Proportional Growth, Modulated Branching Processes and Queueing Duality (II)

E6083: lecture 3

Prof. Predrag R. Jelenković

Dept. of Electrical Engineering
Columbia University , NY 10027, USA
{predrag}@ee.columbia.edu

January 24, 2007

Outline

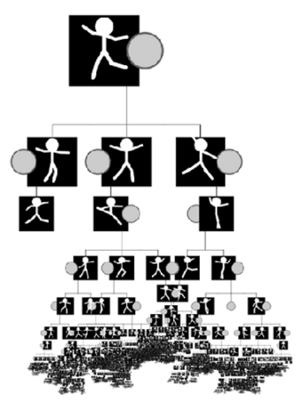
Outline

1. Main Results
   - Definition of RMBP
   - Logarithmic Asymptotics of RMBP
   - Exact Asymptotics of RMBP
2. Modulated Branching Processes with Absorbing Barriers
   - Hotspot Traffic
3. Related Phenomena and Models
   - Double Pareto
   - Truncated Power Law and Randomly Stopped Processes
4. Concluding Remarks

Proportional growth is everywhere

1. Branching process
2. Modulated by environmental parameters
3. Often repelled from a small value

City population



Reflected Modulated Branching Processes (RMBP)

- $\{J_n \in \mathbb{N}\}_{n>-\infty}$ models the environment dynamics.
- $\{B_n^i(j) \in \mathbb{N}\}$ is the number of children of the object i at time n when the environment is in state j.
- l is the lower barrier.

Modulated Branching Processes

For $Z_0 \in \mathbb{N}$, define

$$Z_{n+1} = \sum_{i=1}^{Z_n} B_n^i(J_n).$$

Reflected Modulated Branching Processes (RMBP)

For $l, \Lambda_0 \in \mathbb{N}$,

$$\Lambda_{n+1} = \max\left(\sum_{i=1}^{\Lambda_n} B_n^i(J_n),\ l\right).$$

Reflected Modulated Branching Processes (RMBP)

- $\{J_n \in \mathbb{N}\}_{n>-\infty}$ models the environment dynamics.
- $\{B_n^i(j) \in \mathbb{N}\}$ is the number of children of the object i at time n when the environment is in state j.
- l is the lower barrier.

Modulated Branching Processes

For $Z_0 \in \mathbb{N}$, define

$$Z_{n+1} = \sum_{i=1}^{Z_n} B_n^i(J_n).$$

Reflected Modulated Branching Processes (RMBP)

For $l, \Lambda_0 \in \mathbb{N}$,

$$\Lambda_{n+1} = \max\left(\sum_{i=1}^{\Lambda_n} B_n^i(J_n),\ l \right).$$

Outline

Before stating the Results

Notation

$\{J_n\}$ - modulating process;
$\mu(j) \triangleq \mathbb{E}[B_n^i(j)]$ - replication rate when in state j;
$\Pi_n = \prod_{i=-n}^{-1} \mu(J_i),\ n \geq 1,\ \Pi_0 = I$ and $M = \sup_{n\geq 0} \Pi_n$.

Polynomial Gärtner-Ellis conditions

1. $n^{-1} \log \mathbb{E}[(\Pi_n)^\alpha] \to \Psi(\alpha)$ as $n \to \infty$ for $|\ \alpha - \alpha^*\ | < \varepsilon^*$,
2. Ψ is finite in a neighborhood of α^* and differentiable at α^* with $\Psi(\alpha^*) = 0,\ \Psi'(\alpha^*) > 0$.

- When $\{J_n\}$ is i.i.d., $n^{-1} \log \mathbb{E}[(\Pi_n)^\alpha] = \Psi(\alpha) = \log \mathbb{E}[\mu(J_{-1})^\alpha]$.
- Condition 1) allows some dependency along $\{J_n\}$, e.g. functions of Markov chain, but not too much $\cdots$.

