Problem Set #7 Solutions

1. a. The outage probability at the output of a maximal-ratio combining receiver can be derived as follows:

$$\begin{aligned}\mathbb{P}[out] &= \mathbb{P}\left\{\frac{E_{b,inst}}{N_0} < \frac{E_{b,req}}{N_0}\right\} \\ &= \mathbb{P}\left\{\frac{xE_b}{N_0} < \frac{E_{b,req}}{N_0}\right\} \\ &= \mathbb{P}\left\{x < \frac{E_{b,req}/N_0}{E_b/N_0}\right\},\end{aligned}$$

if we denote $\frac{E_{b,req}/N_0}{E_b/N_0} \triangleq K$, then

$$\begin{aligned}\mathbb{P}[out] &= \int_0^K \frac{1}{(2\delta^2)^2} x e^{-x/2\delta^2} dx \\ &= 1 - \left(1 + \frac{K}{2\delta^2}\right) e^{-k/2\delta^2},\end{aligned}$$

and recall $\bar{E}_b = 2\delta^2 E_b$, we obtain

$$\mathbb{P}[out] = 1 - \left[1 + \frac{E_{b,req}/N_0}{\bar{E}_b/N_0}\right] e^{-\frac{E_{b,req}/N_0}{\bar{E}_b/N_0}}.$$

b. Since $E_{total} = xE_b$, its average is

$$\begin{aligned}\int_0^\infty xE_b f(x)dx &= \int_0^\infty xE_b \frac{1}{(2\delta^2)^2} x e^{-x/2\delta^2} dx \\ &= 4\delta^2 E_b,\end{aligned}$$

so $\bar{E}_{b,t} = 2\bar{E}_b$. By plugging this into the result we got in [a], we get

$$\mathbb{P}[out] = 1 - \left[1 + \frac{E_{b,req}/N_0}{\bar{E}_{b,t}/2N_0}\right] e^{-\frac{E_{b,req}/N_0}{\bar{E}_{b,t}/2N_0}}.$$

c. Desired instantaneous $\mathbb{P}_b[\varepsilon] = 10^{-5}$, then $E_{b,req}/N_0 = 9.6db$. By

$$\frac{1 - \mathbb{P}[out]}{1 + \frac{E_{b,req}/N_0}{\bar{E}_b/N_0}} = e^{-\frac{E_{b,req}/N_0}{\bar{E}_b/N_0}}.$$

By using approximation $\log(1+x) \approx x - x^2/2$, we obtain

$$\frac{E_{b,req}/N_0}{\bar{E}_{b,t}/2N_0} \approx \sqrt{-2\log(1 - \mathbb{P}[out])}. \tag{1}$$

- For $\mathbb{P}[out] = 10^{-3}$, by using (1), we obtain $\bar{E}_b/N_0 = 23.1db$, compared with SISO 39.6db, we get 17.5db less. similarly, for (b), we get $\bar{E}_{b,t}/N_0 = 26.1db$, 14.5db less.
- For $\mathbb{P}[out] = 10^{-1}$, $\bar{E}_b/N_0 = 12.98db$, and $\bar{E}_{b,t}/N_0 = 16db$, compared with SISO $\bar{E}_B/N_0 = 19.3db$.

2. For $x_i = r_i^2$, its density probability is

$$f_{x_i}(x_i) = \frac{1}{2\delta^2} e^{-x_i/2\delta^2}.$$

Let's prove it using induction method. First, when $L = 2$, we have

$$\begin{aligned} f_x(x) &= f_{x_1}(x_1) * f_{x_2}(x_2) \\ &= \int_0^x \frac{1}{(2\delta^2)^2} e^{-x_1/2\delta^2} e^{-(x-x_1)/2\delta^2} dx \\ &= \frac{1}{(2\delta^2)^2} e^{-x/2\delta^2} \int_0^x dx_1 \\ &= \frac{1}{(2\delta^2)^2} x e^{-x/2\delta^2}. \end{aligned}$$

Suppose the for L antennas,

$$f(x) = \frac{1}{(L-1)!(2\delta^2)^L} x^{L-1} e^{-x/2\delta^2},$$

then for $L+1$ antennas, the probability density function can be calculated as

$$\begin{aligned} f(x) &= \int_0^x \frac{1}{(2\delta^2)} e^{-x_1/2\delta^2} \frac{1}{(L-1)!(2\delta^2)^L} (x-x_1)^{L-1} e^{-(x-x_1)/2\delta^2} dx_1 \\ &= \frac{1}{L!(2\delta^2)^{L+1}} x^L e^{-x/2\delta^2}, \end{aligned}$$

thus, the equation holds for all L.