Interval Packing: The Vacant Interval Distribution

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June 6, 1998

Abstract

Starting at time 0, unit-length intervals arrive and are placed on the positive real line by a unit-intensity Poisson process in two dimensions; the left endpoints of intervals appear at the rate of 1 per unit time per unit distance. An arrival is accepted if and only if, for some given x, the interval is contained in $[0, x]$ and overlaps no interval already accepted. This stochastic, on-line interval packing problem generalizes the classical parking problem, the latter corresponding only to the absorbing states of the interval packing process, where successive packed intervals are separated by gaps less than or equal to 1 in length.

In earlier work [3], the authors studied the distribution of the number of intervals accepted during $[0, t]$. In this paper, we give an explicit formula for the large-x asymptotics of the distribution of the sizes of the vacant intervals (or gaps) between consecutive intervals packed during $[0, t]$. The limiting expected gap size as $x \to \infty$ is shown to be

$$m(t) = \frac{1 - \alpha(t)}{\alpha(t)},$$

where

$$\alpha(t) := \int_0^t e^{-2 \int_0^v \frac{1-e^{-x}}{x} dx} dv,$$

with $\alpha(\infty) = 0.748\ldots$ (Renyi's constant), and $m(\infty) = 0.337\ldots$.

We briefly discuss the recent importance acquired by interval packing models in connection with resource-reservation systems. In these applications, our vacant intervals correspond to times between consecutive reservation intervals. Thus, the results of this paper improve our understanding of the fragmentation of time that occurs in reservation systems.

1 Introduction

Unit intervals arrive at random times and at random locations in $\mathbf{R}_+$. The arrival times and left endpoints comprise a unit-intensity Poisson process in two dimensions. Thus, the probability of an arrival in the time interval $[t, t + dt]$ with left endpoint in $[y, y + dy]$ is $dtdy + o(dtdy)$. The number packed during $[0, t]$ in $[0, x]$ is denoted by $N(t, x)$. In [3], it is shown that, for any given $T > 0$,

$$\sup_{0 \leq t \leq T} |\mathrm{E}N(t,x) - (\alpha(t)x + \alpha(t) + \beta(t) - 1)| = O\left(e^{-\xi x \log x}\right) \tag{1}$$

for all $\xi \in (0, 1)$, where

$$\alpha(t) = \int_0^t \beta(v)dv, \quad \beta(t) = \exp\left[-2\int_0^t \frac{1-e^{-v}}{v}dv\right].$$

And for the variance, it is shown in [3] that, for any given $T > 0$,

$$\sup_{0 \leq t \leq T} \left|\sigma^2(N(t,x)) - (\mu(t)x + \mu_1(t))\right| = O(e^{-\xi x \log x}), \tag{2}$$

for all $\xi \in (0, 1)$, where $\mu(t)$ and $\mu_1(t)$ are explicitly computable (see [3, eqs. (75) and (76)]), and where $\mu(t) > 0$ for all $t > 0$.

In this paper, we study the vacant intervals, or gaps, between successive packed intervals at time t. We derive an explicit formula for the limit of the distribution function as $x \to \infty$. Our results are presented and discussed in Section 2. Most of the proof details are left to subsequent sections. In the remainder of this section, we cover the background on a closely related problem, and briefly discuss modern applications that have rekindled interest in interval packing.

The intimate relationship between the classical parking problem of Renyi [7] and our on-line interval packing problem is easy to see. In the former problem, unit-length cars are parked sequentially along a curb (interval) $[0, x]$, $x > 1$. Each car chooses a parking place independently and uniformly at random from those available, i.e., from those where it will not overlap cars already parked or the curb boundaries. This uniform parking of cars continues until every unoccupied gap is less than 1 in length, i.e., no further cars can be parked. It is verified in [3] that, as might be expected, $N(t, x)$ tends to $\tilde{N}(x)$ in distribution as $t \to \infty$, where $\tilde{N}(x)$ is the number parked at the conclusion of the parking process. Extending results of Renyi [7], Dvoretzky and Robbins [5] showed that the mean of $\tilde{N}(x)$ has the estimate in (1), once the limit $t \to \infty$ is taken, where $\alpha(\infty) = 0.748\ldots$. Similarly, the combined results of Dvoretzky and Robbins [5] and Mackenzie [6] showed that the variance of $\tilde{N}(x)$ has the estimate in (2), once the limit $t \to \infty$ is taken; in this limit

$$\begin{aligned} \mu(\infty) = \mu_1(\infty) &= 4\int_0^\infty \left[e^{-y}(1-e^{-y})\frac{\tilde{\alpha}(y)}{y} - \frac{e^{-2y}(e^{-y}-1+y)\tilde{\alpha}^2(y)}{\beta(y)y^2}\right] dy - \alpha(\infty), \\ &= 0.0381\ldots, \end{aligned}$$

where $\tilde{\alpha}(y) := \alpha(\infty) - \alpha(y)$.

Although one expects some strong form of convergence of $N(t,x)$ to $\tilde{N}(x)$, it is surprising at first to find that, for all $x > 2$, the expected time to absorption of the interval packing process is infinite. It is shown in [3] that, if T_x denotes the time-to-absorption of the interval packing process, then for any fixed x, $\mathbf{P}(T_x > t)$ tends to 0 like $1/t$, and hence, T_x is finite almost surely, but $\mathrm{E}T_x = \infty$.

Dvoretzky and Robbins [5] gave a central limit theorem for the parking problem, basing the second of their two proofs on various properties of $\tilde{N}(x)$ also shared by $N(t,x)$. In [3], it is observed that one can adapt the technique to the interval packing problem so as to obtain a central limit theorem for any fixed t. The details of a rigorous proof can be found in [2]. Bankovi [1] investigated the distribution of vacant intervals for the parking problem, as did Mackenzie [6] for a discretized version of the problem. This paper generalizes their results to the interval packing problem. For additional literature on the parking problem, see [3].