Before stating the Results

Notation

$\{J_n\}$ - modulating process;
$\mu(j) \triangleq \mathbb{E}[B^i_n(j)]$ - replication rate when in state j;
$\Pi_n = \prod_{i=-n}^{-1} \mu(J_i),\ n \geq 1,\ \Pi_0 = I$ and $M = \sup_{n\geq 0} \Pi_n$.

Polynomial Gärtner-Ellis conditions

1. $n^{-1} \log \mathbb{E}[(\Pi_n)^\alpha] \to \Psi(\alpha)$ as $n \to \infty$ for $|\ \alpha - \alpha^*\ | < \varepsilon^*$,
2. Ψ is finite in a neighborhood of α^* and differentiable at α^* with $\Psi(\alpha^*) = 0$, $\Psi'(\alpha^*) > 0$.

- When $\{J_n\}$ is i.i.d., $n^{-1} \log \mathbb{E}[(\Pi_n)^\alpha] = \Psi(\alpha) = \log \mathbb{E}[\mu(J_{-1})^\alpha]$.
- Condition 1) allows some dependency along $\{J_n\}$, e.g. functions of Markov chain, but not too much $\cdots$.

Before stating the Results

Notation

$\{J_n\}$ - modulating process;
$\mu(j) \triangleq \mathbb{E}[B_n^i(j)]$ - replication rate when in state j;
$\Pi_n = \prod_{i=-n}^{-1} \mu(J_i),\ n \geq 1,\ \Pi_0 = I$ and $M = \sup_{n\geq 0} \Pi_n$.

Polynomial Gärtner-Ellis conditions

1. $n^{-1} \log \mathbb{E}[(\Pi_n)^\alpha] \to \Psi(\alpha)$ as $n \to \infty$ for $\mid \alpha - \alpha^* \mid < \varepsilon^*$,
2. Ψ is finite in a neighborhood of α^* and differentiable at α^* with $\Psi(\alpha^*) = 0$, $\Psi'(\alpha^*) > 0$.

- When $\{J_n\}$ is i.i.d., $n^{-1} \log \mathbb{E}[(\Pi_n)^\alpha] = \Psi(\alpha) = \log \mathbb{E}[\mu(J_{-1})^\alpha]$.
- Condition 1) allows some dependency along $\{J_n\}$, e.g. functions of Markov chain, but not too much $\cdots$.

Main Results

$\bar{B}_n^i \triangleq \sup_k B_n^i(k)$, $M = \sup_{n \geq 0} \Pi_n$.

Theorem

- *If* $\{\Pi_n\}$ *satisfies the polynomial Gärtner-Ellis conditions, and* $\mathbb{E}(\Pi_n)^{\alpha^*+\varepsilon} < \infty$, $\mathbb{E}[e^{\theta \bar{B}_n^i}] < \infty$ $(\varepsilon, \theta > 0, n \geq 1)$*, then,*

$$\lim_{x \to \infty} \frac{\log \mathbb{P}[\Lambda > x]}{\log x} = \lim_{x \to \infty} \frac{\log \mathbb{P}[M > x]}{\log x} = -\alpha^*.$$

- *If* $\sup_j \mu(j) < 1$ *and* $\mathbb{E}[e^{\theta \bar{B}_n^i}] < \infty$ $(\theta > 0)$*, then,* $\mathbb{P}[\Lambda > x] = o(e^{-\xi x})$ *for some* $\xi > 0$*, implying*

$$\overline{\lim_{x \to \infty}} \frac{\log \mathbb{P}[\Lambda > x]}{\log x} = -\infty.$$

What can we learn from the theorem?

What causes power laws? (expansions and contractions...)

Logarithmically asymptotic equivalence between the tails of M and Λ.

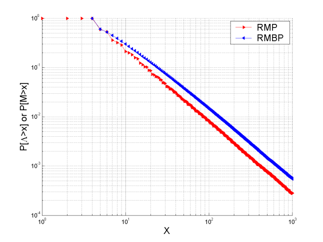
Reflected Multiplicative Process (RMP) $\to$ M

Define for $n \geq 0$ and $M_0 < \infty$

$$M_{n+1} = \max(M_n \cdot \mu(J_n), I),$$

If $\mathbb{E} \log J_n < 0$, then $M_n \xrightarrow{d} M$, $M = \sup_{n \geq 0} \prod_{i=-n}^{-1} \mu(J_i)$.

