p),\ldots,F_n(x,p))^T$ and $G(x(t-\tau),r)=[G_1(x(t-\tau),r), G_2(x(t-\tau),r),\ldots,G_n(x(t-\tau),r)]^T$ are nonlinear and delayed nonlinear functions, respectively, with

$$F_i(x,p) = c_i(x) + \sum_{j=1}^{m_1} p_{ij} f_{ij}(x), \tag{2}$$

and

$$G_i(x(t-\tau),r) = d_i(x(t-\tau)) + \sum_{j=1}^{m_2} r_{ij} g_{ij}\big(x(t-\tau)\big), \tag{3}$$

in which $c_i(x)$, $f_{ij}(x)$, $d_i(x(t-\tau))$, $g_{ij}(x(t-\tau))$ are nonlinear functions, and $p=(p_{ij})\in R^{nm_1}$, $r=(r_{ij})\in R^{nm_2}$ are unknown

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parameters to be estimated. The initial conditions are given by $x_i(t)=\phi_i(t)\in\mathcal{C}([-\tau,0],R)$, the set of all continuous functions from $[-\tau,0]$ to R.

Assume that all nonlinear functions satisfy the uniform Lipschitz condition, i.e., there exist positive constants $\bar{f}$ and $\bar{g}$ such that, for all $i=1,2,\ldots,n$,

$$|F_i(x,p)-F_i(y,p)|\le \bar{f}\max_j|x_j-y_j|,\ \forall\, x,y\in R^n, \tag{4}$$

and

$$|G_i(x,p)-G_i(y,p)|\le \bar{g}\max_j|x_j-y_j|,\ \forall\, x,y\in R^n. \tag{5}$$

Note that conditions (4) and (5) are very mild: if $\partial f_{ij}/\partial x_j$ and $\partial g_{ij}/\partial x_j$ $(i,j=1,2,\ldots,n)$ are bounded, then these two conditions are satisfied. So, system (1) includes many well-known systems, such as the Lorenz system [18], Chen system, Lü system, various neural networks, Chua's circuit, and so on, to name just a few.

To estimate the unknown parameters in p based on synchronization principle using time series $x(t)$ from system (1), the following response system and adaptive laws are introduced [14]:

$$\dot{y}(t)=F(y,q)+G(y(t-\tau),s)-k(y-x), \tag{6}$$

$$\dot{q}_{ij}=-\alpha_{ij}e_i f_{ij}(y),\quad i=1,2,\ldots,n,\quad j=1,2,\ldots,m_1, \tag{7}$$

$$\dot{s}_{ij}=-\beta_{ij}e_i g_{ij}\big(y(t-\tau)\big),\quad i=1,2,\ldots,n,\quad j=1,2,\ldots,m_2, \tag{8}$$

$$\dot{k}_i=\gamma_i(y_i-x_i)^2, \tag{9}$$

where α_{ij}, β_{ij}, and γ_i are positive constants, $e_i=y_i-x_i$, q_{ij}, s_{ij}, and k_i are adaptive law parameters in the response system (6), and $k=\mathrm{diag}(k_1,k_2,\ldots,k_n)$. Here, q_{ij} and s_{ij} are also estimates of parameters p_{ij} and r_{ij} in the master system (1), respectively.

In order to state our main results, the following definition is needed:

Linearly independent. The functions $l_i(t)$ $(i=1,2,\ldots,N)$ are said to be linearly independent if there do not exist nonzero constants η_i $(i=1,2,\ldots,N)$, such that

$$\eta_1 l_1(t)+\eta_2 l_2(t)+\cdots+\eta_N l_N(t)=0.$$

In what follows, it will be shown that if the master system (1) and the slave system (6)–(9) are globally synchronized, and moreover if $f_{ij}[y(t)]$ $(j=1,2,\ldots,m_1)$ and $g_{ij}[y(t-\tau)]$ $(j=1,2,\ldots,m_2)$ are linearly independent (in the sense of linear algebra) on the synchronization manifold $x(t)=y(t)$, then $q_{ij}\to p_{ij}$ and $s_{ij}\to r_{ij}$ as $t\to\infty$.