We conclude this section with a few comments on applications. Modeling reservation protocols in communication systems was the source of our interval packing problem (see e.g., [3] and the references therein). In the baseline model, there is a single resource and there are randomly arriving requests, each specifying a future time interval during which it wants to use the resource. A request arriving at time t identifies the desired interval $[t_1, t_2]$ by giving the advance notice $t_1 - t$ and the duration $t_2 - t_1$. Scheduling decisions are made on-line: a requested reservation is approved/accepted if and only if the specified interval does not overlap an interval already reserved for some earlier request. In the stochastic set-up, requests are Poisson arrivals at rate λ, advance notices are independently and uniformly distributed over $[0,a]$ for some given a, and intervals have unit durations. Suppose that, at some time t in equilibrium, we look at the pattern of vacant intervals that were created by the reservations made during $[t-x,t]$ for some large x. If a is large relative to x, one expects that, except for negligible edge effects, this pattern is approximately the same stochastically as the pattern of vacant intervals created in $[0,x]$ according to the Poisson model of on-line interval packing during the time interval $[0,\lambda]$. This statement is made rigorous and a corresponding limit law proved in [4]; certain generalizations of the above model are also accommodated.

2 The Main Results

Define the function

$$p(t,y,x) := \frac{V_y(t,x)}{K(t,x)+1},$$

where $V_y(t,x)$ is the expected number of vacant intervals of length at most y in $[0,x]$ at time t, and $K(t,x)$ is the expected number of intervals packed in $[0,x]$ at time t (so $K(t,x)+1$ is the expected number of vacant intervals in $[0,x]$ at time t). We will prove

Theorem 1 *The limit $p(t,y) := \lim_{x\to\infty} p(t,y,x)$ exists for all $t, y \geq 0$, and is given by*

$$p(t,y) = \begin{cases} \frac{2\int_0^t \left(1-e^{-vy}\right)\beta(v)dv}{\alpha(t)}, & y < 1, \\ p(t,1) + \frac{\left(1-e^{-t(y-1)}\right)t\beta(t)}{\alpha(t)}, & y \geq 1. \end{cases} \tag{3}$$

For fixed t, (3) shows that $p(t,y)$ is a continuous, strictly increasing function of y, with $p(t,0)=0$. In Corollary 5, we also verify that $p(t,\infty) := \lim_{y\to\infty} p(t,y) = 1$, so $p(t,y)$ is in fact a distribution function in y. A differentiation of (3) shows that the derivative of $p(t,y)$ with respect to y is also continuous, except for a jump of $t^2\beta(t)/\alpha(t)$ at $y=1$. In Corollary 6, we will prove that $p(t,y)$ is strictly increasing in t for every y, and in Corollary 7, we will compute the mean gap size $m(t)$ from the distribution in Theorem 1.

As defined above, $p(t,y)$ is the limit, as $x\to\infty$, of the ratio of two expectations. But we will go further and prove that $p(t,y)$ is the stochastic limit of the empirical distribution of the gap sizes. This is done in Theorem 4, but in preparation, we need Theorems 2 and 3 giving large-x asymptotic results for the first and second moments of the number $N_y(t,x)$ of gaps of length at most y in $[0,x]$ at time t. For the first moment, we have

Theorem 2 *As $x\to\infty$,*

$$V_y(t,x) = \alpha(t)p(t,y)x + O(1).$$

And for the second moment, $M_y(t,x) := \mathrm{E}[N_y^2(t,x)]$, we have

Theorem 3 *As $x\to\infty$,*

$$M_y(t,x) = [\alpha(t)p(t,y)]^2x^2 + O(x),$$

and hence for the variance, we obtain $\sigma_y^2(t,x) = O(x)$, as $x\to\infty$.

We remark that the asymptotics of Theorems 2 and 3 can be improved, but the improvements require cumbersome calculations and do not seem interesting. Our interest in Theorems 2 and 3 is restricted to their use in deriving Theorem 4.

Let $N(t,x)$ be the number of intervals packed in $[0,x]$ at time t, and define the empirical distribution of gap sizes,

$$F^*_{t,x}(y) := \frac{N_y(t,x)}{N(t,x)},$$

the proportion of gaps of length at most y. Let $\xrightarrow{P}$ denote stochastic convergence (i.e., convergence in probability).

Theorem 4 *For all $y\geq 0$, we have*

$$F^*_{t,x}(y) \xrightarrow{P} p(t,y), \text{ as } x\to\infty.$$

Proof. From Theorems 2 and 3 and Chebyshev's inequality, we obtain

$$\frac{N_y(t,x)}{\alpha(t)p(t,y)x} \xrightarrow{P} 1, \text{ as } x\to\infty.$$

Similarly, from Theorems 4 and 13 in [3] and Chebyshev's inequality, we get

$$\frac{N(t,x)}{\alpha(t)x} \xrightarrow{P} 1, \text{ as } x\to\infty,$$

which when divided into the previous equation gives the theorem. ∎

Note that, from (1), it is easy to argue what the formula for the mean gap size $m(t)$ should be. The amount of an interval $[0,x]$ that is vacant at time t is $[1-\alpha(t)]x+O(1)$ for large x, and the number of packed intervals, and hence gaps, in $[0,x]$ at time t is $\alpha(t)x+O(1)$, for large x, so the mean gap size is

$$\frac{[1-\alpha(t)]x+O(1)}{\alpha(t)x+O(1)} \sim m(t) = \frac{1-\alpha(t)}{\alpha(t)}, \text{ as } x \to \infty.$$

The fast convergence of $m(t)$ as $t \to \infty$ is shown in Figure 1. The limiting value is $\lim_{t\to\infty} m(t) = 0.337\ldots$.

the pde; and (4) applying Karamata's theorem to the formula for $\mathcal{V}_y$ obtained in step (3).