Compare RMBP & RMP



What can we learn from the theorem?

What causes power laws? (expansions and contractions...)

Logarithmically asymptotic equivalence between the tails of M and Λ.

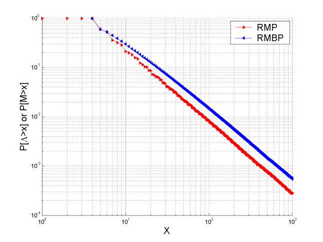
Reflected Multiplicative Process (RMP) $\to$ M

Define for $n \geq 0$ and $M_0 < \infty$

$$M_{n+1} = \max(M_n \cdot \mu(J_n), I),$$

If $\mathbb{E} \log J_n < 0$, then $M_n \xrightarrow{d} M$, $M = \sup_{n \geq 0} \prod_{i=-n}^{-1} \mu(J_i)$.

Compare RMBP & RMP



What can we learn from the theorem?

What causes power laws? (expansions and contractions...)

Logarithmically asymptotic equivalence between the tails of M and Λ.

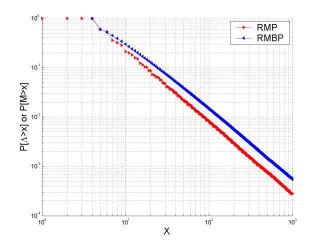
Reflected Multiplicative Process (RMP) $\rightarrow$ M

Define for $n \geq 0$ and $M_0 < \infty$

$$M_{n+1} = \max(M_n \cdot \mu(J_n), I),$$

If $\mathbb{E} \log J_n < 0$, then $M_n \xrightarrow{d} M$,
$M = \sup_{n \geq 0} \prod_{i=-n}^{-1} \mu(J_i)$.

Compare RMBP & RMP



Sketch of the proof

General points

- Based on sample path arguments.
- Representation lemma:

$$\Lambda_n \stackrel{d}{=} \max_{0 \le i \le n} Z_{-i} \to \max_{i \ge 0} Z_{-i} \stackrel{d}{=} \Lambda.$$

- Identify the critical time scale within which $\max_{n \ge 0} Z_{-n}$ reaches a big value. For all $\beta > 0$,

$$\sum_{n > x}^{\infty} \mathbb{P}[Z_n^l > x] = o\left(\frac{1}{x^{\beta}}\right)$$

$$\Rightarrow \mathbb{P}[\Lambda > x] \sim \mathbb{P}[\Lambda_{\lfloor x \rfloor} > x].$$

Sketch of the proof

General points

- Based on sample path arguments.
- Representation lemma:

$$\Lambda_n \stackrel{d}{=} \max_{0 \le i \le n} Z_{-i} \to \max_{i \ge 0} Z_{-i} \stackrel{d}{=} \Lambda.$$

- Identify the critical time scale within which $\max_{n \ge 0} Z_{-n}$ reaches a big value. For all $\beta > 0$,

$$\sum_{n>x}^{\infty} \mathbb{P}[Z_n^l > x] = o\left(\frac{1}{x^\beta}\right)$$

$$\Rightarrow \mathbb{P}[\Lambda > x] \sim \mathbb{P}[\Lambda_{\lfloor x \rfloor} > x].$$

Sketch of the proof

General points

- Based on sample path arguments.
- Representation lemma:

$$\Lambda_n \stackrel{d}{=} \max_{0\le i\le n} Z_{-i} \to \max_{i\ge 0} Z_{-i} \stackrel{d}{=} \Lambda.$$

- Identify the critical time scale within which $\max_{n\ge 0} Z_{-n}$ reaches a big value. For all $\beta > 0$,

$$\sum_{n>x}^{\infty} \mathbb{P}[Z_n^l > x] = o\left(\frac{1}{x^\beta}\right)$$

$$\Rightarrow \mathbb{P}[\Lambda > x] \sim \mathbb{P}[\Lambda_{\lfloor x \rfloor} > x].$$

Sketch of the proof for the upper bound

Increase the lower barrier to get an upper bound.