Consider the following Lyapunov functional candidate:

$$\begin{aligned}V(e,q,s,k)={}&\frac{1}{2}\sum_{i=1}^{n}e_i^2+\sum_{i=1}^{n}\sum_{j=1}^{m_1}\frac{1}{2\alpha_{ij}}(q_{ij}-p_{ij})^2\\&+\sum_{i=1}^{n}\sum_{j=1}^{m_2}\frac{1}{2\beta_{ij}}(s_{ij}-r_{ij})^2+\sum_{i=1}^{n}\frac{1}{2\gamma_i}(k_i-L)^2\\&+\delta\sum_{i=1}^{n}\int_{t-\tau}^{t}e_i^2(s)ds,\end{aligned} \tag{10}$$

where δ and L are positive constants. Taking the derivative of $V(t)$ along the trajectories of (1) and (6)–(9), one obtains

$$\begin{aligned}\dot{V}(t)={}&\sum_{i=1}^{n}e_i^T(t)[F_i(y,q)-F_i(x,p)+G_i(y(t-\tau),s)\\&-G_i(x(t-\tau),r)-k_ie_i]-\sum_{i=1}^{n}\sum_{j=1}^{m_1}e_i(q_{ij}-p_{ij})f_{ij}(y)\\&-\sum_{i=1}^{n}\sum_{j=1}^{m_2}e_i(s_{ij}-r_{ij})g_{ij}(y(t-\tau))+\sum_{i=1}^{n}(k_i-L)e_i^2\\&+\delta\sum_{i=1}^{n}(e_i^2-e_i^2(t-\tau)).\end{aligned} \tag{11}$$

It is easy to verify that

$$\begin{aligned}&F_i(y,q)-F_i(x,p)+G_i(y(t-\tau),s)-G_i(x(t-\tau),r)\\&\quad=F_i(y,q)-F_i(y,p)+F_i(y,p)-F_i(x,p)+G_i(y(t-\tau),s)\\&\qquad-G_i(y(t-\tau),r)+G_i(y(t-\tau),r)-G_i(x(t-\tau),r).\end{aligned} \tag{12}$$

Therefore, one has

$$\begin{aligned}\dot{V}(t)={}&\sum_{i=1}^{n}e_i^T(t)[F_i(y,p)-F_i(x,p)+G_i(y(t-\tau),r)\\&-G_i(x(t-\tau),r)]-(L-\delta)\sum_{i=1}^{n}e_i^2-\delta\sum_{i=1}^{n}e_i^2(t-\tau)\\&\le-(L-n\bar{f}-n\bar{g}-\delta)\sum_{i=1}^{n}e_i^2-(\delta-n\bar{g})\sum_{i=1}^{n}e_i^2(t-\tau).\end{aligned} \tag{13}$$

Now, choosing $L=n\bar{f}+n\bar{g}+\delta+1$ and $\delta=n\bar{g}$, one obtains

$$\dot{V}(t)\le-\sum_{i=1}^{n}e_i^2. \tag{14}$$

Thus, from the Lyapunov theorem ([21] Chap. 5, Theorem 2.1), $y(t)\to x(t)$ as $t\to\infty$, and consequently the LaSalle invariance principle ([21] Chap. 5, Theorem 3.1) implies that the set S satisfying $\dot{V}=0$ is given by

