Effect of Access Probabilities on the Delay Performance of Q-CSMA Algorithms

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***Abstract*—It has been recently shown that queue-based CSMA algorithms can be throughput optimal. In these algorithms, each link of the wireless network has two parameters: a transmission probability and an access probability. The transmission probability of each link is chosen as an appropriate function of its queue-length, however, the access probabilities are simply regarded as some random numbers since they do not play any role in establishing the network stability. In this paper, we show that the access probabilities control the mixing time of the CSMA Markov chain and, as a result, affect the delay performance of the CSMA. In particular, we derive formulas that relate the mixing time to access probabilities and use these to develop the following guideline for choosing access probabilities: each link i should choose its access probability equal to $1/(d_i + 1)$, where d_i is the number of links which interfere with link i. Simulation results show that this choice of access probabilities results in good delay performance.**

I. Introduction

Scheduling in wireless networks is of fundamental importance due to the inherent broadcast property of the wireless medium. Two radios might not be able to transmit simultaneously because they create too much interference for each other causing the *SINR* (Signal-to-Noise-plus-Interference-Ratio) at their corresponding receivers to go below the required threshold for successful decoding of the packets. Therefore, at each time, a *scheduling algorithm* (*MAC protocol*) is needed to schedule a subset of users that can transmit successfully at the same time.

The performance metrics used to evaluate a scheduling algorithm are *throughput* and *delay*. Throughput is characterized by the largest set of arrival rates under which the algorithm can stabilize the queues in the network. The delay performance of a scheduling algorithm can be characterized by the average delay experienced by the packets transmitted in the network. The design of efficient scheduling algorithms, to achieve maximum throughput and low delay, is the main objective of this paper. It is also essential for the scheduling algorithms to be distributed and have low complexity/overhead, since in many wireless networks there is no centralized entity and the resources at the nodes are very limited.

The wireless network is often modeled by its *conflict graph* (or interference model) to capture the interference constraints or technological ones (For example, a node cannot transmit and receive at the same time.). In the conflict graph, two communication links form two neighboring nodes of the graph if they cannot transmit simultaneously. Therefore, at each time slot, the active links should form an independent set of the conflict graph, i.e., no two scheduled nodes can share an edge in the conflict graph. The well-known result of Tassiulas and Ephremides [1] states that the *Maximum Weight Scheduling* (MWS) algorithm, where weights are functions of queue-lengths, is throughput optimal in the sense that it can stabilize the queues in the network for all arrival rates in the capacity region of the network (without explicitly knowing the arrival rates). However, for a general network, MWS involves finding the maximum weight independent set of the conflict graph, with time varying weights, in each time slot which requires the network to solve a complex combinatorial problem in each time slot and hence, is not implementable in practice. This has led to a rich amount of literature on design of approximate algorithms to alleviate the computational complexity of the MWS algorithm.

CSMA (*Carrier Sense Multiple Access*) type algorithms are an important class of scheduling algorithms due to their simplicity of implementation, and have been widely used in practice, e.g., in WLANs (IEEE 802.11 Wi-Fi) or emerging wireless mesh networks. In these protocols, each user listens to the channel and can transmit, with some probability, only when the channel is not busy. In this paper, we consider design of CSMA algorithms in order to maximize throughput and improve delay performance.

Recently, it has been shown that it is possible to design CSMA algorithms that are throughput-optimal, e.g., see [3], [5] for the continuous-time CSMA, and [4], [7] for the discrete-time CSMA. The common component in all these works is a Markov chain (called CSMA Markov chain) over the space of feasible schedules. The transition probabilities of the CSMA chain are controlled, by queue lengths or the differences between the average arrival rates and the average departure rates of the links, to make sure that a suitable schedule is selected at each time. Similar algorithms with fixed link weights were developed earlier in [2] and [8].

Essentially, the prior works on CSMA are mostly concerned with ensuring network stability. Their main focus is often on solving the maximum weight independent set problems in a

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distributed manner by using the so-called Glauber Dynamics. In CSMA algorithms, each user has two parameters: an *access probability* that controls how often the user tries to access the channel and a *transmission probability* that controls the length of the data transmission once the user acquires access to the channel. In the traditional *ALOHA* protocol, for a network of N users, the access probabilities $\{a_l\}_{l=1}^N$ are chosen to be $\frac{1}{N}$ in order to maximize the throughput and the maximum throughput per user is approximately $\frac{1}{Ne}$. However, in the CSMA schemes, as we will see in section II, one of the parameters is fixed and the other parameter is controlled, as a function of the user's local information to achieve the maximum throughput. In particular, in Q-CSMA (*Queue-based CSMA*) schemes, the access probabilities do not play a role in showing the stability/throughput optimality of CSMA because they do not appear in the steady state distribution of the CSMA chain. Hence, they have been simply regarded as some constants between zero and one. However, we will see that, they do have a significant impact on the *mixing time* of the chain, i.e., the amount of time that it takes to reach close to the steady state starting from some initial condition. Therefore, the access probabilities control the rate at which CSMA responds to the queue dynamics and hence, have a significant effect on the delay performance of the network. The relationship between the delay of the scheduling algorithm and the mixing time of the CSMA chain has been characterized in [10].

A. Main Contributions and Organization

In this paper, we analyze the mixing time of the Q-CSMA Markov chain and develop guidelines to choose access probabilities that result in small mixing times. The main contributions of the paper are the following:

(i) In the case of *collocated networks*, we show that access probabilities of the form $1/N$ yield mixing times that are within a constant factor of the optimal mixing time, i.e, the minimum mixing time assuming the global knowledge of the queues/weights of the network.

(ii) In d*-regular networks*, we show that access probabilities of the form $1/\chi$, when χ is the *chromatic number* of the graph, have the same kind of property when we replace the mixing time with a suitable upper-bound on it. In general, $\chi \leq d+1$, nevertheless, replacing the chromatic number with the $d+1$ still yields similar result but for a larger constant gap.

(iii) Based on these observations, in general graphs, We conjecture that access probabilities of the form $\{a_l = \frac{1}{d_l+1}\}_{l=1}^N$ should yield good performance, where d_l is the degree of the link l. Our simulation results show that the conjectured access probabilities have a good delay performance, indeed, they seems to yield average queue lengths that are very close to the smallest queue lengths that can be obtained with any fixed access probabilities.

