

Problem 1 (LCCDE)

$y[n] - .7y[n-1] + .1y[n-2] = x[n] + .5x[n-1]$, with initial conditions $y[-1] = 0$, $y[-2] = 1$ and input $x[n] = 2^n \mu[n]$.

First, we find the complementary solution $y_c[n]$ by setting $x[n] = 0$ and $y[n] = \lambda^n$:

$$\begin{aligned}\lambda^n - .7\lambda^{n-1} + .1\lambda^{n-2} &= 0 \\ \lambda^{n-2}(\lambda^2 - .7\lambda + .1) &= 0 \\ \lambda^{n-2}(\lambda - .5)(\lambda - .2) &= 0\end{aligned}$$

The roots of the characteristic polynomial are $\lambda_1 = .5$ and $\lambda_2 = .2$, so we have

$$y_c[n] = \alpha_1\left(\frac{1}{2}\right)^n + \alpha_2\left(\frac{1}{5}\right)^n.$$

The particular solution has the form of the input $y_p[n] = \beta 2^n$, and we find β by substituting this into the original LCCDE for large n ($n \geq 1$) as follows:

$$\begin{aligned}\beta 2^n - .7\beta 2^{n-1} + .1\beta 2^{n-2} &= 2^n \mu[n] + .52^{n-1} \mu[n-1] \\ 2^{n-2}(\beta 2^2 - .7\beta 2^1 + .1\beta) &= 2^n + .52^{n-1} \\ \beta 2^2 - .7\beta 2^1 + .1\beta &= 2^2 + .5(2) \\ \beta &= \frac{5}{2.7} = 1.8519\end{aligned}$$

So the total solution is

$$y_t[n] = y_c[n] + y_p[n] = \alpha_1\left(\frac{1}{2}\right)^n + \alpha_2\left(\frac{1}{5}\right)^n + 1.8519(2^n).$$

To find the coefficients, we look at the LCCDE at small n , using the total solution for $n \geq 0$, and the initial conditions for $n < 0$. For $n = 0$:

$$\begin{aligned}y[0] - .7y[-1] + .1y[-2] &= x[0] + .5x[-1] \\ \alpha_1\left(\frac{1}{2}\right)^0 + \alpha_2\left(\frac{1}{5}\right)^0 + 1.8519(2^0) - .7(0) + .1(1) &= 1 + .5(0) \\ \alpha_1 + \alpha_2 &= -0.9150\end{aligned}$$

[Note that we used $\beta = 1.8519$ which supposedly only holds for $n > 0$. However, I tried using $\beta = 1.4815$ which holds when $n = 0$, but the final solution didn't match the filtered solution in matlab as closely...] And for $n = 1$:

$$\begin{aligned}y[1] - .7y[0] + .1y[-1] &= x[1] + .5x[0] \\ \alpha_1\left(\frac{1}{2}\right)^1 + \alpha_2\left(\frac{1}{5}\right)^1 + 1.8519(2^1) - .7\left(\alpha_1\left(\frac{1}{2}\right)^0 + \alpha_2\left(\frac{1}{5}\right)^0 + 1.8519(2^0)\right) + .1(0) &= 2 + .5(1) \\ -.2\alpha_1 - .5\alpha_2 &= -.1405\end{aligned}$$

Now we have a linear system with two unknowns, so we solve for α_1 and α_2 :

$$\begin{bmatrix} 1 & 1 \\ .2 & .5 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -.9150 \\ -.1405 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -1.0567 \\ 0.1417 \end{bmatrix}$$

Putting it all together, we have

$$y[n] = -1.0567\left(\frac{1}{2}\right)^n + .1417\left(\frac{1}{5}\right)^n + 1.815(2^n).$$

See the matlab plot to verify that this is correct.

2.60

A cascade of two stable systems is stable. If $h_1[n]$ and $h_2[n]$ are the impulse responses of two stable systems, then by definition

$$S_i = \sum_{n=-\infty}^{\infty} |h_i[n]| < \infty$$

for $i = 1, 2$. The impulse response of the cascade of these two systems is

$$h[n] = h_1[n] * h_2[n] = \sum_{k=-\infty}^{\infty} h_1[k]h_2[n-k].$$

Therefore, the absolute sum of $h[n]$ is

$$\begin{aligned} S &= \sum_{n=-\infty}^{\infty} |h[n]| \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |h_1[k]h_2[n-k]| \\ &= \sum_{k=-\infty}^{\infty} |h_1[k]| \sum_{n=-\infty}^{\infty} |h_2[n-k]| \\ &= \sum_{k=-\infty}^{\infty} |h_1[k]| \sum_{n=-\infty}^{\infty} |h_2[n]| \\ &= S_1 S_2 < \infty, \end{aligned}$$

so the system is stable.