$$S=\{(e,q,s,k)\in R^{2n+n(m_1+m_2)}:e=0\}. \tag{15}$$

It follows from (1) and (6) that, for $i=1,2,\ldots,n$,

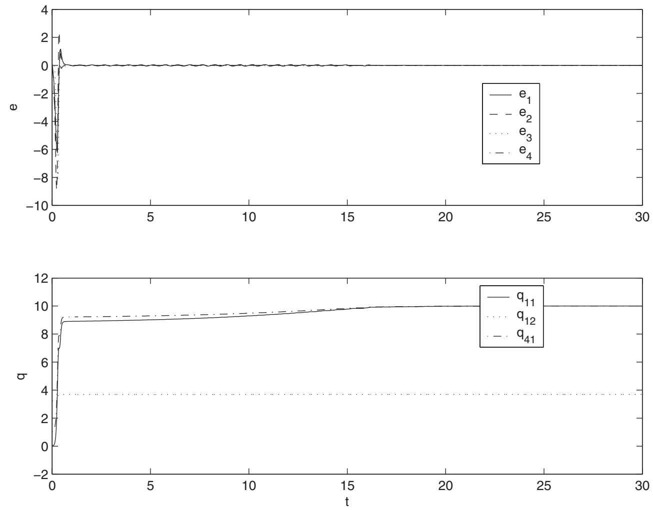

FIG. 1. Orbits of e_i and q_{ij}.

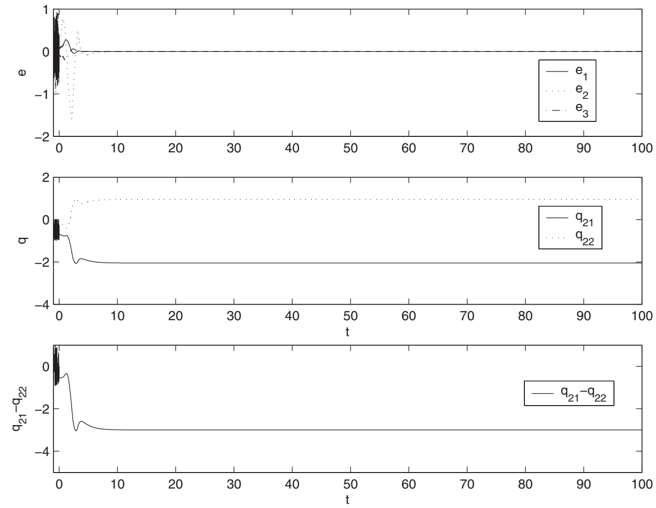

FIG. 2. Orbits of e_i, q_{ij}, and $q_{21}-q_{22}$.

$$\dot{e}_i(t) = F_i(y,q) - F_i(x,p) + G_i(y(t-\tau),s) - G_i(x(t-\tau),r) - k_i e_i, \tag{16}$$

so that, on the set S, i.e., when $e_i=0$, one has

$$0 = F_i(y,q) - F_i(y,p) + G_i(y(t-\tau),s) - G_i(y(t-\tau),r) = \sum_{j=1}^{m_1} (q_{ij}-p_{ij})f_{ij}(y) + \sum_{j=1}^{m_1} (s_{ij}-r_{ij})g_{ij}(y(t-\tau)). \tag{17}$$

Since $f_{ij}(y(t))$ $(j=1,2,\ldots,m_1)$ and $g_{ij}(y(t-\tau))$ $(j=1,2,\ldots,m_2)$ are linearly independent on the synchronization manifold, one has $q_{ij}\to p_{ij}$ and $s_{ij}\to r_{ij}$ as $t\to\infty$.

This synchronization thus results in $y(t)\to x(t)$, the same as reported before [8–14] where, unlike the above rigorous analysis, only simulations were given without a complete mathematical proof.

It should be emphasized that the above theoretical analysis shows that one cannot estimate the system parameters without the condition of linear independence of the functions on the synchronization manifold. In fact, for stable systems, parameters cannot be estimated when $f_{ij}(y)=c$ (constant) on the synchronization manifold, since if $f_{ij}(y)$ are not linearly independent then there will be many nonzero parameters, $\alpha\neq 0$ and $\beta\neq 0$, such that $(q_{ij}-p_{ij})=\alpha$ and $(s_{ij}-r_{ij})=\beta$, and they both satisfy (17); as a result, the parameters are not distinguishable (by any parameter identification method).

Next, examples are given to show that if the linear independence condition is not satisfied, one may not be able to estimate the system parameters. These counterexamples in effect disprove previous reports [8–14] claiming that chaotic dynamics is sufficient for parameter estimation.