The remainder of the paper is organized as follows. In section II, we give an overview the CSMA-type algorithms. In section III, we briefly explain some preliminaries and definitions used in the proofs of the results. Section IV is devoted to the results for collocated networks. We extend the results to the general networks in Section V. Section VII contains proofs of some of the results. Section VI contains the simulation results. Finally, we will end the paper with some concluding remarks.

II. Description of CSMA-type Algorithms

Let $G(V,E)$ denote the conflict graph of the wireless network consisting of N communication links. Formally, a schedule can be represented by a vector $X = [x_s : s = 1,...,N]$ such that $x_s \in \{0,1\}$ and $x_i + x_j \leq 1$ for all $(i,j) \in E$. Let $\mathcal{M}$ denote the set of all feasible schedules and $C(i)$ denote the set of neighbors of i. Then, the basic idea of CSMA is to use *Glauber Dynamics* (to be described below) to sample the independent sets of such a graph.

A. Continuous-Time CSMA

In the continuous time CSMA, each link l has two parameters λ_l and μ_l. The parameter λ_l determines the attempt rate and μ_l determines the transmission length. In other words, the link l senses the channel at the end of exponentially distributed back off intervals with the parameter λ_l and if it detects no ongoing transmissions (the channel is idle), it will transmit for an exponentially distributed amount of time with the mean μ_l. It is easy to check that such a Markov chain is reversible with the stationary distribution

$$\pi(X) = \frac{\prod_{l \in X} \lambda_l \mu_l}{\sum_{Y \in \mathcal{M}} \prod_{l \in Y} \lambda_l \mu_l}, \ \forall X \in \mathcal{M}.$$

By choosing $\lambda_l \mu_l = e^{w_l}$ where w_l is the weight of the link l, e.g., an appropriate function of its queue length, the stationary distribution will be in the form of

$$\pi(X) = \frac{1}{Z} \exp(\sum_{i \in X} w_i); \quad X \in \mathcal{M}, \tag{1}$$

where Z is the normalizing constant. Hence, when the weights are large, the algorithm picks the maximum weight schedule with high probability in steady state. Therefore the algorithm is throughput optimal [9] if we make sure that the instantaneous probability distribution and the stationary distribution are close enough. To get a faster mixing time, one can let λ_l grow very large (and $\mu_l = \exp(w_l)/\lambda_l$). But this does not make sense since, in practice, the carrier sensing is performed using energy detection (and hence, cannot be instantaneous) and the back off interval cannot be smaller than a certain mini-slot. Similarly, the data transmission slot cannot be made arbitrarily small. Moreover, this model is based on a perfect carrier sense assumptions and does not consider the collisions due to propagation delays. Thus, in the rest of the paper, we consider the discrete time CSMA algorithm proposed in [4] called *Q-CSMA*.

B. Q-CSMA

In Q-CSMA, each link l has two parameters a_l and p_l. The parameter a_l is the access probability and chosen to be constant and p_l is the transmission probability and chosen to be

$$p_l(t) = \frac{e^{w_l(t)}}{1 + e^{w_l(t)}}, \tag{2}$$

where w_l is an appropriate function of q_l (the queue length at link l). Each time slot is divided into a control slot and a data slot. In the control slot, each link l that wishes to become part of the data transmission schedule transmits an INTENT message with probability a_l. Those links that transmit INTENT messages and do not hear any INTENT messages from the neighboring links consist a decision schedule. In the data slot, each link l that is included in the decision schedule can transmit a data packet with probability p_l only if none of its neighbors have been transmitting in the previous data slot (see the description of the algorithm below).

Algorithm 1 : Q-CSMA in Time Slot t

1: In the control slot, randomly select a decision schedule $m(t) \subseteq \mathcal{M}$ by using access probabilities $\{a_l\}_{l=1}^N$.
2: - $\forall$ i in $m(t)$:
If no links in $C(i)$ were active in the previous data slot, i.e., $\sum_{j \in C(i)} x_j(t-1) = 0$:
- $x_i(t) = 1$ with probability $p_i(t)$, $0 < p_i(t) < 1$;
- $x_i(t) = 0$ with probability $\bar{p}_i(t) = 1 - p_i(t)$.

Else $x_i(t) = 0$.
- $\forall i \notin m(t)$: $x_i(t) = x_i(t-1)$.
3: In the data slot, use $X(t)$ as the transmission schedule.

If the weights are constant, then the above algorithm is the discrete-time version of the Glauber dynamics with multiple-site updates that generates the independent sets of G. So, the state space $\mathcal{M}$ consists of all independent sets of G. Q-CSMA algorithm uses a time-varying version of the Glauber dynamics, where the weights change with time. This yields a time inhomogeneous Markov chain but, for the proper choice of weights, it behaves similarly to the Glauber dynamics. It is easy to check that the stationary distribution with fixed transmission probabilities, i.e., when weights are fixed and do not change with time, is given by

$$\pi(X) = \frac{1}{Z} \prod_{i \in X} \frac{p_i}{\bar{p}_i}, \ \forall X \in \mathcal{M}.$$

By choosing the transmission probabilities to be in the form of (2), the stationary distribution will be the same as (1), and therefore can pick the maximum weight schedule with high probability as the queues in the network grows.

C. Throughput Optimality

The proof of throughput optimality of CSMA algorithms follows based on a time-scale separation assumption, i.e., the Markov chain evolves much faster than the rate of changes in the weights (due to queue dynamics in the network) such that the chain always remains close to its stationary distribution. This time-scale separation is justified in [6] and [5]. More precisely, for Q-CSMA, it is shown in [6] that, throughput optimality is preserved with weight functions of the form $w(q) = \log(1+q)/g(q)$, where $g(q)$ can be a function that increases arbitrarily slowly, e.g., $w(q) = (\log(1+q))^{1-\epsilon}$ for any small positive ϵ. Roughly speaking, by choosing any function slower than such a $w(.)$, the rate of changes in the weights will be much smaller than the rate at which the CSMA chain responds to these changes, although, for the sake of delay, we will always choose the fastest weight function possible.

III. Preliminaries

Before we state the main results, some preliminaries regarding the mixing time of Markov chains is needed.

Consider a time-homogenous discrete-time Markov chain over the finite state-space $\mathcal{M}$. For simplicity, we index the elements of $\mathcal{M}$ by $1, 2, ..., r$, where $r = |\mathcal{M}|$. Assume the Markov chain is irreducible and aperiodic, so that a unique stationary distribution $\pi = [\pi(1), ..., \pi(r)]$ always exists.

