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EE 4810, DSP
HW-1

2.2(a) By introducing the variables a and b, we can write the relationships

$$\begin{aligned} a[n] &= x[n] + \alpha b[n] \\ b[n] &= a[n-1] \\ y[n] &= (\beta+\gamma)b[n]. \end{aligned}$$

Substituting, we get

$$a[n] = x[n] + \alpha a[n-1]$$

and

$$\begin{aligned} b[n] = a[n-1] &= x[n-1] + \alpha a[n-2] \\ &= x[n-1] + \alpha(x[n-2] + \alpha(x[n-3] + \alpha(\dots))) \\ &= x[n-1] + \alpha x[n-2] + \alpha^2 x[n-3] + \alpha^3 x[n-4]\dots \\ &= \sum_{i=1}^{\infty} \alpha^{i-1} x[n-i]. \end{aligned}$$

Therefore,

$$y[n] = (\beta+\gamma)\sum_{i=1}^{\infty} \alpha^{i-1} x[n-i].$$

We can get rid of the infinite sum as follows. First, we break the first terms out of the sum to get

$$y[n] = (\beta+\gamma)x[n-1] + (\beta+\gamma)\sum_{i=2}^{\infty} \alpha^{i-1} x[n-i].$$

Now we note that

$$y[n-1] = (\beta+\gamma)\sum_{i=1}^{\infty} \alpha^{i-1} x[n-(i+1)],$$

and substituting $j = i+1$,

$$\begin{aligned} y[n-1] &= (\beta+\gamma)\sum_{j=2}^{\infty} \alpha^{j-2} x[n-j] \\ \alpha y[n-1] &= (\beta+\gamma)\sum_{i=2}^{\infty} \alpha^{i-1} x[n-i] \\ y[n] &= (\beta+\gamma)x[n-1] + \alpha y[n-1]. \end{aligned}$$

Another method to develop this relation between delayed versions of the output and input is to use a table of coefficients. First, we write the input and output in terms of delayed versions of a single variable:

$$x[n] = a[n] - \alpha a[n-1]$$
$$y[n] = (\beta + \gamma) b[n]$$

Then, we create two tables where the columns represent delayed versions of the variable $a[n]$ and rows represent delayed versions of the input or output. The entries are the coefficients of the delayed versions of $a[n]$ in the linear combination that equals the corresponding delayed input or output for that row. By multiplying rows by a scalar, we look for linear combinations of the input and output that cancel out the intermediate variable, in other words where the sum of the columns of the output and input tables are equal. For example, in this problem:

	$a[n]$	$a[n-1]$	$a[n-2]$
$x[n]$	1	$-\alpha$	
$(\beta+\gamma)x[n-1]$		$(\beta+\gamma)$	$-\alpha(\beta+\gamma)$
$y[n]$		$(\beta+\gamma)$	
$-\alpha y[n-1]$			$-\alpha(\beta+\gamma)$

From this table, we see that as before,

$$y[n] - \alpha y[n-1] = (\beta + \gamma) x[n-1].$$

2.2(b)

$$y[n] = \alpha(x[n] + x[n-4]) + \beta(x[n-1] + x[n-3]) + \gamma x[n-2]$$

2.2(c) Introducing the variable $a[n]$, we have

$$x[n] = a[n] + d a[n-1]$$
$$y[n] = d a[n] + a[n-1]$$

Using the table method, we write:

	$a[n]$	$a[n-1]$	$a[n-2]$
$dx[n]$	d	d^2	
$x[n-1]$		1	d
$y[n]$	d	1	
$dy[n-1]$		d^2	d

Reading the table gives

$$y[n] + d y[n-1] = d x[n] + x[n-1].$$

2.2(d) We start with three new variables,

$$\begin{aligned} a[n] &= x[n] + b[n] \\ b[n] &= d_1 a[n-1] + d_2 c[n] \\ c[n] &= x[n] + a[n-2] \\ y[n] &= a[n-2] + b[n] \end{aligned}$$