1. At time $t+\Delta t$, consider what happened to the initial vacant interval $[0,x]$, where $x \geq 1$, during the first Δt time units. If there was no arrival, then $V_y(t+\Delta t, x) = V_y(t,x)$, and if there was, say at $[z, z+1]$, $z \leq x-1$, then $V_y(t+\Delta t, x) = V_y(t,z) + V_y(t, x-1-z)$. Thus, according to the Poisson arrival process, we have

$$V_y(t+\Delta t, x) = [1-(x-1)\Delta t]V(t,x) + \Delta t \int_0^{x-1} [V_y(t,z) + V_y(t, x-1-z)]dz + o(\Delta t), \ x \geq 1, \ t \geq 0.$$

Dividing by Δt, letting $\Delta t \to 0$, and exploiting symmetry, we obtain

$$\partial V_y$$

3. A formula for $\mathcal{V}_y$ can be found by using results in [3]. To this end, rewrite (8) as follows. Let $\mathcal{V}_y(t,u)=\mathcal{V}_y^{(1)}(t,u)+\mathcal{V}_y^{(2)}(t,u)$, where $\mathcal{V}_y^{(1)},\ \mathcal{V}_y^{(2)}$ are defined to be the respective solutions to

$$\begin{aligned}\frac{\partial \mathcal{V}_y^{(1)}}{\partial t} &= \frac{\partial \mathcal{V}_y^{(1)}}{\partial u}+\left(1+\frac{2e^{-u}}{u}\right)\mathcal{V}_y^{(1)}+\frac{2e^{-u}}{u^2}g_y(u), \quad \mathcal{V}_y^{(1)}(0,u)=0, \quad u>0, &(10)\\ \frac{\partial \mathcal{V}_y^{(2)}}{\partial t} &= \frac{\partial \mathcal{V}_y^{(2)}}{\partial u}+\left(1+\frac{2e^{-u}}{u}\right)\mathcal{V}_y^{(2)}, \quad \mathcal{V}_y^{(2)}(0,u)=h_y(u), &(11)\end{aligned}$$

Equation (10) is identical to (51) in [3], with $2g_y(u)$ replacing $\mathcal{A}(t,u)$. Hence, by (54) in [3],

$$\mathcal{V}_y^{(1)}=\frac{2e^{-u}}{u^2\beta(u)}\int_u^{u+t} g_y(v)\beta(v)dv. \tag{12}$$

To solve (11), let $y>1$, since for $y\le 1$, we have $\mathcal{V}_y^{(2)}=0$. Make the change of variables $r=u+t,\ s=t$ which transforms (11) into a pde for $\mathcal{V}_y^{(2)}=\mathcal{V}_y^{(2)}(s,r-s)$,

$$\frac{\partial \mathcal{V}_y^{(2)}}{\partial s}=\left(1+2\frac{e^{s-r}}{r-s}\right)\mathcal{V}_y^{(2)}(s,r-s), \quad \mathcal{V}_y^{(2)}(0,r-s)=h_y(r-s),\ r>0,\ 0\le s\le r.$$

For fixed r, this is an ordinary linear differential equation for $\mathcal{V}_y^{(2)}$ in s, and has the solution

$$\mathcal{V}_y^{(2)}(s,r-s)=h_y(r-s)\exp\left[\int_0^s\left(1+\frac{2e^{v-r}}{r-v}\right)dv\right].$$

Then in terms of the t,u variables, elementary manipulations show that

$$\mathcal{V}_y^{(2)}(t,u)=\frac{e^{-u}-e^{-yu-(y-1)t}}{u^2\beta(u)}(u+t)\beta(u+t). \tag{13}$$

which together with (12) yields the desired formula

$$\mathcal{V}_y(t,u) = \frac{2e^{-u}}{u^2\beta(u)}\int_u^{u+t} g_y(v)\beta(v)dv+\chi(y)\frac{e^{-u}-e^{-yu-(y-1)t}}{u^2\beta(u)}(u+t)\beta(u+t), \tag{14}$$

where $\chi(y)=0$ for $y<1$, and $\chi(y)=1$ for $y\ge 1$.

4. To complete the proof of Theorem 1, consider first the limit $u\to 0$ of (14). We find that, as $u\to 0$,

$$\mathcal{V}_y(t,u)\sim\begin{cases}\frac{2\int_0^t g_y(v)\beta(v)dv}{u^2}, & \text{if } y\le 1\\ \frac{2\int_0^t g_y(v)\beta(v)dv+\left(1-e^{-(y-1)t}\right)t\beta(t)}{u^2}, & \text{if } y>1\end{cases}$$

so by Karamata's Tauberian theorem,

$$\lim_{x\to\infty}\frac{\int_0^x V_y(t,z)dz}{x^2/2}=\begin{cases}2\int_0^t g_y(v)\beta(v)dv, & \text{if } y\le 1\\ 2\int_0^t g_y(v)\beta(v)dv+\left(1-e^{-(y-1)t}\right)t\beta(t), & \text{if } y>1.\end{cases} \tag{15}$$

Duplicating the reasoning of Theorem 2 in [3], we can replace the left-hand side of (15) by $\lim_{x\to\infty} V_y(t,x)/x$. Theorem 2 in [3] shows that

$$\lim_{x\to\infty} \frac{K(t,x)}{x} = \alpha(t),$$

which together with (15) proves Theorem 1. ∎

We now prove the corollaries to Theorem 1 that were mentioned in Section 2.

