Homework Set no. 8

1. **Convexity and Binary Channels.**

   (a) Prove that $I(X;Y)$ is a convex function of $p(y|x)$ for fixed $p(x)$. You may use the fact that $D(p||q)$ is convex in the pair (p,q).

   (b) For this part you may assume the result of the previous part. Consider the channel shown in Figure 1. Assume $P(X=0)=\lambda$ where $0<\lambda<1$. Show that replacing both p and q

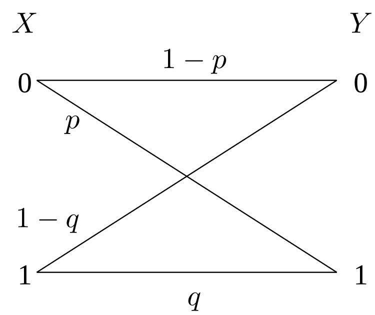

Figure 1: Binary (but not necessarily symmetric) channel.

   with $\frac{p+q}{2}$ will either lower the mutual information or leave it alone. This new channel has a certain symmetry, but it is *not* generally the standard binary symmetric channel.

   (c) (1 pt) For fixed λ, compute the mutual information for the reduced-mutual-information channel you created in the previous part.

   (d) 1 pt Argue that for fixed p, one choice of q that certainly minimizes $I(X;Y)$ is $q=p$.

2. **Maximum number of configurations with assigned minimum distance** Consider a binary code (i.e., the alphabet is $\{o,1\}$ of length n. An important question in coding is to determine the maximum number of configurations (codewords) having minimum pairwise distance d. Call this number $A(n,d)$. This is a difficult problem, but it is easy to come up with bounds and equalities. Among these equalities, two are of interest and are not too difficult to prove.

   - Show that $A(n,d) \le 2A(n-1,d)$;
   - Show that $A(n+1,2s) = A(n,2s-1)$;

   Note: though it is not too difficult to come up with the answers, the problem requires a bit of thought. Writing up an example might give you an insight of what a proof might be.

3. **Ternary Channel** Consider the ternary channel (i.e., where both inputs and outputs are ternary) defined by the conditional probability matrix

$$P_{i,j} = P(Y=i \mid X=j) = \begin{bmatrix} 2/3 & 0 & 1/3 \\ 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \end{bmatrix}.$$

Compute the capacity of this channel.

4. **Differential entropy (EIT 9.1).** Evaluate the differential entropy $h(X) = -\int f \ln f$ for the following:

   (a) The exponential density, $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$.

   (b) The Laplace density, $f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}$.

   (c) The sum of X_1 and X_2, where X_1 and X_2 are independent normal random variables with means μ_i and variances $\sigma_i^2, i = 1, 2$.

   Hint: this is essentially a mechanical problem.