Substituting, we can write x and y in terms of a as follows:

$$\begin{aligned} b[n] &= d_2 x[n] + d_1 a[n-1] + d_2 a[n-2] \\ a[n] &= x[n] - b[n] \\ &= (1-d_2)x[n] - d_1 a[n-1] - d_2 a[n-2] \\ (1-d_2)x[n] &= a[n] + d_1 a[n-1] + d_2 a[n-2] \end{aligned}$$

and for the output,

$$\begin{aligned} y[n] &= b[n] + a[n-2] \\ &= d_1 a[n-1] + (1+d_2)a[n-2] + d_2 x[n] \\ &= d_1 a[n-1] + (1+d_2)a[n-2] + d_2[\frac{a[n] + d_1 a[n-1] + d_2 a[n-2]}{(1-d_2)}] \\ &= \frac{d_2 a[n] + d_1 a[n-1] + a[n-2]}{(1-d_2)} \end{aligned}$$

So now we have

$$\begin{aligned} (1-d_2)x[n] &= a[n] + d_1 a[n-1] + d_2 a[n-2] \\ (1-d_2)y[n] &= d_2 a[n] + d_1 a[n-1] + a[n-2]. \end{aligned}$$

Now we use the table method. The scaling factor $(1-d_2)$ can be ignored because it scales both equations equally.

	$a[n]$	$a[n-1]$	$a[n-2]$	$a[n-3]$	$a[n-4]$
$d_2 x[n]$	d_2	$d_1 d_2$	$(d_2)^2$		
$d_1 x[n-1]$		d_1	$(d_1)^2$	$d_1 d_2$	
$x[n-2]$			1	d_1	d_2
$y[n]$	d_2	d_1	1		
$d_1 y[n-1]$		$d_1 d_2$	$(d_1)^2$	d_1	
$d_2 y[n-2]$			$(d_2)^2$	$d_1 d_2$	d_2

Reading the table gives

$$y[n] + d_1 y[n-1] + d_2 y[n-2] = d_2 x[n] + d_1 x[n-1] + x[n-2].$$

2.6(a) If $x[n] = A\alpha^n$ where $A = |A|e^{j\phi}$ and $\alpha = e^{(\sigma+j\omega)}$, then

$$\begin{aligned} x[n] &= |A|e^{j\phi}e^{(\sigma+j\omega)n} \\ &= |A|e^{\sigma n}e^{j(\omega n+\phi)}, \\ x^*[-n] &= |A|e^{\sigma(-n)}e^{-j(\omega(-n)+\phi)} \end{aligned}$$

Therefore,

$$\begin{aligned} x_{pcs}[n] &= \frac{1}{2}(x[n] + x^*[-n]) \\ &= |A|\frac{1}{2}[e^{\sigma n+j(\omega n+\phi)} + e^{\sigma(-n)-j(\omega(-n)+\phi)}] \end{aligned}$$

and

$$\begin{aligned} x_{pcs}[n] + x^*_{pcs}[n] &= |A|\frac{1}{2}[e^{\sigma n+j(\omega n+\phi)} + e^{\sigma n-j(\omega n+\phi)} + e^{\sigma(-n)+j(\omega(-n)+\phi)} + e^{\sigma(-n)-j(\omega(-n)+\phi)}] \\ &= |A|e^{\sigma n}cos(\omega n+\phi) + |A|e^{-\sigma n}cos(-\omega n+\phi) \end{aligned}$$

I'm not quite sure how to get $x_{pcs}[n]$ alone from this...

Similarly, the conjugate asymmetric part can be obtained as follows:

$$\begin{aligned} x_{pca}[n] &= \frac{1}{2}(x[n] - x^*[-n]) \\ &= |A|\frac{1}{2}[e^{\sigma n+j(\omega n+\phi)} - e^{\sigma(-n)-j(\omega(-n)+\phi)}] \end{aligned}$$

and

$$\begin{aligned} x_{pca}[n] - x^*_{pca}[n] &= |A|\frac{1}{2}[e^{\sigma n+j(\omega n+\phi)} + e^{\sigma n-j(\omega n+\phi)} - e^{\sigma(-n)+j(\omega(-n)+\phi)} - e^{\sigma(-n)-j(\omega(-n)+\phi)}] \\ &= |A|e^{\sigma n}cos(\omega n+\phi) - |A|e^{-\sigma n}cos(-\omega n+\phi). \end{aligned}$$