Corollary 5 *For fixed t, $p(t,y)$ is a distribution function in y, i.e., $p(t,\infty) = 1$.*

Proof: First, observe that $\beta'(v) = -2(1-e^{-v})\beta(v)/v$, so an integration by parts gives

$$2\int_0^t (1-e^{-v})\beta(v)dv = -\int_0^t v\beta'(v)dv = -t\beta(t) + \alpha(t),$$

Now substitute this into

$$p(t,\infty) = \frac{2\int_0^t (1-e^{-v})\beta(v)dv + t\beta(t)}{\alpha(t)},$$

which we obtain from Theorem 1. The corollary follows. ∎

Corollary 6 *The following holds.*

(i) *For fixed $y > 0$, $p(t,y)$ is strictly increasing in t, and*

(ii) $\lim_{t\to\infty} p(t,y) = 1$ *uniformly in $y \geq 1$.*

Proof: For part (i), consider first $0 < y \leq 1$. Differentiate (3) and get

$$\begin{aligned}
\frac{\partial p}{\partial t} &= \frac{\alpha(t)\cdot 2(1-e^{-ty})\beta(t) - \int_0^t (1-e^{-vy})\beta(v)dv \cdot \beta(t)}{\alpha^2(t)} \\
&= \frac{2\beta(t)}{\alpha^2(t)}\left[\int_0^t (1-e^{-ty})\beta(v)dv - \int_0^t (1-e^{-vy})\beta(v)dv\right] \\
&= \frac{2\beta(t)}{\alpha^2(t)}\int_0^t (e^{-vy} - e^{-ty})\beta(v)dv
\end{aligned}$$

But e^{-vy} is decreasing in v, so $\frac{\partial p}{\partial t} > 0$ and part (i) is proved for $0 < y \leq 1$.

For $y > 1$, let $y \to \infty$ in (3), apply $p(t,\infty) = 1$, and obtain

$$p(t,1) + t\beta(t)/\alpha(t) = 1, \tag{16}$$

which together with (3) gives

$$1 - p(t,y) = e^{-t(y-1)}[1 - p(t,1)], \text{ for } y \geq 1.$$

Since both $e^{-t(y-1)}$ and $[1-p(t,1)]$ are positive, strictly decreasing functions of t, this proves (i) for $y>1$.

For part (ii), it suffices to show that $\lim_{t\to\infty} p(t,1)=1$, since $p(t,y)$ increases in y. But $\beta(t)=O(1/t^2)$ as $t\to\infty$, so we obtain the desired result by letting $t\to\infty$ in (16). ∎

Corollary 7 *The limiting mean gap length (i.e., the mean of $p(t,y)$) is*

$$m(t)=\frac{1-\alpha(t)}{\alpha(t)}.$$

Proof: By (1) for large x, there are asymptotically $\alpha(t)x$ vacant intervals with a total size of $[1-\alpha(t)]x$, so $m(t)=[1-\alpha(t)]/\alpha(t)$ follows from a simple argument. But let us determine $m(t)$ by differentiating (3), as a check on (3). Accordingly, we have the density

$$\frac{\partial p}{\partial y}=\frac{2\int_0^t v\beta(v)e^{-vy}dv+\chi(y)t^2e^{-t(y-1)}\beta(t)}{\int_0^t\beta(v)dv}$$

where $\chi(y)=1$ if $y>1$ and is 0 otherwise. Thus, the expected value is given by

$$\begin{aligned} m(t) &= \frac{2\int_0^1\int_0^t vy\beta(v)e^{-vy}dvdy+t^2\beta(t)\int_1^\infty ye^{-t(y-1)}\beta(t)}{\int_0^t\beta(v)dv} \\ &= \frac{2\int_0^t\beta(v)dv\int_0^1 vye^{-vy}dy+t^2\beta(t)\int_0^\infty(y+1)e^{-ty}dy}{\int_0^t\beta(v)dv} \end{aligned}$$

Now substitute from

$$\int_0^1 vye^{-vy}dy=\frac{1}{v}\int_0^v ye^{-y}dy=\frac{1-e^{-v}}{v}-e^{-v}$$

and

$$\int_0^1(y+1)e^{-ty}dy=\frac{1}{t}+\frac{1}{t^2}$$

to obtain

$$m(t)=\frac{2\int_0^t\beta(v)\left(\frac{1-e^{-v}}{v}\right)dv-2\int_0^t\beta(v)e^{-v}dv+(t+1)\beta(t)}{\int_0^t\beta(v)dv}.$$

Now $\beta'(v)=-2(1-e^{-v})\beta(v)/v$, so after routine manipulations we find that $2\int_0^t\beta(v)\left(\frac{1-e^{-v}}{v}\right)dv=1-\beta(t)$, and $-2\int_0^t\beta(v)e^{-v}dv=-t\beta(t)-\int_0^t\beta(v)dv$; after substitution into the above expression for $m(t)$, we get the desired formula. ∎

4 The Second-Moment Transform

We derive a formula for $\mathcal{M}_y(t,u)$ using the method of the previous section. Let $t\geq 0$, and $u>0$. We begin by computing an integro-differential equation for $\mathcal{M}_y$. Let $x\geq 1$ and let

$N_y(t,x,z)$ denote the conditional number of gaps at time t in $[0,x]$ with lengths at most y, given that at time 0 an interval was packed in $[z,z+1]$ with $0 \le z \le x-1$. Clearly, $N_y(t,x,z) = N_y(t,z) + N_y(t,x-1-z)$. But $N_y(t,z)$ and $N_y(t,x-1-z)$ are independent, so we can write

$$\begin{aligned} M_y(t+\Delta t, x) &= [1-(x-1)\Delta t] M_y(t,x) + \Delta t \int_0^{x-1} \mathrm{E}[N_y(t,z) + N_y(t,x-1-z)]^2 dz + o(\Delta t) \\ &= [1-(x-1)\Delta t] M_y(t,x) + \\ &\qquad 2\Delta t \left[\int_0^{x-1} M_y(t,z) dz + \int_0^{x-1} V_y(t,z) V_y(t,x-1-z) dz \right] + o(\Delta t) \end{aligned}$$

and then divide by Δt and take the limit $\Delta t \to 0$ to get

$$\frac{\partial M_y}{\partial t} = -(x-1)M_y + 2\int_0^{x-1} M_y(t,z)dz + 2\int_0^{x-1} V_y(t,z)V_y(t,x-1-z)dz, \quad x \ge 1, \ t \ge 0. \tag{17}$$