2.71

For the linear system $y[n] + .5y[n - 1] = x[n]$, the characteristic polynomial is

$$\begin{aligned}\lambda^n + .5\lambda^{n-1} &= 0 \\ \lambda^{n-1}(\lambda + .5) &= 0\end{aligned}$$

so the characteristic root is $\lambda = -.5$, and the complementary solution is $y_c[n] = \alpha(-\frac{1}{2})^n$. For the input $x[n] = \delta[n]$, the particular solution is 0, so we find α by looking at the LCCDE at $n = 0$:

$$\begin{aligned}h[0] + .5h[-1] &= 1 \\ \alpha(-\frac{1}{2})^0 &= 1 \\ \alpha &= 1\end{aligned}$$

So the impulse response is

$$h[n] = \left(-\frac{1}{2}\right)^n.$$

2.76(a)

The periodic sequence $x[n] = \cos \frac{\pi}{M}n$ has period $2M$, and therefore the auto-correlation $r_{xx}[l]$ also has period $2M$. We find $r_{xx}[l]$ as follows:

$$r_{xx}[l] = \frac{1}{2M} \sum_{n=0}^{2M-1} \cos \left[\frac{\pi}{M}n \right] \cos \left[\frac{\pi}{M}n - \frac{\pi}{M}l \right]$$

and since $\cos(A - B) = \cos A \cos B + \sin A \sin B$, this gives

$$\begin{aligned}&= \frac{1}{2M} \sum_{n=0}^{2M-1} \cos \frac{\pi}{M}n \left[\cos \frac{\pi}{M}n \cos \frac{\pi}{M}l + \sin \frac{\pi}{M}n \sin \frac{\pi}{M}l \right] \\ &= \frac{1}{2M} \left[\cos \frac{\pi}{M}l \sum_{n=0}^{2M-1} \cos^2 \frac{\pi}{M}n + \sin \frac{\pi}{M}l \sum_{n=0}^{2M-1} \cos \frac{\pi}{M}n \sin \frac{\pi}{M}n \right]\end{aligned}$$

and using the identities $\cos^2 A = \frac{1+\cos 2A}{2}$ and $\sin A \cos B = \frac{1}{2} \sin(A - B) + \frac{1}{2} \sin(A + B)$ where $A = B$, so $\sin A \cos A = \frac{1}{2} \sin 2A$,

$$= \frac{1}{2M} \left[\cos \frac{\pi}{M}l \sum_{n=0}^{2M-1} \left(\frac{1}{2} + \frac{1}{2} \cos \frac{2\pi}{M}n \right) + \sin \frac{\pi}{M}l \sum_{n=0}^{2M-1} \frac{1}{2} \sin \frac{2\pi}{M}n \right]$$

now, $\sum_{n=0}^{2M-1} \sin \frac{2\pi}{M}n = 0$ because it's the sum of two full periods of a sinusoid. Similarly for $\sum_{n=0}^{2M-1} \cos \frac{2\pi}{M}n$, so we have

$$\begin{aligned} r_{xx}[l] &= \frac{1}{2M} \left[\cos \frac{\pi}{M}l \left[\sum_{n=0}^{2M-1} \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{2M-1} \cos \frac{2\pi}{M}n \right] \right] \\ &= \frac{1}{2M} \cos \frac{\pi}{M}l \left(\frac{1}{2} 2M \right) \\ &= \frac{1}{M} \cos \frac{\pi}{M}l. \end{aligned}$$

(b) For $x[n] = n \bmod 6$, the period is $N = 6$. The autocorrelation sequence is

$$\begin{aligned} r_{xx}[l] &= \frac{1}{6} \sum_{n=0}^5 n(n-l) \\ &= \frac{1}{6} \{55, 40, 31, 38, 31, 40\}. \end{aligned}$$

(c) For $x[n] = (-1)^n = \{-1, 1, -1, 1, \dots\}$, the period is $N = 2$. The autocorrelation sequence is

$$\begin{aligned} r_{xx}[l] &= \frac{1}{2} \sum_{n=0}^1 (-1)^n (-1)^{n-l} \\ &= \frac{1}{2} \sum_{n=0}^1 (-1)^{2n-l} \\ &= \frac{1}{2} [(-1)^{-l} + (-1)^{2-l}] \\ &= \{1, -1, 1, -1, \dots\} = (-1)^l. \end{aligned}$$