2.6(b)

$$\begin{aligned} h[n] &= \{-2+j5, 4-j3, 5+j6, 3+j, -7+j2\} \\ h^*[-n] &= \{-7-j2, 3-j, 5-j6, 4+j3, -2-j5\} \\ h_{pcs}[n] = \frac{1}{2}(h[n]+h^*[-n]) &= \frac{1}{2}\{-9+j3, 7-j4, 10, 7+j4, -9-j3\} \\ h_{pca}[n] = \frac{1}{2}(h[n]-h^*[-n]) &= \frac{1}{2}\{5+j7, 1-j2, 12j, -1-j2, -5+j7\} \end{aligned}$$

By inspection, we verify that the periodic conjugate symmetric part indeed follows $h_{pcs}[n] = h^*_{pcs}[-n]$, and $h_{pca}[n] = -h^*_{pca}[-n]$ holds for the asymmetric part.

2.19

We show that $\delta[n] = \mu[n] - \mu[n-1]$. From the definition $\mu[n] = \sum_{k=-\infty}^{n} \delta[k]$, we see that $\mu[n-1] = \sum_{k=-\infty}^{n-1} \delta[k]$. Therefore,

$$\begin{aligned} \mu[n] - \mu[n-1] &= \sum_{k=-\infty}^{n} \delta[k] - \sum_{k=-\infty}^{n-1} \delta[k] \\ &= \delta[n] + \sum_{k=-\infty}^{n-1} \delta[k] - \sum_{k=-\infty}^{n-1} \delta[k] \\ &= \delta[n]. \end{aligned}$$

2.21

(a) $x_1[n] = e^{-j.4\pi n}$ has fundamental frequency $.4 * \pi$. Therefore, the period is the smallest integer multiple of $\frac{2\pi}{.4\pi}$, so $N = 5$.

(b) The fundamental frequency of $\sin(.6\pi n + .6\pi)$ is $.6\pi$, and $\frac{2\pi}{.6\pi} = \frac{10}{3}$, so $N = 10$.

(c) The fundamental frequency of $2\cos(1.1\pi n - .5\pi) + 2\sin(.7\pi n)$ is the least common multiple of 1.1π and $.7\pi$, or 7.7π. In normalized frequency, this is equivalent to $-.3\pi$, and and $\frac{2\pi}{-.3\pi} = \frac{-20}{3}$, so $N = 20$.

(d) The fundamental frequency of $3\sin(1.3\pi n) - 4\cos(.3\pi n + .45\pi)$ is the least common multiple of 1.3π and $.3\pi$, or 3.9π. In normalized frequency, this is equivalent to $-.1\pi$, and and $\frac{2\pi}{-.1\pi} = -20$, so $N = 20$.

(e) The fundamental frequency of $5\sin(1.2\pi n + .65\pi) + 4\sin(.8\pi n) - \cos(.8\pi n)$ is the least common multiple of 1.2π and $.8\pi$, or 2.4π (the second two terms have the same frequency, so can be treated as a single sinusoid). In normalized frequency, this is equivalent to $.4\pi$, and and $\frac{2\pi}{.4\pi} = 5$, so $N = 5$.

(f) The period of $x_6[n] = n \mod 6$ is $N = 6$.

2.23

$x[n] = cos\Omega_0 nT$ is periodic when $\frac{2\pi}{\Omega_0 T}$ is rational, i.e.

$$\frac{2\pi}{\Omega_0 T} = \frac{N}{r}$$

or

$$T = (\frac{2\pi}{\Omega_0})q$$

where q is any rational number.

If $\Omega_0 = 18$ and $T = \frac{\pi}{6}$, then

$$\frac{2\pi}{18\pi/6} = \frac{2}{3},$$

so $N = 2$.