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EE 4810, DSP
HW-4

3.1
First, note that

$$\mathfrak{Re}\{x[n]e^{-j\omega n}\} = x[n]\cos(-\omega n)$$
$$\mathfrak{Im}\{x[n]e^{-j\omega n}\} = x[n]\sin(-\omega n)$$

So we have

$$X_{\rm re}(e^{j\omega}) = \sum_{-\infty}^{\infty} \mathfrak{Re}\{x[n]e^{-j\omega n}\} = \sum_{-\infty}^{\infty} x[n]\cos(-\omega n)$$
$$= \sum_{-\infty}^{\infty} x[n]\cos(\omega n) = \sum_{-\infty}^{\infty} \mathfrak{Re}\{x[n]e^{j\omega n}\} = X_{\rm re}(e^{-j\omega}),$$

so $X_{\rm re}(e^{j\omega})$ is even. Similarly,

$$X_{\rm im}(e^{j\omega}) = \sum_{-\infty}^{\infty} \mathfrak{Im}\{x[n]e^{-j\omega n}\} = \sum_{-\infty}^{\infty} x[n]\sin(-\omega n)$$
$$= -\sum_{-\infty}^{\infty} x[n]\sin(\omega n) = -\sum_{-\infty}^{\infty} \mathfrak{Im}\{x[n]e^{j\omega n}\} = -X_{\rm im}(e^{-j\omega}),$$

so $X_{\rm im}(e^{j\omega})$ is odd. The magnitude of the DTFT is even as well, as we see from

$$|X(e^{j\omega})| = \sum |x[n]e^{-j\omega n}| = \sum |x[n]|$$
$$= \sum |x[n]e^{j\omega n}| = |X(e^{-j\omega})|$$

Finally, the phase of the DTFT is odd:

$$\angle X(e^{j\omega}) = \sum \angle x[n]e^{-j\omega n} = \sum -\omega n$$
$$= -\sum \omega n = -\sum \angle x[n]e^{j\omega n} = -\angle X(e^{-j\omega}).$$

3.5
a) Time-shift. Substituting $m = n - n_0$ into the equation for the DTFT of $g[n - n_0]$ gives

$$\sum_{n=-\infty}^{\infty} g[n-n_0]e^{-j\omega n} = \sum_{m=-\infty}^{\infty} g[m]e^{-j\omega(m+n_0)}$$
$$= \sum_{m=-\infty}^{\infty} g[m]e^{-j\omega m}e^{-j\omega n_0}$$
$$= e^{-j\omega n_0}G(e^{j\omega}).$$

b) Frequency shift.

$$\begin{aligned}G(e^{j(\omega-\omega_0)}) &= \sum_{n=-\infty}^{\infty} g[n]e^{-j(\omega-\omega_0)n} \\ &= \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}e^{j\omega_0 n} \\ &= \sum_{n=-\infty}^{\infty} e^{j\omega_0 n}g[n]e^{-j\omega n} \\ &= \mathfrak{F}\{e^{j\omega_0 n}g[n]\},\end{aligned}$$

the DTFT of $e^{j\omega_0 n}g[n]$.

c) Differentiation in frequency.

$$\begin{aligned}\frac{d}{d\omega}G(e^{j\omega}) &= \frac{d}{d\omega}\sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} (-jn)g[n]e^{-j\omega n} \\ &= -j\sum_{n=-\infty}^{\infty} ng[n]e^{-j\omega n} \\ &= -j\mathfrak{F}\{ng[n]\},\end{aligned}$$

so we have

$$\mathfrak{F}\{ng[n]\} = -\frac{1}{j}\frac{d}{d\omega}G(e^{j\omega}) = j\frac{d}{d\omega}G(e^{j\omega}).$$

12.

a) Rectangular pulse of width $M = 2N+1$, centered on 0, with height 1:

$$y[n] = \begin{cases} 1 & -N \leq n \leq N \\ 0 & \text{otherwise} \end{cases}$$

In other words,

$$y[n] = \sum_{m=-N}^{N} \delta[n-m]$$

so, using the time-shift property of the DTFT in addition to the fact that $\mathfrak{F}\{\delta[n]\} = 1$, we have (remembering that $N = (M-1)/2$)

$$\begin{aligned}y[n] \Longleftrightarrow Y(e^{j\omega}) &= \sum_{n=-N}^{N} e^{-j\omega n} \\ &= \sum_{-\frac{M-1}{2}}^{\frac{M-1}{2}} (e^{-j\omega})^n\end{aligned}$$

