Problem Set #4 Solutions

1. a. Proof: because of the symmetry of the constellation, we can just consider the first quadrant with $M/4$ points. The total energy is

$$E_t = \sum_{k=0}^{\frac{\sqrt{M}}{2}-1} \left[\frac{\sqrt{M}}{2} \left(\frac{(2k+1)d_{min}}{2} \right)^2 + \sum_{i=0}^{\frac{\sqrt{M}}{2}-1} \left(\frac{(2i+1)d_{min}}{2} \right)^2 \right]$$

$$= \frac{d_{min}^2 \sqrt{M}}{4} \sum_{k=0}^{\frac{\sqrt{M}}{2}-1} (2k+1)^2$$

$$= \frac{d_{min}^2 \sqrt{M}}{4} \frac{M\sqrt{M} - \sqrt{M}}{6},$$

thus,

$$E_s = \frac{E_t}{M/4} = \frac{(M-1)d_{min}^2}{6}.$$

b. The peak energy is

$$E_{peak} = 2(\sqrt{M}-1)^2 d_{min}^2/4 = \frac{d_{min}^2(M - 2\sqrt{M} + 1)}{2},$$

thus the peak-to-average energy ratio of a general square QAM constellation is

$$\frac{E_{peak}}{E_s} = \frac{3(M - 2\sqrt{M} + 1)}{M-1}.$$

c.

$$\frac{E_{peak}}{E_s} = \lim_{M\to\infty} = 3.$$

d. For internal symbols $\Pr_s\{\varepsilon\} = 2Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$, for the edge symbols, $\Pr_s\{\varepsilon\} = Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$.

2. a. For the first ring, $E_{s1} = \left(\frac{d_{min}}{2sin\frac{\pi}{8}}\right)^2 = 2cos^2\frac{\pi}{8}d_{min}^2$, and $E_{s2} = \left(\sqrt{2}d_{min}cos^2\frac{\pi}{8} + \frac{\sqrt{3}d_{min}}{2}\right)^2$, thus the average energy per signal is

$$E_s = \frac{1}{2}(E_{s1} + E_{s2}) = \frac{d_{min}^2}{2}\left(2cos^4\frac{\pi}{8} + \frac{3}{4} + \sqrt{6}cos^2\frac{\pi}{8} + 2cos^2\frac{\pi}{8}\right) = 3d_{min}^2.$$

This one requires more energy than square 16 QAM, which is only $2.5d_{min}^2$.

b. The peak-to-average energy ratio for square one is 1.8, and this one is $(2cos^4(\pi/8) + \sqrt{6}cos^2(\pi/8) + 0.75)/3 = 1.4326$, so this one has smaller peak-to-average ratio.

3. Check your course notes.

4. a. Because $Pr_s\{\varepsilon\} \leq (M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right)$, for $M=2$, $E_b/N_0 = 18.06$, which is 12.57db; for $M=4$, $E_b/N_0 = 10.125$, which is 10.05db; for $M=8$, $E_b/N_0 = 7.3$, which is 8.634db; for $M=16$, $E_b/N_0 = 5.835$, which is 7.66db; for $M=64$, $E_b/N_0 = 4.35$, which is 6.39db; for $M=256$, $E_b/N_0 = 3.6$, which is 5.568db

   b. Proof:

$$\begin{aligned} Pr_b\{\varepsilon\} &= \frac{1}{M-1}\left\{\frac{1}{n}\binom{n}{1}Pr_s\{\varepsilon\} + \frac{2}{n}\binom{n}{2}Pr_s\{\varepsilon\} + \cdots \frac{n}{n}\binom{n}{n}Pr_s\{\varepsilon\}\right\} \\ &= \frac{1}{M-1}\{\frac{1}{n}\sum_{k=1}^{n} k\binom{n}{k}\}Pr_s\{\varepsilon\} \\ &= \frac{M/2}{M-1}Pr_s\{\varepsilon\}. \end{aligned}$$

   c. For large M, the ratio goes to 1/2.