We must solve (17) subject to the boundary conditions in (5) and (6). Taking the Laplace transform of (17), we obtain

$$\begin{aligned} \frac{\partial \mathcal{M}_y}{\partial t} = \mathcal{M}_y + \frac{\partial \mathcal{M}_y}{\partial u} \quad &+ \quad \frac{2e^{-u}}{u} \left[\mathcal{M}_y + \int_0^1 M(t,z) e^{-uz} dz \right] \\ &+ 2e^{-u} \left[\mathcal{V}_y + \int_0^1 \mathcal{V}_y(t,z) e^{-uz} dz \right]^2 . \end{aligned}$$

Now, $\int_0^1 M(t,z)e^{-uz}dz = \int_0^1 V(t,z)e^{-uz}dz = g_y(u)/u$, where $g_y(u)$ is given by (7), so the pde can be rewritten

$$\frac{\partial \mathcal{M}_y}{\partial t} = \frac{\partial \mathcal{M}_y}{\partial u} + \left(1 + \frac{2e^{-u}}{u}\right) \mathcal{M}_y + \frac{2e^{-u}}{u^2} \mathcal{A}_y(t,u), \ t \ge 0, \ u > 0, \tag{18}$$

where

$$\mathcal{A}_y(t,u) := (u\mathcal{V}_y)^2 + 2g_y(u)u\mathcal{V}_y + g_y(u) + g_y^2(u). \tag{19}$$

The boundary condition that $\mathcal{M}_y$ must satisfy is $\mathcal{M}_y(0,u) = h_y(u)$, where $h_y(u)$ is given by (9). To solve (18), let $\mathcal{M}_y = \mathcal{M}^{(1)} + \mathcal{M}_y^{(2)}$, where $\mathcal{M}^{(1)}$, $\mathcal{M}_y^{(2)}$ are the respective solutions to

$$\frac{\partial \mathcal{M}_y^{(1)}}{\partial t} = \frac{\partial \mathcal{M}_y^{(1)}}{\partial u} + \left(1 + \frac{2e^{-u}}{u}\right) \mathcal{M}_y^{(1)} + \frac{2e^{-u}}{u^2} \mathcal{A}_y(t,u), \tag{20}$$

$$\frac{\partial \mathcal{M}_y^{(2)}}{\partial t} = \frac{\partial \mathcal{M}_y^{(2)}}{\partial u} + \left(1 + \frac{2e^{-u}}{u}\right) \mathcal{M}_y^{(2)}, \text{ with } \mathcal{M}_y^{(2)}(0,u) = h_y(u), \ u > 0. \tag{21}$$

Note that (20) is similar in form to (10). Applying (54) in [3], as in the last section, we obtain

$$\begin{aligned} \mathcal{M}_y^{(1)}(t,u) &= e^{-u} \int_0^t \frac{\mathcal{A}_y(t-v, u+v)}{(u+v)^2} \exp\left[2\int_u^{u+v} \frac{e^{-z}}{z} dz \right] dv \\ &= \frac{e^{-u}}{u^2 \beta(u)} \int_u^{u+v} \mathcal{A}_y(t+u-v, v) \beta(v) dv \end{aligned} \tag{22}$$

Now (21) is (9) with $\mathcal{V}_y^{(2)}$ replaced by $\mathcal{M}_y^{(2)}$. Hence, $\mathcal{M}_y^{(2)}(t,u) = \mathcal{V}_y^{(2)}(t,u)$, and we get

$$\mathcal{M}_y(t,u) = \frac{e^{-u}}{u^2\beta(u)} \int_u^{u+v} \mathcal{A}_y(t+u-v,v)\beta(v)dv + \mathcal{V}_y^{(2)}(t,u),$$

where $\mathcal{V}_y^{(2)}(t,u)$ is given by (14).

5 Properties of $\mathcal{V}_y$ and $\mathcal{M}_y$

As pointed out earlier, we obtain the estimates of Theorems 2 and 3 for V_y and M_y by analyzing the inverses of the transforms $\mathcal{V}_y$ and $\mathcal{M}_y$. For this purpose, we need analyticity properties and estimates on the growth of the transforms.

Analyticity. We begin with $\mathcal{V}_y$.

Lemma 8 *The transform $\mathcal{V}_y(t,u)$ is analytic in u except at $u=0$ where it has the estimate*

$$\mathcal{V}_y(t,u) = \frac{k_y(t)}{u^2} + O\left(\frac{1}{u}\right) \text{ as } u \to 0,$$

with

$$k_y(t) := 2\int_0^t g_y(v)\beta(v)dv + \chi(y)\left(1 - e^{-(y-1)t}\right)t\beta(t).$$

Proof: Inspection of (12) shows that $\mathcal{V}_y^{(1)}(t,u)$ is analytic for $u \neq 0$, and since $\beta(0)=1$, it also shows that

$$\mathcal{V}_y^{(1)}(t,u) = \frac{2\int_0^t g_y(v)\beta(v)dv}{u^2} + O\left(\frac{1}{u}\right) \text{ as } u \to 0.$$

For $y \leq 1$, $\mathcal{V}_y^{(2)}(t,u) = 0$, and for $y > 1$, (13) shows that $\mathcal{V}_y^{(2)}(t,u)$ is analytic for $u \neq 0$ and that

$$\mathcal{V}_y^{(2)}(t,u) = \frac{\left(1-e^{-(y-1)t}\right)t\beta(t)}{u^2} + O\left(\frac{1}{u}\right) \text{ as } u \to 0.$$

The lemma follows immediately, since $\mathcal{V}_y = \mathcal{V}_y^{(1)} + \mathcal{V}_y^{(2)}$. ∎

Our study of $\mathcal{M}_y$ first requires the analytic properties of $\mathcal{A}_y(t-u,u)$ and $\mathcal{B}_y(t,u) := \mathcal{A}_y(t-u,u)\beta(u)$

