Problem Set #1 Solutions

1a) Rayleigh probability density function:

$$f(r) = \frac{r}{\delta^2} e^{-r^2/2\delta^2}; r \geq 0,$$

(a) Average value:

$$E(r) = \int_0^\infty r f(r) dr = \int_0^\infty r \cdot \frac{r}{\delta^2} e^{-r^2/2\delta^2} \cdot dr,$$

Let $x = r^2/2\delta^2$, we get

$$\begin{aligned} E(r) &= \int_0^\infty 2x \cdot e^{-x} \frac{\delta^2}{r} \cdot dx \\ &= \int_0^\infty 2x \cdot e^{-x} \frac{\delta^2}{\sqrt{2\delta^2 x}} \cdot dx \\ &= -\sqrt{2} \cdot \delta \sqrt{x} \cdot e^{-x} + \int_0^\infty e^{-x} \sqrt{2} \delta d\sqrt{x} \\ &= \int_0^\infty e^{-r^2/2\delta^2} \cdot dr = \sqrt{\frac{\pi}{2}} \delta. \end{aligned}$$

(b) Average of the square of the random variable

$$E(r^2) = \int_0^\infty r^2 \cdot f(r) \cdot dr = \int_0^\infty 2\delta^2 \cdot x \cdot e^{-x} dx = 2\delta^2.$$

(c) Variance is

$$E(r^2) - E(r)^2 = 2\delta^2 - \frac{\pi}{2}\delta^2 = \delta^2 (2 - \frac{\pi}{2}).$$

(d) Cumulative PDF is

$$\int_0^r f(r) dr = \int_0^r \frac{r}{\delta^2} e^{-r^2/2\delta^2} \cdot dr = \int_0^r -e^{-r^2/2\delta^2} \cdot d - r^2/2\delta^2 = 1 - e^{-r^2/2\delta^2}.$$

Rician probability density function:

$$f(r) = \frac{r}{\delta^2} e^{-r^2+A^2/2\delta^2} \cdot I_0 \left[r(A/\delta^2) \right]; r \geq 0$$

$$I_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x \cos\theta} d\theta$$

(a) Average value:

$$E(r) = \int_{-\infty}^\infty r f(r) dr = \delta \sqrt{\frac{\pi}{2}} L_{1/2}(\frac{-A^2}{2\delta^2}),$$

where

$$L_{1/2}(x) = e^{x/2} \left[(1-x) I_0 \left(\frac{-x}{2} \right) - x I_1 \left(\frac{-x}{2} \right) \right].$$

(b)
$$E(r^2) = \int_0^\infty r^2 f(r) dr = 2\delta^2 + A^2.$$

(c) Variance
$$E(r^2) - (E(r))^2 = 2\delta^2 + A^2 - \frac{\pi\delta^2}{2} L_{1/2}^2 (\frac{-A^2}{2\delta^2}).$$

(d) Cumulative PDF is
$$1 - Q_1(\frac{A}{\delta}, \frac{x}{\delta}),$$
where Q_1 is the Marcum Q-Function.

1b) (a) Cumulative distribution function for $x = r^2$
$$F(x) = \Pr[X \le x] = \Pr[r \le \sqrt{x}] = 1 - e^{-r^2/2\delta^2} = 1 - e^{-x/2\delta^2}.$$

(b) The probability density function
$$f(x) = \frac{dF(x)}{dx} = \frac{e^{-x/2\delta^2}}{2\delta^2}.$$