First, consider the following system, constructed based on the classical Lorenz system:

$$\dot{x}_1 = a(x_2-x_1) + \bar{a}(x_4-x_1),$$

$$\dot{x}_2 = cx_1 - x_2 - x_1x_3,$$

$$\dot{x}_3 = -bx_3 + x_1x_2,$$

$$\dot{x}_4 = -(x_4-x_1) + a(x_2-x_1). \tag{18}$$

System (18) is chaotic when parameters $a=10$, $b=8/3$, $c=28$, $\bar{a}=6$. Obviously, since $\dot{x}_4-\dot{x}_1=-(1+\bar{a})(x_4-x_1)$, one has $x_1\to x_4$. Suppose only a and $\bar{a}$ are unknown for simplicity. Here, $p_{11}=a$, $p_{12}=\bar{a}$, $p_{41}=a$, $f_{11}=f_{41}=x_2-x_1$, and $f_{12}=x_4-x_1$. The time series can be received by the slave system (6)–(9). The error states e_i and the estimated parameters q_{ij} are shown in Fig. 1.

It is clear that $q_{11}\to p_{11}=10$, $q_{41}\to p_{41}=10$, but $q_{12}\nrightarrow p_{12}=6$. Actually, from (17) above, one has

$$(q_{11}-p_{11})f_{11}(y) + (q_{12}-p_{12})f_{12}(y) = 0 \tag{19}$$

on the synchronization manifold $x=y$ and $x_1=x_4$. Thus, $f_{12}=0$ and (19) is valid for any pair q_{12}, p_{12}.

Next, consider the following neural network model, as discussed in [8,9,22]:

$$\dot{x}(t) = -Cx(t) + A\tilde{f}(x(t)) + B\tilde{f}(x(t-1)), \tag{20}$$

where $x=(x_1,x_2,x_3)^T$, $\tilde{f}(x)=(\tanh(x_1),\tanh(x_2),\tanh(x_3))^T$,

$$C = \begin{pmatrix} -0.6 & 0 & -0.4 \\ 0 & -1 & 0 \\ 0.5 & 0 & 0.3 \end{pmatrix},$$

$$A = \begin{pmatrix} 2.0 & -0.1 & 0 \\ p_{21} & 3.0 & p_{22} \\ -2.0 & 0.1 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} -1.5 & -0.1 & 0 \\ -0.2 & -2.5 & 0 \\ 1.5 & 0.1 & 0 \end{pmatrix},$$

in which $f_{21}=\tanh(x_1)$, $f_{22}=\tanh(x_3)$ and $p_{21}=-4.0$, $p_{22}=-1.0$ are unknown to the slave system (6)–(9).

It is easy to verify that $\dot{x}_1+\dot{x}_3=-0.1(x_1+x_3)$ and $x_1(t)\to -x_3(t)$ as $t\to\infty$. It follows from (17) that, on the synchronization manifold $x=y$ and $x_1=-x_3$, one has

$$(q_{21}-p_{21})\tanh(y_1)+(q_{22}-p_{22})\tanh(y_3)=0, \qquad (21)$$

so that

$$[(q_{21}-p_{21})-(q_{22}-p_{22})]\tanh(y_1)=0, \qquad (22)$$

which implies that $q_{21}-q_{22}=p_{21}-p_{22}$ and p_{21} and p_{22} cannot be estimated, respectively. Here, only $p_{21}-p_{22}$ can be estimated by $q_{21}-q_{22}$.

The errors e_i and the estimated parameters q_{ij} and $q_{21}-q_{22}$ are shown in Fig. 2. It can be seen that $q_{21}-q_{22} \to p_{21}-p_{22}=-3$.

In fact, the parameter estimation method will fail if there is linear dependence because a violation of the linear independence condition indicates an overdetermination of the model which means that there are more parameters in the determine than one would need to model the observed dynamical behavior (redundant parametrization). The above analysis and counterexamples show the importance of the linear independence condition that was not paid attention to in previous work on (synchronization based) parameter estimation methods. In effect, it provides a necessary and sufficient condition for proper parameter identification which is crucial for many practical applications.

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