Lemma 9 *For fixed $t \geq 0$, $\mathcal{A}_y(t-u,u)$ and $\mathcal{B}_y(t,u)$ are analytic for $u \neq 0$. At $u=0$, these functions have the expansions*

$$\mathcal{A}_y(t-u,u) = \frac{k_y^2(t)}{u^2} + \frac{2k_y^2(t)}{u} + \cdots \quad (23)$$

$$\mathcal{B}_y(t,u) = \frac{k_y^2(t)}{u^2} + \frac{0}{u} + \cdots. \quad (24)$$

Proof: By (12) and (13), we have

$$\mathcal{V}_y^{(1)}(t-u,u) = \frac{e^{-u}}{u^2\beta(u)}\left[2\int_u^t g_y(v)\beta(v)dv\right] \tag{25}$$

$$\mathcal{V}_y^{(2)}(t-u,u) = \chi(y)\left(1-e^{-(y-1)t}\right)t\beta(t)\frac{e^{-u}}{u^2\beta(u)} \tag{26}$$

from which we see that $\mathcal{V}_y(t-u,u)$ is analytic for $u \neq 0$. Then (19) shows that both $\mathcal{A}_y(t-u,u)$ and $\mathcal{B}_y(t,u)$ are analytic for $u \neq 0$. We have the expansion

$$\frac{e^{-u}}{u^2\beta(u)} = \frac{1}{u^2} + \frac{1}{u} + \cdots,$$

and, since $g_y(0) = 0$, the expansion

$$\int_u^t g_y(v)\beta(v)dv = \int_0^t g_y(v)\beta(v)dv + 0\cdot u + \cdots.$$

Equations (25) and (26) together with the above expansions give

$$\mathcal{V}_y(t-u,u) = \frac{k_y(t)}{u^2} + \frac{k_y(t)}{u} + \cdots.$$

Since $g_y(u)$ is entire with $g_y(0) = 0$, this last expansion implies (23), and since $\beta(u) = 1 - 2u + \cdots$, (23) in turn implies (24). ∎

Lemma 10 *For fixed t, the transform $\mathcal{M}_y(t,u)$ is analytic in u for $u \neq 0$ and satisfies*

$$\mathcal{M}_y(t,u) = \frac{2k_y^2(t)}{u^3} + O\left(\frac{1}{u^2}\right) \text{ as } u \to 0.$$

Proof: Define $\kappa_y(t,u) := \mathcal{B}_y(t,u) - 2k_y^2(t)/u^2$. From the formula for $\mathcal{V}_y(u)$ and elementary manipulations, we may rewrite $\kappa_y(t,u)$ as

$$\kappa_y(t,u) = k_y^2(t)f_{1,y}(u) + k_y f_{2,y}(u) + f_{3,y}(u)$$

where $f_{i,y}, 1 \le i \le 3$, are specified, entire functions of u. This formula shows that $\kappa_y(t,u)$ is entire in t,u. Rewrite (22) as

$$\begin{aligned}\mathcal{M}_y^{(1)}(t,u) &= \frac{e^{-u}}{u^2\beta(u)}\int_u^{t+u}\left[\kappa_y(t+u,v) + \frac{2k_y^2(t+u)}{v^2}\right]dv \\ &= \frac{e^{-u}}{u^2\beta(u)}\left[\int_u^{t+u}\kappa_y(t+u,v)dv + \frac{2k_y^2(t+u)}{u} - \frac{2k_y^2(t+u)}{t+u}\right].\end{aligned}$$

In this form, we see that $\mathcal{M}_y^{(1)}$ is analytic for $u \neq 0$ and satisfies $\mathcal{M}_y^{(1)}(t,u) = \frac{2k_y^2(t)}{u^3} + O(\frac{1}{u^2})$ as $u \to 0$. Next, by (13), $\mathcal{M}_y^{(2)} = \mathcal{V}_y^{(2)}$ is analytic for $u \neq 0$, and has a second order pole at $u = 0$. The lemma follows immediately, since $\mathcal{M}_y = \mathcal{M}_y^{(1)} + \mathcal{M}_y^{(2)}$. ∎

Estimates. We estimate $\mathcal{V}_y = \mathcal{V}_y^{(1)} + \mathcal{V}_y^{(2)}$ and $\mathcal{M}_y = \mathcal{M}_y^{(1)} + \mathcal{M}_y^{(2)}$, but since $\mathcal{V}_y^{(2)} = \mathcal{M}_y^{(2)}$, we need only estimate $\mathcal{V}_y^{(1)}$, $\mathcal{V}_y^{(2)}$, and $\mathcal{M}_y^{(1)}$. For $y > 1$, we shall also require an estimate for $\mathcal{V}_y^{(2,2)}$, where $\mathcal{V}_y^{(2)} = \mathcal{V}_y^{(2,1)} + \mathcal{V}_y^{(2,2)}$ with

$$\begin{aligned} \mathcal{V}_y^{(2,1)} &= \frac{e^{-u} - e^{-yu-(y-1)t}}{u+t} && (27) \\ \mathcal{V}_y^{(2,2)} &= \mathcal{V}_y^{(2,1)} \left(\exp\left[2\int_u^{u+t} \frac{e^{-x}}{x} dx \right] - 1 \right). && (28) \end{aligned}$$

It proves convenient to rewrite (12), (22), and (13) using the identity

$$\frac{\beta(v)}{\beta(u)} = \frac{u^2}{v^2} \exp\left[2\int_u^v \frac{e^{-x}}{x} dx \right].$$

We obtain in so doing

$$\begin{aligned} \mathcal{V}_y^{(1)} &= 2e^{-u} \int_0^t \frac{g_y(u+v)}{(u+v)^2} \exp\left[2\int_u^{u+v} \frac{e^{-x}}{x} dx \right] dv && (29) \\ \mathcal{M}_y^{(1)} &= 2e^{-u} \int_0^t \frac{\mathcal{A}_y(t-v, u+v)}{(u+v)^2} \exp\left[2\int_u^{u+v} \frac{e^{-x}}{x} dx \right] dv && (30) \\ \mathcal{V}_y^{(2)} &= \mathcal{M}_y^{(2)} = \chi(y) \frac{e^{-u} - e^{-yu-(y-1)t}}{u+t} \exp\left[2\int_u^{u+v} \frac{e^{-x}}{x} dx \right] dv. && (31) \end{aligned}$$

In the lemma to follow, $c(\sigma)$ denotes a generic constant depending only on σ.