Choose $l_x = \lfloor x^\epsilon \rfloor \geq l$, then, $\mathbb{P}[\Lambda^{l_x} > x] \geq \mathbb{P}[\Lambda^l > x]$, and

$$\mathbb{P}[\Lambda^l > x] = \mathbb{P}\left[\sup_{j\geq 1} Z^l_{-j} > x\right] \leq \mathbb{P}\left[\Lambda^{l_x}_{\lfloor x \rfloor} > x\right] + \sum_{j>x} \mathbb{P}[Z^l_j > x]$$

$\curvearrowleft$ now, l_x large $\Rightarrow Z^{l_x}_i \approx \Pi^{l_x}_i$

$$\leq \mathbb{P}\left[\sup_{j\geq 1} \Pi_j (1+\epsilon)^j > x^{1-\epsilon}\right] + x\mathbb{P}[\mathcal{B}^{l_x,\epsilon}_1] + \sum_{j>x} \mathbb{P}[Z^l_j > x],$$

where $\mathcal{B}^{l_x,\epsilon}_1 = \bigcup_{j\geq l_x} \{ \sum_{i=1}^j B^i_1(J_1) > j\mu(J_1)(1+\epsilon) \}$, $\Pi_j = \prod_{i=-1}^{-j} \mu(J_i)$.

Sketch of the proof for the lower bound

How to increase the lower barrier but still obtain a lower bound?
Given $\{J_n\}_{n\geq 0}$, if $\{\Lambda_n^{y_1}\}$ and $\{\Lambda_n^{y_2}\}$ are conditional independent, then,

$$\Lambda_n^{y_1+y_2} \overset{d}{\leq} \Lambda_n^{y_1} + \Lambda_n^{y_2}. \qquad (1)$$

$\mathbb{P}[\Lambda_n^l > x] \geq \mathbb{P}[\Lambda_n^1 > x] = \frac{y\cdot\mathbb{P}[\Lambda_n^1 > x]}{y} \geq \frac{\mathbb{P}[\sum_{j=1}^{y}\Lambda_{n,j}^1 > y\cdot x]}{y} \geq \frac{\mathbb{P}[\Lambda_n^y > yx]}{y}$.
Choose $y = \lfloor x^\delta \rfloor$, $n = \lfloor x \rfloor$. Then, use similar arguments for the upper bound $\cdots$
Many details are in the paper.

Complete proof can be found here

Predrag R. Jelenković and Jian Tan
Proportional growth, queueing duality and heavy-tails.
Technique report (in preparation).

Predrag R. Jelenković and Jian Tan
Modulated branching processes and power-laws.
In *Proceedings of The Forty-Fourth Annual Allerton Conference on Communication, Control, and Computing*, September 2006.

Outline

Exact Asymptotics of RMBP

Difficult! In the scaling region, barrier grows very slowly $\cdots$

$\{J_n : n \geq 1\}$ i.i.d; $\{S_n = \sum_{i=1}^{n} \log J_i : n \geq 1\}$ is nonlattice with the ladder height distribution G_+; $\|G_+\| = \mathbb{P}[S_n \leq 0 \text{ for all } n \geq 1] < 1$.

Theorem

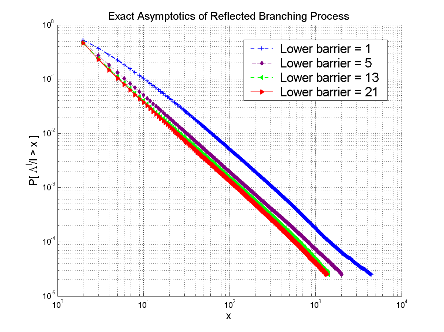
If there exits $\epsilon > 0$ such that $\mathbb{E}[\mu(J_1)^{\alpha^*}] = 1$, $\mathbb{E}\left[\mu(J_1)^{\alpha}\right] < \infty$ for $\alpha^* - \epsilon < \alpha < \alpha^* + \epsilon$, $\mathbb{E}[\log \mu(J_1)] < 0$, and $\mathbb{E}\left[e^{\theta \sup_k \{|B_1^1(k) - \mu(k)|\}}\right] < \infty$ in a neighborhood of the origin, then for any $\gamma > 0$,