A. Distance Between Probability Distributions

First, we introduce two convenient norms on $\mathbb{R}^r$ that are linked to the stationary distribution. Let $\ell^2(\pi)$ be the real vector space $\mathbb{R}^r$ endowed with the scalar product

$$\langle z, y \rangle_\pi = \sum_{i=1}^r z(i) y(i) \pi(i).$$

Then, the norm of z with respect to π is defined as

$$\|z\|_\pi = \left(\sum_{i=1}^r z(i)^2 \pi(i) \right)^{1/2}.$$

We shall also use $\ell^2(\frac{1}{\pi})$, the real vector space $\mathbb{R}^r$ endowed with the scalar product

$$\langle z, y \rangle_{\frac{1}{\pi}} = \sum_{i=1}^r z(i) y(i) \frac{1}{\pi(i)},$$

and its corresponding norm. For any two strictly positive probability vectors μ and π, the following relationship holds

$$\|\mu - \pi\|_{\frac{1}{\pi}} = \|\frac{\mu}{\pi} - 1\|_\pi \geq 2\|\mu - \pi\|_{TV},$$

where $\|\pi - \mu\|_{TV}$ is the total variation distance

$$\|\pi - \mu\|_{TV} = \frac{1}{2} \sum_{i=1}^r |\pi(i) - \mu(i)|.$$

B. Mixing Times of Markov Chains

Starting from some initial distribution μ_0, the convergence to steady state distribution is geometric with a rate equal to the *second largest eigenvalue modulus* (SLEM) of the transition matrix [11] as it is described next.

Lemma 1. *Let P be an irreducible, aperiodic, and reversible transition matrix on the finite state space $\mathcal{M}$ with the stationary distribution π. Then, the eigenvalues of P are ordered in such a way that*

$$\lambda_1 = 1 > \lambda_2 \geq ... \geq \lambda_r > -1,$$

and for any initial probability distribution μ_0 on $\mathcal{M}$, and for all $n \geq 1$

$$\|\mu_0 \mathbf{P}^n - \pi\|_{\frac{1}{\pi}} \leq \sigma^n \|\mu_0 - \pi\|_{\frac{1}{\pi}}, \tag{3}$$

where $\sigma = \max\{\lambda_2, |\lambda_r|\}$ is the SLEM of P.

Therefore, if we define the mixing time as

$$\tau(\epsilon) = \inf\{n : \|\mu_0 \mathbf{P}^n - \pi\|_{1/\pi} \leq \epsilon\},$$

then a simple calculation reveals that

$$(T-1)\log(\|\mu_0 - \pi\|_{1/\pi}/\epsilon) \leq \tau(\epsilon) \leq T \log(\|\mu_0 - \pi\|_{1/\pi}/\epsilon).$$

where $T = \frac{1}{1-\sigma}$. We will see that for Q-CSMA algorithm, T is exponential in number of links or the maximum weight of the network. Therefore, T is approximately proportional to $\tau(\epsilon)$, and by abusing terminology, we will also sometimes refer to T as the mixing time.

C. Characterization of The Eigenvalues

Let $\beta_i = 1 - \lambda_i$, so $0 = \beta_1 < \beta_2 \cdots \leq \beta_r < 2$. For any vector $\theta \in \mathbb{R}^{|\mathcal{M}|}$, define the Dirichlet form $\mathcal{E}_\pi(\theta, \theta)$ as

$$\mathcal{E}_\pi(\theta, \theta) = \langle (I-P)\theta, \theta \rangle_\pi,$$

and also the variance

$$Var_\pi(\theta) = \|\theta\|_\pi^2 - \langle \theta, 1 \rangle_\pi^2.$$

Lemma 2 (Raleigh Theorem [11])**.** *Let P be an irreducible, aperiodic and reversible transition matrix on a finite state space $\mathcal{M}$, then for $j \geq 2$,*

$$\beta_j = \inf_{\theta \neq 0} \left\{ \frac{\mathcal{E}_\pi(\theta,\theta)}{Var_\pi(\theta)} : \langle \theta, v_i \rangle_\pi = 0 \text{ for } 1 \leq i \leq j-1 \right\},$$

where v_is are the right eigenvectors of P. Moreover, any vector θ achieving the infimum is an eigenvector of P corresponding to the eigenvalue $\lambda_j = 1 - \beta_j$.

Expanding the inner product, and using reversibility of the Markov chain, reveals that

$$\mathcal{E}_\pi(\theta, \theta) \quad = \quad = \frac{1}{2} \sum_{i,j \in \mathcal{M}} \pi(i) p_{ij} (\theta_j - \theta_i)^2.$$

To characterize the SLEM σ, we need to find λ_2 and λ_r. When solving the minimization in Lemma 2 is difficult, one can still use the result of the geometric convergence rate, Lemma 1, by finding good bounds on λ_2 and λ_r. In these cases, the following Lemmas are useful [12], [11]. First, for a nonempty set $B \subset E$, define the followings:

$$\pi(B) = \sum_{i \in B} \pi(i),$$

and

$$F(B) = \sum_{i \in B, j \in B^c} \pi(i) p_{ij}.$$

Then, the *conductance* of an irreducible, aperiodic, and reversible transition matrix P is defined as

$$\phi(P) = \inf_{B:\pi(B) \leq 1/2} \frac{F(B)}{\pi(B)}.$$

Lemma 3. *(Cheeger's inequality)*

$$1 - 2\phi(P) \leq \lambda_2 \leq 1 - \frac{\phi^2(P)}{2}.$$

Lemma 4 (Gershgorin's bound)**.** *Let P be a finite $r \times r$ matrix. Then for any eigenvalue λ, and all $k \in [1, r]$,*

$$|\lambda - a_{kk}| \leq \min(r_k, s_k),$$

where $r_k = \sum_{j \neq k} |a_{kj}|$ and $s_k = \sum_{j \neq k} |a_{jk}|$.