Using the summation formula

$$\sum_{n=M}^{N} r^n = \frac{r^{M+1} - r^N}{r-1},$$

we get

$$\begin{aligned} Y(e^{j\omega}) &= \frac{e^{-j\omega(\frac{M+1}{2})} - e^{j\omega(\frac{M-1}{2})}}{e^{-j\omega} - 1} \\ &= \frac{e^{-j\frac{\omega}{2}}}{e^{-j\frac{\omega}{2}}} \left(\frac{e^{-j\omega\frac{M}{2}} - e^{j\omega\frac{M}{2}}}{e^{-j\frac{\omega}{2}} - e^{j\frac{\omega}{2}}} \right) \\ &= \frac{\frac{1}{2j}}{\frac{1}{2j}} \frac{(e^{-j\omega\frac{M}{2}} - e^{j\omega\frac{M}{2}})}{(e^{-j\frac{\omega}{2}} - e^{j\frac{\omega}{2}})} \\ &= \frac{\sin \omega\frac{M}{2}}{\sin\frac{\omega}{2}} \end{aligned}$$

b) Triangular pulse with width $M = 2N+1$, centered on 0, height 1:

$$y[n] = \begin{cases} 1 - \frac{|n|}{N} & -N \le n \le N \\ 0 & \text{otherwise} \end{cases}$$

It is easier to evaluate the DTFT if we realize that a triangular pulse is the convolution of two rectangular pulses. Since the length of the convolution of two sequences of length N is $2N+1$, we see that $y[n] = g[n] * g[n]$ where

$$g[n] = \begin{cases} \frac{1}{N} & -\frac{(N-1)}{2} \le n \le \frac{N-1}{2} \\ 0 & otherwise. \end{cases}$$

Based on the results of part (a), the transform of $g[n]$ is

$$G(e^{j\omega}) = \frac{1}{N}\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}.$$

Therefore, by the convolution property of the DTFT, we have

$$y[n] = g[n] * g[n] \Leftrightarrow Y(e^{j\omega}) = G(e^{j\omega})G(e^{j\omega}) = \frac{1}{N^2}\frac{\sin^2\frac{\omega N}{2}}{\sin^2\frac{\omega}{2}}.$$

c) $cos\frac{\pi}{2}\frac{n}{N} = \frac{1}{2}(e^{j\frac{\pi}{2}\frac{n}{N}} + e^{-j\frac{\pi}{2}\frac{n}{N}})$, so

$$\begin{aligned} Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} \\ &= \sum_{n=-N}^{N} \frac{1}{2}(e^{j\frac{\pi}{2}\frac{n}{N}} + e^{-j\frac{\pi}{2}\frac{n}{N}})e^{-j\omega n} \\ &= \sum_{n=-N}^{N} \frac{1}{2}\left(e^{-jn(\omega - \frac{\pi}{2N})} + e^{-jn(\omega + \frac{\pi}{2N})}\right) \end{aligned}$$

19.

The DTFT of a sequence $x[n]$ is real if the sequence is even, and imaginary if the sequence is odd.

a) $x[n] = |n|, -N \leq n \leq N$ is even, so $X(e^{j\omega})$ is real.

b) $n^3 = -(-n)^3$, so $x[n]$ is odd and the DTFT is imaginary.

c)

$$x[n] = \frac{\sin \omega_c n}{\pi n} = \frac{\pi}{\omega_c} \operatorname{sinc} \omega_c n$$

, and the sinc function is even, so the DTFT of this sequence is real.

d)

$$x[n] = \begin{cases} 0 & n \text{even} \\ \frac{2}{\pi n} & n \text{odd.} \end{cases}$$

Writing the sequence out as $x[n] = \{\cdots - \frac{2}{3\pi} 0 - \frac{2}{\pi} 0 \frac{2}{\pi} 0 \frac{2}{3\pi} \ldots\}$, we see that the sequence is odd so the DTFT is imaginary.

e)

$$x[n] = \begin{cases} 0 & n = 0 \\ \frac{\cos \pi n}{n} & |n| > 0. \end{cases}$$

Since $\frac{\cos \pi(-n)}{-n} = -\frac{\cos \pi n}{n}$, the sequence is odd and the DTFT is imaginary.

43.

The DFT of a periodic-even sequence is real, and that of a periodic-odd sequence is imaginary.

a) For $x[n] = \{1, 1, 1, 0, 0, 0, 1, 1\}$, $x[\langle -n \rangle_N] = x[N - n] = \{1, 1, 1, 0, 0, 0, 1, 1\} = x[n]$, so the sequence is periodic-even and its 8-point DFT is real.

b) $x[n] = \{1, 1, 0, 0, 0, -1, -1\}$ is neither periodic-even nor periodic-odd, so the DFT has both real and imaginary parts.

c) $x[n] = \{0, 1, 1, 0, 0, 0, -1, -1\}$ is periodic-odd, so the DFT is imaginary.

d) $x[n] = \{0, 1, 1, 0, 0, 0, 1, 1\}$ is periodic-even, so the DFT is real.