Problem Set #2 Solutions

1 Remark: We work under the condition $\Phi_1(t) = X_1(t)/\sqrt{E_b}$, since $||\Phi(t)|| = 1$. (It is a minor typo in the problem that $\Phi_1(t) = X_1(t)/\sqrt{E}$).

a)

$$\begin{aligned}\Phi_1(t) &= \frac{X_1(t)}{\sqrt{E_b}} \\ &= \sqrt{\frac{1}{T}}, 0 \le t \le T.\end{aligned}$$

By Gram-Schmidt procedure, we can compute

$$c_{21} =< \Phi_1(t), X_2(t) >= \sqrt{\frac{E_b}{2}},$$

and therefore,

$$\begin{aligned}\Phi_2(t) &= \frac{X_2(t) - c_{21}\Phi_1(t)}{||X_2(t) - c_{21}\Phi_1(t)||} \\ &= \begin{cases} \sqrt{\frac{1}{T}}, & 0 \le t \le T/2 \\ -\sqrt{\frac{1}{T}}, & T/2 \le t \le T. \end{cases}\end{aligned}$$

It is easy to compute

$$\rho = \frac{1}{E_b}\int_0^{T/2} \frac{E_b\sqrt{2}}{T} = \frac{\sqrt{2}}{2}.$$

b) Note that $X_1 = \sqrt{E_b}\Phi(t)$, and it is also easy to compute

$$X_2(t) = \sqrt{\frac{E_b}{2}}\Phi_1(t) + \sqrt{\frac{E_b}{2}}\Phi_2(t),$$

therefore, $b_1 = b_2 = \rho\sqrt{E_b}$.

2 Assume that $X(t) = \sum_{i=1}^{\infty} a_i\Phi_i(t)$, $Y(t) = \sum_{i=1}^{\infty} b_i\Phi_i(t)$ and $a = (a_1, a_2, \cdots)$, $b = (b_1, b_2, \cdots)$. We obtain, by using the orthonormality of $\{\Phi_i\}$,

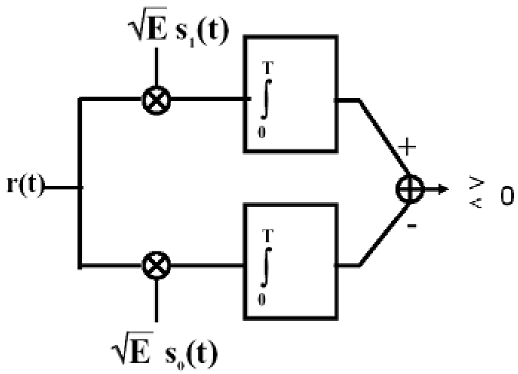
$$\begin{aligned}d_{X,Y}^2 &= \int_0^T (X(t) - Y(t))^2 dt \\ &= \int_0^T \left[\sum_{i=1}^{\infty} a_i\Phi_i(t) - \sum_{i=1}^{\infty} b_i\Phi_i(t)\right]^2 dt \\ &= \sum_{i=1}^{\infty}(a_i - b_i)^2 \int_0^T \Phi_i^2(t)dt \\ &= |a - b|^2.\end{aligned}$$

3 Using the fact that s_1 and s_0 are two signals with equal unit energy, it is easy to see that

$$\int_0^T [r(t) - \sqrt{E_1}S_1(t)]^2 dt < \int_0^T [r(t) - \sqrt{E_0}S_0(t)]^2 dt$$
$$\Longleftrightarrow \int_0^T [r^2(t) - 2r(t)\sqrt{E_1}S_1(t) + E_1 S_1^2(t)]dt < \int_0^T [r^2(t) - 2r(t)\sqrt{E_0}S_0(t) + E_0 S_0^2(t)]dt$$
$$\Longleftrightarrow \frac{E_1 - E_0}{2} < \int_0^T \sqrt{E_1} r(t) S_1(t) dt - \int_0^T \sqrt{E_0} r(t) S_0(t) dt,$$

which is the output of the correlation receiver.

4 a) It is easy to check that s_1 and s_0 are two orthonormal basis functions. The optimum two-branch correlation receiver is



b) Suppose that the received signal is $r(t) = X(t) + n(t)$ where $X(t)$ is one of the two signals shown in the problem and $n(t)$ is the white noise. If $X(t)$ is equal to $\sqrt{E}s_1$, then the output of the correlation receiver is

$$\begin{aligned}
&\int_0^T [r(t) + n(t)]\sqrt{E}s_1(t)dt - \int_0^T [r(t) + n(t)]\sqrt{E}s_0(t)dt \\
&= E + \int_0^T \sqrt{E}n(t)s_1(t)dt - \int_0^T \sqrt{E}n(t)s_0(t)dt \\
&\overset{define}{=} E + S_1 - S_0,
\end{aligned}$$

where S_1 and S_0 are two independent Gaussian random variables (please check that the correlation of S_1 and S_0 is equal to zero by recalling that s_1 and s_0 are two orthonormal basis functions) satisfying

$$\begin{aligned}
\mathbb{V}ar(S_1) = \mathbb{V}ar(S_0) &= \mathbb{E}\int_0^T \int_0^T E n(u)s_1(u)n(v)s_1(v)dudv \\
&= E\int_0^T \int_0^T \mathbb{E}[n(u)n(v)]s_1(u)s_1(v)dudv \\
&= E\int_0^T \int_0^T \frac{N_0}{2}\delta(u-v)s_1(u)s_1(v)dudv \\
&= \frac{EN_0}{2},
\end{aligned}$$

therefore, by noting that $S_1 - S_0$ is a Gaussian random variable with variance $\frac{EN_0}{2} + \frac{EN_0}{2} = EN_0$, we can compute the conditional probability

$$\mathbb{P}[error|X(t) = s_1(t)] = \mathbb{P}[E + S_1 - S_0 > 0] = Q\left(\sqrt{\frac{E}{N_0}}\right).$$

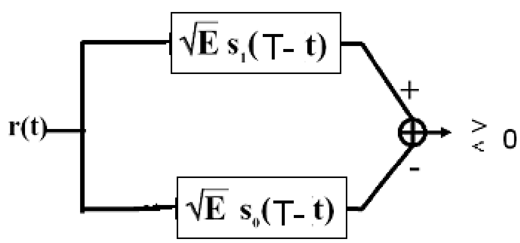
Using a similar argument, we can calculate

$$\mathbb{P}[error|X(t) = s_0(t)] = Q\left(\sqrt{\frac{E}{N_0}}\right),$$

which implies

$$\begin{aligned}\mathbb{P}[error] &= \mathbb{P}[X(t) = s_0(t)]\mathbb{P}[error|X(t) = s_0(t)] + \mathbb{P}[X(t) = s_1(t)]\mathbb{P}[error|X(t) = s_1(t)] \\ &= Q\left(\sqrt{\frac{E}{N_0}}\right).\end{aligned}$$

c) The two-branch matched filter for this modulation technique is shown below. Sample at T.



d) Using the same argument as in b), we know that the error probability is $Q\left(\sqrt{\frac{E}{N_0}}\right)$.