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EE 4810, DSP
HW-2

2.27 The system described by $y[n] = x[n+1] - 2x[n] + x[n-1]$ is linear, time-invariant, but not causal. To prove linearity, if $x_1[n]$ leads to the output $y_1[n]$ and $x_2[n] \to y_2[n]$, then the output to a linear combination of the inputs is

$$\begin{aligned}\alpha x_1[n] + \beta x_2[n] &\to \alpha x_1[n+1] + \beta x_2[n+1] - 2\alpha x_1[n] - 2\beta x_2[n] + \alpha x_1[n-1] + \beta x_2[n] \\ &= \alpha(x_1[n+1] + -2x_1[n] + x_1[n-1]) + \beta(x_2[n+1] - 2x_2[n]x_2[n]) \\ &= \alpha y_1[n] + \beta y_2[n].\end{aligned}$$

To show time-invariance, we note that on input $x[n-k]$, the output is

$$x[n-k] \to x[n-k+1] - 2x[n-k] + x[n-k-1] \qquad = y[n-k].$$

Finally, the system is not causal because the output at time n depends on input at time $n+1$.

2.28 The median filter in not linear. Although the median operation does not change order when multiplied by a scalar, the median of a sum of two signals is not necessarily equal to the sum of the medians of the two signals. Proof by counterexample:

$$\begin{gathered} x_1[n] = \{1,0,0\}, x_2[n] = \{0,1,2\} \\ med(x_1[n]) = 0, med(x_2[n]) = 1 \\ 3med(x_1[n]) + med(x_2[n]) = 3 \times 0 + 1 = 1 \\ 3x_1[n] + x_2[n] = \{3,0,0\} + \{0,1,2\} = \{3,1,2\} \\ med(3x_1[n] + x_2[n]) = 2 \neq 1 \end{gathered}$$

The median filter is time-invariant because

$$\begin{aligned} x[n] &\to y[n] = med(x[n-k] \ldots x[n] \ldots x[n+k]) \\ x[n-l] &\to med(x[n-l-k] \ldots x[n-l] \ldots x[n-l+k] = y[n-l] \end{aligned}$$

33(a).

$$\begin{aligned} y_1[1] &= 2 \times 2 = 4 \\ y_1[2] &= 2 \times 1 = 2 \\ y_1[3] &= -.5 \times 2 = -1 \\ y_1[4] &= -.5 \times 1 + 2 \times -3 = -6.5 \\ y_1[5] &= 2 \times 2 = 0 \\ y_1[6] &= -.5 \times -3 = 1.5 \end{aligned}$$

33(b)

$$\begin{aligned} y_2[-1] &= -.5 \\ y_2[0] &= -1 \\ y_2[1] &= 3 \\ y_2[2] &= 1.5 \\ y_2[3] &= 3 \\ y_2[4] &= -9 \end{aligned}$$

33(c)

$$\begin{aligned} y_3[2] &= -1 \\ y_3[3] &= -2 \\ y_3[4] &= 6.25 \\ y_3[5] &= .5 \\ y_3[6] &= -1.5 \end{aligned}$$

33(d)

$$\begin{aligned} y_4[-2] &= 2 \\ y_4[-1] &= 1 \\ y_4[-0] &= 0 \\ y_4[1] &= -9 \\ y_4[2] &= -3 \\ y_4[3] &= 0 \\ y_4[4] &= 9 \end{aligned}$$

40. We show that the convolution of two sequences of length N and M has length $N + M - 1$. Without loss of generality, assume that sequence $x[n]$ of length M is defined for $0 \leq n < M$, in other words $x[n] = 0$ for $n < 0, n \geq M$. Similarly, $h[n]$ is defined over $0 \leq n < N$. Furthermore, this implies that $x[0] \neq 0, x[M-1] \neq 0$, and $h[0] \neq 0, h[N-1] \neq 0$. The convolution is

$$y[n] = x[n] * h[n] = \sum_k x[k]h[n-k].$$

If $n < 0$, then $y[n] = \sum_k x[k]h[n-k] = 0$ because $x[k] = 0$ for $k < 0$, and $h[n-k] = 0$ for $k > 0$. If $n \geq N + M - 1$, then $y[n] = 0$ because $x[k] = 0$ for $k \geq M$, and $h[n-k] = 0$ for $k < M$ since $n - k > N - 1$ when $n \geq N + M - 1$ and $k < M$. Finally, we show that $y[n] \neq 0$ when $n = 0, N + M - 2$. When $n = 0$,

$$y[0] = \sum_k x[k]h[-k] = x[0]h[0] + \sum_{k=1}^{\infty} x[k]h[-k] \neq 0.$$

Similarly, when $n = N + M - 2$,

$$\begin{aligned} y[N+M-2] &= \sum_k x[k]h[N-M-2-k] \\ &= \sum_{k=-\infty}^{M-1} x[k]h[N-M-2-k] + \sum_{k=M+1}^{\infty} x[k]h[N-M-2-k] \\ &= x[M-1]h[N-M-2-(M-1)] + \sum_{k=M+1}^{\infty} x[k]h[N-M-2-k] \\ &= x[M-1]h[N-1] \neq 0 \end{aligned}$$

68. $y[n] + .5y[n-1] = 2\mu[n]$, $y[-1] = 2$. The complementary solution has the form

$$y_c[n] = \alpha_1(\lambda_1)^n.$$

We find λ_1 by looking at the roots of the characteristic polynomial

$$\begin{aligned} \lambda^n + .5\lambda^{n-1} &= 0 \\ \lambda^{n-1}(\lambda + .5) &= 0 \end{aligned}$$

so $\lambda_1 = -\frac{1}{2}$. The form of the particular solution matches the form of the input, which is simply a constant, so $y_p[n] = \beta$, and substituting into the difference equation we get

$$\begin{aligned} \beta + \frac{1}{2}\beta &= 2\mu[n] \\ \frac{3}{2}\beta &= 2 \\ \beta &= \frac{4}{3} \end{aligned}$$

To find the coefficient α_1, we use the initial conditions at $n = 0$ as follows:

$$\begin{aligned} y_c[0] + y_p[0] + .5y[-1] &= 2\mu[0] \\ \alpha_1(\lambda_1)^0 + \frac{4}{3} + .5y[-1] &= 2\mu[0] \\ \alpha_1(-\frac{1}{2})^0 + \frac{4}{3} + .5(2) &= 2 \\ \alpha_1 &= -\frac{1}{3}. \end{aligned}$$

Note that we substituted the total solution for $y[0]$, but used the initial conditions for $y[-1]$, because the total solution holds for $n \geq 0$. So we have finally

$$y[n] = -\frac{1}{3}(-\frac{1}{2})^n + \frac{4}{3}.$$