Lemma 11 *Assume that $t > 0$ and $\Re u \geq \sigma$. Then*

(i) $\left| \mathcal{V}_y^{(2,2)}(t,u) \right| \leq \frac{c(\sigma)t}{|u|^2}, \ |u| \geq 2t$

(ii) $\left| \mathcal{V}_y^{(2)}(t,u) \right| \leq \frac{c(\sigma)}{|u|}, \ |u| \geq 2t$

(iii) $\left| \mathcal{V}_y^{(1)}(t,u) \right| \leq \frac{c(\sigma)t}{|u|^2}, \ |u| \geq 2t$

(iv) $\left| \mathcal{M}_y^{(1)}(t,u) \right| \leq \frac{c(\sigma)t}{|u|^2}, \ |u| \geq 3t.$

Proof: We use repeatedly the elementary estimates

$$|e^{-u}| \leq e^{-\sigma}, \text{ for } \Re u \geq \sigma \qquad |u+t| \geq \frac{|u|}{2} \geq t, \text{ for } |u| \geq 2t. \tag{32}$$

Note that the conditions in the above estimates apply to the inequalities in the lemma.

Proof of (i): From (32), we get

$$\left| \int_u^{u+t} \frac{e^{-x}}{x} dx \right| = \left| \int_0^t \frac{e^{-(u+v)}}{u+v} dv \right| \leq \frac{2te^{-\sigma}}{|u|},$$

which together with the inequality $|e^z - 1| \le e^{|z|} - 1$ yields

$$\left|\exp\left(2\int_u^{u+t}\frac{e^{-x}}{x}dx\right) - 1\right| \le \exp\left(\left|2\int_u^{u+t}\frac{e^{-x}}{x}dx\right|\right) - 1 \le \exp\left(\frac{4te^{-\sigma}}{|u|}\right) - 1, \tag{33}$$

Then, by the fact that $e^z - 1 \le ze^z$, for $z \ge 0$, we get

$$\left|\exp\left(2\int_u^{u+t}\frac{e^{-x}}{x}dx\right) - 1\right| \le \frac{4te^{-\sigma}}{|u|}\exp\left(\frac{4te^{-\sigma}}{|u|}\right) \le \frac{4t\exp(-\sigma + 2e^{-\sigma})}{|u|}.$$

It also follows from (32) that, for $y > 1$,

$$\left|\frac{e^{-u} - e^{-yu-(y-1)t}}{u+t}\right| \le 2\frac{e^{-\sigma} + e^{-y\sigma}}{|u|}. \tag{34}$$

Then (27) and (28) together with the above inequalities show that (i) holds with the choice

$$c(\sigma) = 8(e^{-\sigma} + e^{-y\sigma})\exp(-\sigma + 2e^{-\sigma}).$$

∎

Proof of (ii): The estimates in (33) imply

$$\left|\exp\left(2\int_u^{u+t}\frac{e^{-x}}{x}dx\right)\right| \le \exp\left(\frac{4te^{-\sigma}}{|u|}\right) \le \exp(2e^{-\sigma}), \tag{35}$$

which combined with (31) and (34) gives (ii) with the choice

$$c(\sigma) = 2(e^{-\sigma} + e^{-y\sigma})\exp(2e^{-\sigma}).$$

∎

Proof of (iii): We have

$$|g_y(u)| \le a_y(\sigma) := 1 + e^{-\min(1,y)\sigma}, \tag{36}$$

so by (29), (32), and (35), we obtain (iii) with the choice

$$c(\sigma) = 8a_y(\sigma)\exp(-\sigma + 2e^{-\sigma}).$$

∎

Proof of (iv): We first estimate $\mathcal{A}_y(t,u)$ for $\Re u \ge \sigma$ and $|u| \ge 2t$. We see from (iii) that

$$|u\mathcal{V}_y^{(1)}| \le \frac{8a_y(\sigma)t\exp(-\sigma + 2e^{-\sigma})}{|u|} \le 4a_y(\sigma)\exp(-\sigma + 2e^{-\sigma}) \tag{37}$$

and from (ii)

$$|u\mathcal{V}_y^{(2)}| \le 2(e^{-\sigma} + e^{-y\sigma})\exp(2e^{-\sigma}),$$

so

$$|u\mathcal{V}_y| \le b_y(\sigma) := 4a_y(\sigma)\exp(-\sigma + 2e^{-\sigma}) + 2(e^{-\sigma} + e^{-y\sigma})\exp(2e^{-\sigma}). \tag{38}$$

Assembling (19), (36), and (38), we find that

$$|\mathcal{A}_y(t,u)| \le c_1(\sigma) := b_y^2(\sigma) + 2a_y(\sigma)b_y(\sigma) + a_y(\sigma) + a_y^2(\sigma), \text{ for } \Re u \ge \sigma,\ |u| \ge 2t. \tag{39}$$

Now if we keep with $\Re u \ge \sigma$, but require $|u| \ge 3t$, then for $0 \le v \le t$, (39) implies $|\mathcal{A}_y(t-v, u+v)| \le c_1(\sigma)$, which together with (30), (32), and (35) yields the desired bound in (iv) with the choice

$$c(\sigma) = 8c_1(\sigma)\exp(-\sigma + 2e^{-\sigma}).$$

∎

6 Proofs of Theorems 2 and 3

The proofs are based on the estimates of Lemma 8. We work out the details for Theorem 2 but omit them for Theorem 3; except for minor details, the proof of Theorem 3 mirrors that of Theorem 2. The only essential difference is that the residue at $u = 0$ of $\mathcal{V}_y e^{ux}$ is linear in x (cf. (40) below), while that of $\mathcal{M}_y e^{ux}$ is quadratic in x.