$$\lim_{\substack{l \geq (\log x)^{3+\gamma} \\ x \to \infty}} \mathbb{P}[\Lambda^l / l > x] x^{\alpha^*} = \frac{1 - \|G_+\|}{\alpha^* \int_0^\infty u e^{\alpha^* u} G_+(du)}.$$

Exact Asymptotics of RMBP

Difficult! but, in the scaling region, when barrier grows very slowly . . .

$\{J_n : n \geq 1\}$ i.i.d.
barrier $l \geq (\log x)^{3+\epsilon}$



Outline

Modulated BP with an Absorbing Barrier

New Model of Hotspot Visitors

Hotspot dynamics for Web sites.
A tells B, C to visit the web, and later B may tell D.

Model

- $\{J_n\}$ - i.i.d. modulating process; A_t - triggering events.
- Define stopping time $P \triangleq \inf\{n > 0 : Z_n^l \leq l\}$ for l; after P the process is killed/absorbed.
- At time t, Poisson A_t $\#$ of objects are created, and each evolves according to an i.i.d. copy of J_P.

Modulated BP with an Absorbing Barrier

New Model of Hotspot Visitors

Hotspot dynamics for Web sites.
A tells B, C to visit the web, and later B may tell D.

Theorem

Assume that $\mathbb{E}[J_n^\alpha] < \infty$, $\alpha - \epsilon < \alpha^* < \alpha + \epsilon$ *and* $\mathbb{E}[e^{\theta \bar{B}_n^i}] < \infty$ *for some* $\theta > 0$, *then,*

$$\lim_{x \to \infty} \frac{\log \mathbb{P}[Z_s > x]}{\log x} = -\alpha^*.$$

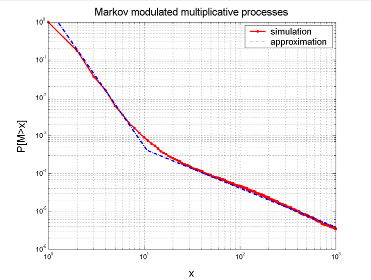
Outline

Double Pareto - frequent empirical observations

- Transition from heavy-traffic region to large deviation region
- Multiple time scales resulting in double Pareto

Example (Jelenković & Lazar(1995), queueing context)

$J_n \equiv J(X(n))$,
$X(n)$- Markov chain
$p_{12} = 1/5000$,
$p_{21} = 1/10$,
$\mathbb{P}[J(1) = 1.2] = 0.5$,
$\mathbb{P}[J(1) = 0.6] = 0.5$,
$\mathbb{P}[J(2) = 1.7] = 0.6$,
$\mathbb{P}[J(2) = 0.25] = 0.4$.



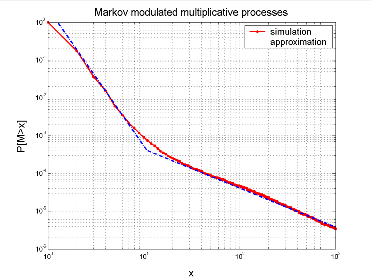
New explanations of double Pareto

Double Pareto - frequent empirical observations

- Transition from heavy-traffic region to large deviation region
- Multiple time scales resulting in double Pareto

Example (Jelenković & Lazar(1995), queueing context)

$J_n \equiv J(X(n))$,
$X(n)$- Markov chain
$p_{12} = 1/5000$,
$p_{21} = 1/10$,
$\mathbb{P}[J(1) = 1.2] = 0.5$,
$\mathbb{P}[J(1) = 0.6] = 0.5$,
$\mathbb{P}[J(2) = 1.7] = 0.6$,
$\mathbb{P}[J(2) = 0.25] = 0.4$.