IV. Main Results for Collocated Networks

Consider a collocated network under Q-CSMA where every link interferes with all the other links, i.e., the conflict graph is complete. In this case, we can index the feasible schedules by $0, 1, 2, \cdots, N$ where 0 shows the empty schedule and nonzero indices show the active link number. Every link i, $1 \leq i \leq N$, can change its state, i.e., becomes active or silent, if and only if it is selected in the decision schedule. Link i is selected in the decision schedule, when it sends an INTENT message and nobody else transmits INTENT messages which happens with probability

$$\alpha_i = a_i \prod_{\substack{j=1 \\ j \neq i}}^{N} (1 - a_j).$$

Therefore, it follows that the transition probabilities of the CSMA Markov chain are given by

$$p_{0i} = a_i p_i \prod_{j \neq i} (1 - a_j), \;\; i \neq 0,$$

$$p_{ii} = 1 - a_i \bar{p}_i \prod_{j \neq i} (1 - a_j), \;\; i \neq 0,$$

$$p_{i0} = a_i \bar{p}_i \prod_{j \neq i} (1 - a_j), \;\; i \neq 0,$$

$$p_{00} = 1 - \sum_{i=1}^{N} p_{0i},$$

where $\bar{p}_i = 1 - p_i$.

Calculating λ_2 and λ_r ($r = |\mathcal{M}|$) directly from the transition probability matrix, especially when N is large and weights are different, is not an easy task. Instead, we use the Raleigh Lemma (Lemma 2) to calculate λ_2. Solving the exact minimization in Lemma 2 is possible, but it does not yield a closed form expression for $\beta_2 = 1 - \lambda_2$ (See [14], β_2 is expressed as a zero of a complex polynomial). Hence, we do not present the exact solution due to space limitations and instead, present the following more useful result about the upper and lower bounds on λ_2. The proof is provided in Section VII.

Lemma 5. *For a collocated network of $N \geq 2$ links, and given a set of access probabilities $\{a_i\}_{i=1}^N$ and a set of transmission probabilities $\{p_i\}_{i=1}^N$, $\beta_2^{low} \leq \beta_2 \leq \beta_2^{up}$, where*

$$\beta_2^{low} = \min_{1 \leq i \leq N} \bar{p}_i a_i \prod_{j \neq i} (1 - a_j). \tag{4}$$

$$\beta_2^{up} = 2 \min_{i:\pi_i \leq 1/2} \bar{p}_i a_i \prod_{j \neq i} (1 - a_j). \tag{5}$$

Note that in the case of $N = 1$, trivially, no scheduling is needed and $a_1^* = 1$. So, we can assume that there are at least two links in the network. Next, we use the Gershgorin's bound (Lemma 4) to find a lower bound on all the eigenvalues. We state the result as a Lemma, whose proof is given in Section VII.

Lemma 6. *For a collocated network of $N \geq 2$ links, under equal access probabilities and any set of transmission probabilities, all the eigenvalues are nonnegative, i,e, $\lambda_r \geq 0$.*

Note that for general access probabilities, $T = \frac{1}{1-\sigma} \geq \frac{1}{\beta_2}$. However, in the case of equal access probabilities, by Lemma 6, SLEM is dominated by λ_2 and, hence, $T = 1/\beta_2$.

We will use the following result in bounding the smallest possible mixing time.

Lemma 7. *The optimal access probabilities that maximize β_2^{low}, in Lemma 5, are in the form of $a_i^* = \frac{k}{k+\bar{p_i}}$, where the constant k is chosen such that $\sum_{i=1}^N a_i^* = 1$.*

See [14] for the proof. As a special case, when all the p_is are equal, i.e., weights are equal, simple calculation reveals that the optimal access probabilities in Lemma 7 are all equal to $1/N$. Therefore, for such a choice of access probabilities, the equality $T = 1/\beta_2$ holds and therefore, $T \leq 1/\beta_2^{low}$. Hence, in the case of equal weights, the access probabilities of the traditional *ALOHA* protocol, i.e, $a_i = \frac{1}{N}$, minimize the upper bound $1/\beta_2^{low}$.

In general, the ALOHA access probabilities are not optimal for the queue-based random access protocols and finding the optimal access probabilities requires the knowledge of all the weights in the network which might not be feasible in practice. In this case, one might be interested in a suboptimal solution that does not require the global knowledge and the *mixing time ratio*, i.e., the ratio of the optimal solution to the suboptimal solution, remains bounded, i.e.,

$$1 < \frac{T(\text{subopt.})}{T(\text{opt.})} < M, \tag{6}$$

for some constant M *independent of the network size* N. It suffices to find a suboptimal solution such that

$$1 < \frac{T^{up}(\text{subopt.})}{T^{low}(\text{opt.})} < M, \tag{7}$$

where $T^{up}(\text{subopt.})$ is the upperbound on the suboptimal solution and $T^{low}(\text{opt.})$ is the lower bound on the optimal solution. Equivalently, for a suboptimal solution with equal access probabilities, we need to show that

$$1 < \frac{\beta^{up}(\text{opt.})}{\beta^{low}(\text{subopt.})} < M. \tag{8}$$

where β^{up} and β^{low} were defined in Lemma 5. To show such a property, we need to consider an appropriate distribution of $\bar{p}_i$s as the number of nodes N grows. Here, we assume that there exist m types of weights, such that a constant fraction α_k of the nodes have the weight $\bar{p}_k$ for $k = 1, ..., m$. Note that in such a setting, if there exists a state/link l with $\pi_l > 1/2$, then since p_l is one of the m possible weights, there must exist $\alpha_k N$ links with the same transmission probability p_l, and all of them should have stationary probabilities greater than $1/2$ which is impossible since $\sum_{i=0}^N \pi_i = 1$. Therefore, all the states have the stationary probability less than $1/2$, and $\beta_2^{up} = 2\beta_2^{low}$[1]. Hence, the access probabilities that maximize β_2^{up}, i.e., yield the smallest lower bound on the mixing time, are given by Lemma 7. Then, it is easy to see that the optimal k in Lemma 7 is in the form of $\frac{c}{N}$ for some constant $\bar{p}^{min} < c < 1/2$, where $\bar{p}^{min} = \min_i \bar{p}_i$. Thus, we have

$$\beta^{up}(\text{opt.}) \quad = \quad 2k^* \prod_i \frac{\bar{p_i}}{k^* + \bar{p_i}} = \frac{2c}{N} \prod_{k=1}^m \left(\frac{\bar{p_k}}{\bar{p_k} + c/N} \right)^{\alpha_k N},$$

Putting everything together, the suboptimal access probabilities of the form $a_i = \frac{1}{N}$ yield a bounded mixing time ratio independent of N, because

$$\beta^{low}(\text{subopt.}) = \frac{\bar{p}^{min}}{N} \left(1 - \frac{1}{N}\right)^{N-1},$$

and

$$\lim_{N \to \infty} \frac{\beta^{up}(\text{opt.})}{\beta^{low}(\text{subopt.})} = \frac{2c}{\bar{p}^{min}} \exp \left(1 - c \sum_{k=1}^m \frac{\alpha_k}{\bar{p}_k}\right) < \infty.$$

Therefore the mixing time ratio is bounded for all values of N. Furthermore, it is easy to check that choosing access probabilities independent of N results in unbounded mixing time ratio. The importance of the above ratio is that it guarantees that the mixing time of the suboptimal solution is within a constant multiple of the optimal mixing time, independent of the network size.

V. Main Results for General Networks

The extension of results to general networks is more difficult since the corresponding CSMA Markov chain is much more complex than the Markov chain of collocated networks, hence finding the second largest eigenvalue by solving the optimization (2) is cumbersome. Instead, we find an upper bound on the SLEM based on the conductance bound (Lemma 3).

Assume the current schedule is $X(t) = X$, for some $X \in \mathcal{M}$, and the CSMA Markov chain makes a transition to the next state/schedule $X(t+1) = Y$. Note that $X \backslash Y = \{l : x_l = 1, y_l = 0\}$ is the set of links that change their states from 1 (active) to 0 (silent). Similarly $Y \backslash X = \{l : x_l = 0, y_l = 1\}$ is the set of links that change their states from 0 to 1. From the scheduling algorithm, it is clear that a link can change its state only when it belongs to the decision schedule. Therefore, X can make a transition to Y when $X \Delta Y \subseteq m$, for some $m \in \mathcal{M}$, where $X \Delta Y = (X \backslash Y) \cup (Y \backslash X)$.

Let $\alpha(m)$ denote the probability of generating a decision schedule m. Then, it is not hard to argue that $P(X, Y)$, the

[1]The same analysis is possible for other kinds of weight assignments. Essentially, since there exists at most one link l with $\pi_l > 1/2$, we can prove that this does not change the asymptotics. The details are omitted here due to the space limit.

probability of transition from the schedule X to the schedule Y, is given by

$$P(X,Y) = \sum_{m\in\mathcal{M}:X\Delta Y\subseteq m} \alpha(m) \prod_{l\in X\setminus Y} \bar{p}_l \prod_{k\in Y\setminus X} p_k \times \prod_{i\in m\cap(X\cap Y)} p_i \prod_{j\in m\setminus(X\cup Y)\setminus C(X\cup Y)} \bar{p}_j. \quad (9)$$

Recall the mechanism for generating a decision schedule m by transmitting the INTENT messages based on access probabilities $\{a_i\}_{i=1}^N$. All the links that are included in m, should have transmitted INTENT messages and have not heard any INTENT messages from their interfering neighbors. So the probability of generating a decision schedule $\alpha(m) = \alpha(m;G)$ in the graph G can be characterized by

$$\alpha(m,G) = \prod_{i\in m} a_i \prod_{j\in C(m)} (1-a_j)\alpha\left(\emptyset; G\setminus(m\cup C(m))\right),$$

where $\alpha\left(\emptyset; G\setminus(m\cup C(m))\right)$ is the probability that no nodes included in the decision schedule in graph $G\setminus(m\cup C(m))$, i.e., the graph obtained by removing all the links in m and $C(m)$ from G. The expression for $\alpha(\emptyset;G')$ could be quite complicated since it has to account for all the events that yield a $\emptyset$ schedule due to either not transmitting INTENT messages or collision between INTENT messages transmitted by the nodes of G'. Nevertheless, the mixing time can be upper bounded by[2]

$$T = \frac{1}{\beta_2} \leq \frac{2}{\tilde{\phi}^2(P)}, \quad (10)$$

where $\tilde{\phi}(P)$ is an approximate conductance defined in the following Lemma. The proof is presented in Section VII.

Lemma 8. *In a general network, under the Q-CSMA with transition probability matrix P, the conductance $\phi(P)$ is lower bounded by $\tilde{\phi}(P)$, where*

$$\tilde{\phi}(P) = \min_{m_0\in\mathcal{M}_0} P(m_0,\emptyset),$$

$\mathcal{M}_0 \subset \mathcal{M}$ is the set of all maximal schedules, and $P(m_0,\emptyset)$ is the probability of transition from the maximal schedule m_0 to the empty schedule $\emptyset$.

Therefore, we can try to find optimal access probabilities that maximize $\tilde{\phi}(P)$. In this case, the optimal access probabilities are the solution to

$$\max_{\{a_i\}} \min_{m_0\in\mathcal{M}_0} \prod_{i\in m_0} a_i\bar{p}_i \prod_{j\notin m_0} (1-a_j). \quad (11)$$

Solving the above optimization needs some global knowledge of the network. Hence, we investigate possible suboptimal solutions with the bounded mixing time ratio (6) when we use the upperbounds on the mixing times, based on (10), instead of the exact values.

As a special case, consider a d-regular network with N links, i.e., each link has exactly d interfering neighbors. Furthermore, assume that the weights are equal, i.e., $\bar{p}_1 = ... = \bar{p}_N$. It is easy to show that, in this case, in the optimization (11), we need to consider the minimization over the maximal schedules with the maximum size, i.e., over the set of nodes with the same color in a valid node coloring of the graph. Let χ denote the chromatic number of the corresponding graph. Note that since there is no unique way of constructing a d-regular graph with N nodes, the chromatic number depends on the construction, but we know that

$$\text{maximum clique size} \leq \chi \leq d+1.$$

Since the graph is symmetric, all the access probabilities must be equal and the maximum size of the maximal schedule is $s = \lceil \frac{N}{\chi} \rceil$. Then, the optimal access probabilities in (11) are all equal and simply the solution to

$$\max_a a^s(1-a)^{N-s},$$

i.e., $a = \frac{s}{N} = \frac{1}{N}\lceil\frac{N}{\chi}\rceil$ or

$$\frac{1}{\chi} \leq a \leq \frac{1}{\chi} + \frac{1}{N}. \quad (12)$$

This suggests using $1/\chi$ as the access probability. Since, in general, the chromatic number of the network might not be known, our conjecture is that $\frac{1}{d+1}$ is a good candidate for the access probabilities when each node only knows the number of its interfering neighbors. We validate this conjecture through simulations later.