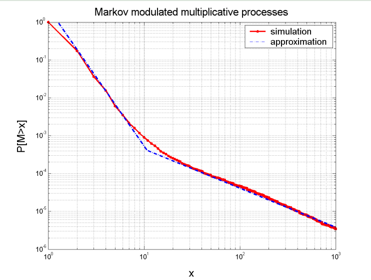
New explanations of double Pareto

Double Pareto - frequent empirical observations

- Transition from heavy-traffic region to large deviation region
- Multiple time scales resulting in double Pareto

Example (Jelenković & Lazar(1995), queueing context)

$J_n \equiv J(X(n))$,
$X(n)$- Markov chain
$p_{12} = 1/5000$,
$p_{21} = 1/10$,
$\mathbb{P}[J(1) = 1.2] = 0.5$,
$\mathbb{P}[J(1) = 0.6] = 0.5$,
$\mathbb{P}[J(2) = 1.7] = 0.6$,
$\mathbb{P}[J(2) = 0.25] = 0.4$.



Outline

Related phenomena and models

- Truncated power laws $\cdots$
  M(B)P with both lower and upper barriers
  Similarly as obtaining truncated geometric distributions in finite buffer queue, e.g., $M/M/1/B$
- Randomly stopped multiplicative processes $\rightleftharpoons$ RMP
  ladder height representation + Pollaczek-Khintchine formula
- Randomly stopped MBP

Related phenomena and models

- Truncated power laws $\cdots$
  M(B)P with both lower and upper barriers
  Similarly as obtaining truncated geometric distributions in finite buffer queue, e.g., $M/M/1/B$

- Randomly stopped multiplicative processes $\rightleftharpoons$ RMP
  ladder height representation + Pollaczek-Khintchine formula

- Randomly stopped MBP

Related phenomena and models

- Truncated power laws $\cdots$
  M(B)P with both lower and upper barriers
  Similarly as obtaining truncated geometric distributions in finite buffer queue, e.g., $M/M/1/B$
- Randomly stopped multiplicative processes $\rightleftharpoons$ RMP
  ladder height representation + Pollaczek-Khintchine formula
- Randomly stopped MBP

Concluding remarks

- RMBP - a new general model of proportional growth
- Under the general polynomial Gärtner-Ellis conditions,

  RMBP $\Rightarrow$ power laws $\Rightarrow$ ubiquitous nature of power law
- Discover the duality:

  additive processes (queueing theory) $\Leftrightarrow$ proportional growth

Amusing question

What is more frequent, power law or exponential distributions?	$\Longleftrightarrow$	What is more frequent, proportional growth or additive growth?

Questions?

Concluding remarks

- RMBP - a new general model of proportional growth
- Under the general polynomial Gärtner-Ellis conditions,

  RMBP $\Rightarrow$ power laws $\Rightarrow$ ubiquitous nature of power law
- Discover the duality:

  additive processes (queueing theory) $\Leftrightarrow$ proportional growth

Amusing question

What is more frequent, power law or exponential distributions? $\Longleftrightarrow$ What is more frequent, proportional growth or additive growth?

Questions?

Concluding remarks

- RMBP - a new general model of proportional growth
- Under the general polynomial Gärtner-Ellis conditions,

  RMBP $\Rightarrow$ power laws $\Rightarrow$ ubiquitous nature of power law
- Discover the duality:

  additive processes (queueing theory) $\Leftrightarrow$ proportional growth

Amusing question

What is more frequent, power law or exponential distributions? $\Longleftrightarrow$ What is more frequent, proportional growth or additive growth?

Questions?

Concluding remarks

- RMBP - a new general model of proportional growth
- Under the general polynomial Gärtner-Ellis conditions,

  RMBP $\Rightarrow$ power laws $\Rightarrow$ ubiquitous nature of power law
- Discover the duality:

  additive processes (queueing theory) $\Leftrightarrow$ proportional growth

Amusing question

What is more frequent, power law or exponential distributions? $\Longleftrightarrow$ What is more frequent, proportional growth or additive growth?

Questions?