Next, consider a more general case of a d-regular network with different weights. Although $1/\chi$ or $1/(d+1)$ are not the optimal access probabilities, we argue that they yield a bounded gap between the upper-bound (10) on the mixing time of the optimal access probabilities and the corresponding upper-bound on the suboptimal solution. To prove such a property, similar to the collocated network, we need to consider an appropriate scaling of the network and a weight assignment as we add more nodes to the network. For the assignment of transmission probabilities/weights, we consider the worst assignment that is possible for the suboptimal solution: consider transmission probabilities $\bar{p}_1 \leq \bar{p}_2 ... \leq \bar{p}_\chi$ and then assign $\bar{p}_i$ to all the links in the i-th maximal schedule, for $i = 1, ..., \chi$. It is clear that the following optimization gives an upper bound on (11)

$$\tilde{\phi}^{up}(P) = \max_{\{a_i\}} \min_{1\leq i\leq\chi} \bar{p}_i^s a_i^s \prod_{j\neq i}(1-a_j)^s, \quad (13)$$

where s is the maximum size of a maximal schedule. The rest of calculations follows in parallel with those of the complete graph. The optimal access probabilities, maximizers of (13), are given by

$$a_i^* = \frac{k}{k+\bar{p}_i}, \ \sum_{i=1}^{\chi} a_i^* = 1. \quad (14)$$

[2]Here, we assume that the SLEM is λ_2. To ensure this, one may have to modify the CSMA Markov chain slightly to make it a lazy chain [13]. But this is not considered here due to page limitations.

Next, we prove that the suboptimal solution has a bounded mixing time property, i.e.,

$$\frac{T^{up}(\text{subopt.})}{T^{up}(\text{opt.})} = \left(\frac{\tilde{\phi}(\text{opt.})}{\tilde{\phi}(\text{subopt.})}\right)^2 \leq \left(\frac{\tilde{\phi}^{up}(\text{opt.})}{\tilde{\phi}(\text{subopt.})}\right)^2 < \infty.$$

To show such a property, we need to consider an appropriate scaling of the network as the number of nodes N grows. We assume that the degree d grows uniformly for all the nodes in G as N increases, i.e., the number of interfering neighbors of each node increases uniformly. Therefore the chromatic number grows linearly in N ($\chi = \chi_{_N}$) and the maximum size of a schedule remains constant s. Moreover, there are m types of weights such that α_k fraction of maximal schedules have the transmission probability p_k for $k = 1, ..., m$. Noting that constant k is in the form of $k^* = \frac{c}{\chi}$ for some $\bar{p}^{min} \leq c \leq 1/2$, the optimal $\tilde{\phi}^{up}$ is given by

$$\begin{aligned}
\tilde{\phi}^{up}(\text{opt.}) &= \left(k^* \prod_{i=1}^{N} \frac{\bar{p_i}}{k^* + \bar{p_i}}\right)^s \\
&= \left(\frac{c}{\chi_{_N}} \prod_{k=1}^{m} \left(\frac{\bar{p_k}}{\bar{p_k} + c/\chi_{_N}}\right)^{\alpha_k \chi_N}\right)^s \\
&\simeq \left(\frac{c}{\chi_{_N}}\right)^s \exp\left(-cs \sum_{k=1}^{m} \frac{\alpha_k}{\bar{p}_k}\right),
\end{aligned}$$

where "$\simeq$" shows the asymptotic as $N \to \infty$. The suboptimal $\tilde{\phi}$ is

$$\tilde{\phi}(\text{subopt.}) = \left(\bar{p}^{min} \frac{s}{N} \left(1 - \frac{s}{N}\right)^{N/s-1}\right)^s \simeq \left(\bar{p}^{min} \frac{s}{eN}\right)^s,$$

and, hence

$$\lim_{N\to\infty} \frac{\tilde{\phi}^{up}(\text{opt.})}{\tilde{\phi}(\text{subopt.})} = \left(\frac{ce}{\bar{p}^{min}}\right)^s \exp\left(-cs \sum_{k=1}^{m} \frac{\alpha_k}{\bar{p}_k}\right). \quad (15)$$

The importance of the above ratio is that it guarantees that the gap between the upperbounds on the mixing times of the suboptimal solution and the optimal solution is a constant independent of the network size.

Similarly, using $1/(d_N + 1)$ as the suboptimal access probability still yields a bounded ratio, but the corresponding ratio will be

$$\begin{aligned}
\lim_{N\to\infty} \frac{\tilde{\phi}^{up}(\text{opt.})}{\tilde{\phi}(\text{subopt.})} &= \left(\frac{c}{\bar{p}^{min}} \frac{d_N + 1}{\chi_{_N}} e^{\frac{\chi_N}{d_N+1}}\right)^s \times \\
& \quad \exp\left(-cs \sum_{k=1}^{m} \frac{\alpha_k}{\bar{p}_k}\right), \quad (16)
\end{aligned}$$

which is greater than (15) (since $\frac{d_N+1}{\chi_N} \geq 1$ and $xe^{1/x} \geq e$ for all $x \geq 1$.).

VI. Simulation Results

In this section, we evaluate the performance of different access probabilities via simulations. For this purpose, we have considered different topologies for the wireless network. In the algorithm, we have selected the transmission probabilities at time t as $p_l(t) = \frac{e^{w_l(t)}}{1+e^{w_l(t)}}$ where $w_l(t) = \log(q_l(t) + 1)$, $1 \leq l \leq N$. This choice makes the network stable [6] and yields the best delay performance. The first example is a collocated network of $N = 8$ links. To choose the arrival rates, choose a point $v = (v_1 \cdots v_8)$ on the boundary of the capacity region which satisfies $\sum_{i=1}^{8} v_i < 1$ and $v_i \geq 0$, and consider the arrival rates of the form $\lambda = \rho v$ for a loading $0 < \rho < 1$. Note that, as $\rho \to 1$, λ approaches a point on the boundary of the capacity region. For example, we have chosen $v_1 = \cdots = v_4 = 3/16$, $v_5 = \cdots = v_8 = 1/16$ and $\rho = 0.8$. Figure 1 shows the average queue length (averaged over time and over the links) for different values of the access probability a (all the links have the same access probability). It can be seen that $a = 1/N (= 1/8)$ yields the smallest average queue size.

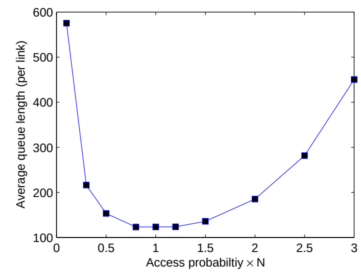

Fig. 1. Average queue size vs Access probability in a collocated network with $N = 8$ links.

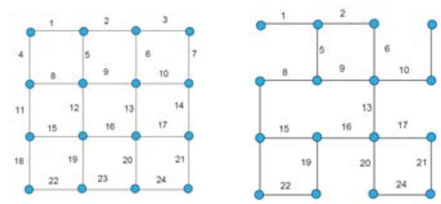

Fig. 2. Left: A grid network consisting of 16 nodes and 24 links. Right: An asymmetric network obtained from removing some links from the grid.

Next, consider the grid network on the left side of Figure 2. Note that this is the actual network not its conflict graph. The network has 16 nodes and 24 links. We consider a one hop interference constraint, i.e., two links interfere if they are adjacent (share a node in the network). Consider the following maximal schedules $M_1 = \{1, 3, 8, 10, 15, 17, 22, 24\}$, $M_2 = \{4, 5, 6, 7, 18, 19, 20, 21\}$, $M_3 = \{1, 3, 9, 11, 14, 16, 22, 24\}$, $M_4 = \{2, 4, 7, 12, 13, 18, 21, 23\}$. With a little abuse of notation, let M_i also be a vector whose i-th element is 1 if $i \in M_i$ and 0 otherwise. We consider arrival rates that are a convex combination of the above maximal schedules scaled by $\rho = 0.8$, e.g., $\lambda = \rho \sum_{i=1}^{4} c_i M_i$, $c = [0.2, 0.3, 0.2, 0.3]$. Our conjecture is that access probabilities of the form $\{1/(d_i + 1)\}_{i=1}^{N}$ (d_i is the number of interfering links of the link i) should result in good performance. We compare the performance of the network under this choice of access probabilities with the performance of the network under equal access probabilities, i.e., $a_i = a$, $1 \leq i \leq 24$, for some constant a between zero and one. Figure 3 verifies our conjecture where the dashed

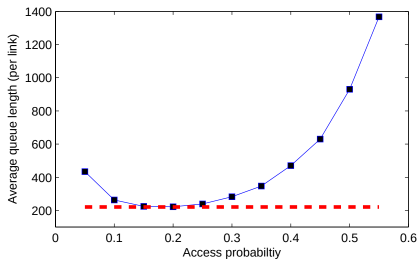

Fig. 3. Average queue size vs Access probability for the grid network in Figure 2.

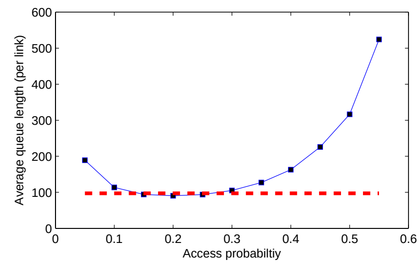

Fig. 4. Average queue size vs Access probability for the asymmetric network in Figure 2.

line at the bottom of Figure is the average queue size resulted by using $1/(d_i+1)$ as the access probabilities.

Next, to make the topology more asymmetric, we remove a random set S of links from the network. In the simulation shown here, $S = \{3, 4, 12, 14, 23\}$. Figure 2 shows the resulting network. For the arrival rates, we consider the same convex combination of 4 maximal schedules M_1 to M_4 with links of the set S removed from the maximal schedules. The average queue size for the equal access probabilities, ranging from 0.05 to 0.55, and also for probabilities of the form $1/(d_i+1)$, the dashed line, have been depicted in Figure 4. Again we see that the choice of $a_i = 1/(d_i+1)$ performs nearly as well as the best choice of fixed access probabilities. However, it is important to note that the choice $1/(d_i+1)$ adapts itself to the topology of the network, whereas it is not clear how one can choose the best fixed access probabilities a priori for a network. See [14] for more simulation results.

VII. Proofs

A. Proof of Lemma 5

Note that $v_1 = 1$, because $\pi P = 1\pi$, and $\mathcal{E}_\pi(\theta,\theta) = \mathcal{E}_\pi(\theta - c\mathbf{1}, \theta - c\mathbf{1})$, and $Var_\pi(\theta) = Var_\pi(\theta - c\mathbf{1})$ for any constant vector $c\mathbf{1}$. Therefore, without loss of optimality, we can only consider zero mean vectors θ. Then, it is easy to show that the minimization in Lemma 2 can be written as

$$\beta_2 = \frac{1}{2}\inf_{\theta} \sum_{i,j=0}^{N} \pi(i) p_{ij} (\theta_j - \theta_i)^2, \tag{17}$$

subject to the constraints

$$\theta \neq 0,\ \sum_{i=0}^{N} \pi(i)\theta_i = 0,\ \sum_{i=0}^{N} \pi(i)\theta_i^2 = 1. \tag{18}$$

Note that, for the complete graph,

$$\pi(i) = \frac{1}{Z}\frac{p_i}{\bar{p}_i} = \frac{1}{Z}\exp(w_i),$$

for $i = 0, 1, .., N$, where we have defined $p_0 = 1/2 = \bar{p}_0$. Hence, the optimization can be written as

$$\beta_2 = \frac{1}{2Z}\inf_{\theta}\left\{\sum_{j=0}^{N} p_{0j}(\theta_j - \theta_0)^2 + \sum_{i=1}^{N}\frac{p_i}{\bar{p}_i} p_{i0}(\theta_0 - \theta_i)^2\right\},$$

subject to the constraints in (18). Using the reversibility of the Markov chain, we have $\frac{p_i}{\bar{p}_i} p_{i0} = p_{0i}$, and therefore

$$\beta_2 = \frac{1}{Z}\inf_{\theta}\sum_{i=1}^{N} p_{0i}(\theta_i - \theta_0)^2, \tag{19}$$

subject to the constraints in (18).

Although solving the above optimization is possible, it does not yield a closed form expression for β_2 (See [14]). Instead, we try to find upper and lower bounds on β_2. By a change of variable $y_i = \theta_i - \theta_0$, $i = 1, ..., N$, the optimization problem simplifies to

$$\beta_2 = \frac{1}{Z}\inf_{y,\theta_0}\sum_{i=1}^{N} y_i^2 a_i p_i \prod_{j\neq i}^{N}(1 - a_j), \tag{20}$$

subject to

$$\sum_{i=1}^{N}\frac{p_i}{\bar{p}_i} y_i^2 = Z(1+\theta_0)^2,\ \sum_{i=1}^{N}\frac{p_i}{\bar{p}_i} y_i = -\theta_0 Z. \tag{21}$$

To get a lower bound on β_2, we ignore the last constraint, i.e., the lower bound is the solution to

$$\beta_2^{low} = \frac{1}{Z}\inf_{u}\sum_{i=1}^{N} u_i^2 \bar{p}_i a_i \prod_{j\neq i}^{N}(1 - a_j), \tag{22}$$

$$\text{s.t.}\quad \sum_{i=1}^{N} u_i^2 = Z(1+\theta_0)^2. \tag{23}$$

where we have used a change of variable $u_i = \sqrt{\frac{p_i}{\bar{p}_i}}y_i$. Then, it is clear that $\theta_0^* = 0$, and the optimal value, which is a lower bound on β_2, is given by (4) for any set of access probabilities.

Next, we prove the upperbound. By Lemma 3, $\beta_2 \leq 2\phi(P)$, so it remains to calculate the conductance

$$\phi(P) = \inf_{B:\pi(B)\leq 1/2}\frac{\sum_{i\in B, j\in B^c}\pi(i)P_{ij}}{\sum_{i\in B}\pi(i)}.$$

First, consider the set B to only contain singletons. The optimal B does not contain 0 since $\pi(0) \leq \pi(i)$ for any $i \neq 0$, and, by the reversibility, $\pi(i)p_{i0} = \pi(0)p_{0i}$ for any $i \neq 0$. So considering only the singletons, we have

$$\phi(P) = \min\{p_{i0} : \pi(i) \leq 1/2, i \neq 0\} \tag{24}$$

$$= \min_i\left\{\bar{p}_i a_i \prod_{j\neq i}(1 - a_j) : \pi(i) \leq 1/2\right\}. \tag{25}$$

Let i^* be the minimizer of (24). Then it is not hard to argue that adding any other state to i^*, only increases $\phi(P)$. So (24) is the actual conductance. Note that if there is only one link in the network, the set (24) is empty.

B. Proof of Lemma 6

By Gershgorin's bound, Lemma 4, $\lambda_r \geq -1 + 2\min_i p_{ii}$, where we have used the fact that $\sum_{j\neq i} p_{ij} = 1 - p_{ii}$. Note that since $\bar{p}_i \leq 1/2$, $p_{ii} \geq 1/2$ for $i \neq 0$. Furthermore,

$$\begin{aligned} p_{00} &= 1 - \sum_{i=1}^{N} a_i p_i \prod_{j\neq i}(1-a_i) \\ &\geq 1 - \sum_{i=1}^{N} a \prod_{j\neq i}(1-a) \\ &\geq 1 - \sum_{i=1}^{N} \frac{1}{N}(1-\frac{1}{N})^{N-1} \\ &\geq 1 - (1-\frac{1}{N})^{N-1} \geq 1/2, \end{aligned}$$

where we used the assumption that there are at least two links in the network. Hence, $p_{ii} \geq 1/2$ for all $0 \leq i \leq N$, and therefore $\lambda_r \geq 0$.

C. Proof of Lemma 8

Let m_0 be a maximal schedule such that $X\Delta Y \subseteq m_0$, then it is clear that

$$\begin{aligned} P(X,Y) \geq& \alpha(m_0) \prod_{l\in X\setminus Y} \bar{p}_l \prod_{k\in Y\setminus X} p_k \times \\ & \prod_{i\in m_0\cap(X\cap Y)} p_i \prod_{j\in m_0\setminus(X\cup Y)\setminus C(X,Y)} \bar{p}_j \\ \geq& \prod_{i\in m_0} a_i \prod_{j\notin m_0}(1-a_j) \prod_{l\in m_0} \bar{p}_l = P(m_0,\emptyset), \end{aligned}$$

where we have used the fact $p_i \geq \bar{p}_i$, for all i, due to (2). The conductance can be lower bounded by

$$\begin{aligned} \phi(P) &= \inf_{B:\pi(B)\leq 1/2} \frac{F(B)}{\pi(B)} \\ &\geq \inf_{B:\emptyset\notin B} \frac{F(B)}{\pi(B)}, \end{aligned}$$

since if $\emptyset \in B$, we can replace B with B^c and get a smaller conductance because $\pi(B^c) \geq \pi(B)$ and $F(B) = F(B^c)$ by reversibility. Note that there is a transition between X and Y whenever $X\Delta Y$ is a valid schedule, therefore a direct transition from the empty schedule $\emptyset$ to any schedule X, and vice versa, is possible. Hence, the conductance can be further lower bounded as follows

$$\begin{aligned} \phi(P) &\geq \inf_{B:\emptyset\notin B} \frac{\sum_{X\in B, Y\in B^c}\pi(X)P(X,Y)}{\pi(B)} \\ &\geq \inf_{B:\emptyset\notin B} \frac{\sum_{X\in B}\pi(X)P(X,\emptyset)}{\sum_{X\in B}\pi(X)} \\ &= \min_{X\neq\emptyset} P(X,\emptyset). \end{aligned}$$

Hence

$$\phi(P) \geq \min_{m_0\in\mathcal{M}_0} P(m_0,\emptyset) = \tilde{\phi}(P), \qquad (26)$$

where $\mathcal{M}_0$ denotes the set of all maximal schedules in $\mathcal{M}$.

VIII. Concluding Reamrks

Access probabilities affect the mixing time of the CSMA Markov chain, which, in turn, has a significant impact on the delay performance of the algorithm. It turns out that formulating the optimal mixing time, as a function of access probabilities, in general, is a formidable task. Even if we are able to formulate the optimal mixing time and find the optimal access probabilities, they will depend on global knowledge of the network, and thus, will not be suitable for the distributed operation of the CSMA algorithm. Instead, sub optimal access probabilities of the form $1/(d_l+1)$, where d_l is the number of interfering neighbors of link l, can yield mixing times that are within a constant gap of the optimal mixing time. This was proved for fully-connected networks and d-regular networks. We conjecture that, in general topologies, such access probabilities should have good delay performance. This conjecture is verified through extensive simulations, some of which are shown in the paper. It would be interesting to prove such a conjecture for general networks as a future work.

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