We use the Laplace inversion formula,

$$V_y(t,x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{V}_y(t,u)e^{ux}du,$$

the integration path being the vertical line $\Re u = \sigma$ directed upward, with σ any positive real. We shift the integration path to the left of the origin and apply the residue theorem. We observe first from Lemma 8 that $\mathcal{V}_y(t,u)e^{ux}$ is analytic everywhere except at the origin where it has a pole. From Lemma 8 and the power series $e^{ux} = 1 + xu + \cdots$, we find that

$$\mathcal{V}_y(t,u)e^{ux} = \frac{k_y(t)}{u^2} + \frac{k_y(t)x + \ell}{u} + \cdots,\ u \ne 0, \tag{40}$$

where ℓ is the coefficient of u^{-1} in the expansion for $\mathcal{V}_y$. By the residue theorem,

$$\frac{1}{2\pi i}\int_{\Gamma(\rho)} \mathcal{V}(t,u)e^{xu}du = k_y(t)x + \ell,$$

where $\Gamma(\rho)$ is the rectangular contour sketched in Figure 2, with $\sigma, \rho > 0$ and $\vartheta \ge 2t$. The estimates in Lemma 11(iii) and (iv) show that the contribution of the horizontal sides of $\Gamma(\rho)$ tends to 0 as $\rho \to \infty$, so as $\rho \to \infty$, the contour integral becomes

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{V}_y(t,u)e^{ux}du = k_y(t)x + \ell + \frac{1}{2\pi i}\int_{-\vartheta-i\infty}^{-\vartheta+i\infty} \mathcal{V}_y(t,u)e^{ux}du.$$

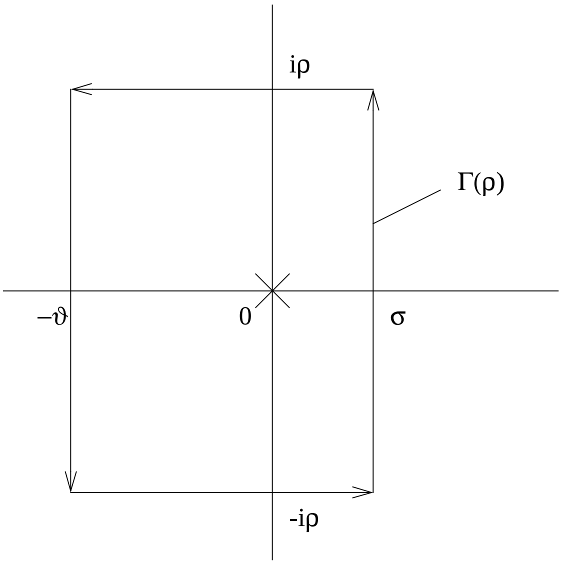

Figure 2: Rectangular contour of integration.

Then Theorem 2 is proved, provided we can show that

$$\frac{1}{2\pi i}\int_{-\vartheta-i\infty}^{-\vartheta+i\infty} \mathcal{V}_y(t,u)e^{ux}du \to 0 \text{ as } x\to\infty. \tag{41}$$

To show this, we write

$$\mathcal{V}_y = \mathcal{V}_y^{(1)} + \mathcal{V}_y^{(2,1)} + \mathcal{V}_y^{(2,2)}$$

and prove that the contribution of each term to the integral tends to 0 as $x\to\infty$. By Lemma 11(iii),

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{-\vartheta-i\infty}^{-\vartheta+i\infty} \mathcal{V}_y(t,u)e^{ux}du\right| &\le c(-\vartheta)e^{-\vartheta x}\int_{-\infty}^{\infty}\frac{dy}{\vartheta^2+y^2}\\ &= \frac{\pi c(-\vartheta)}{v}e^{-\vartheta x},\end{aligned}$$

which also holds for $\mathcal{V}^{(2,2)}$, by Lemma 11(ii). Thus,

$$\frac{1}{2\pi i}\int_{-\vartheta-i\infty}^{-\vartheta+\infty} (\mathcal{V}_y^{(1)}+\mathcal{V}_y^{(2,2)})e^{ux}du \to 0 \text{ as } x\to 0. \tag{42}$$

For the contribution of $\mathcal{V}^{(2,1)}$, we need an additional argument, as we now have only $\mathcal{V}^{(2,1)} = O(1/|u|)$ (not $O(1/|u|^2)$). The argument reduces to integration by parts. We first write

$$\begin{aligned}\int_{-\vartheta-i\infty}^{-\vartheta+i\infty} \mathcal{V}_y^{(2,1)}(t,u)e^{ux}du &= \int_{-\vartheta-i\infty}^{-\vartheta+i\infty}\frac{e^{(x-1)u}du}{u+t} - \int_{-\vartheta-i\infty}^{-\vartheta+i\infty}\frac{e^{(x-y)u-(y-1)t}du}{u+t}\\ &:= I_1+I_2.\end{aligned}$$

For $x>1$, integration by parts gives

$$I_1 = \frac{1}{x-1}\int_{-\vartheta-i\infty}^{-\vartheta+i\infty}\frac{e^{(x-1)u}du}{(u+t)^2}$$

and so

$$
\begin{aligned}
|I_1| &\leq \frac{1}{x} e^{-\vartheta(x-1)} \int_{-\infty}^{\infty} \frac{dy}{(-\vartheta+t)^2+y^2} \\
&= \frac{\pi}{x} \frac{e^{-\vartheta(x-1)}}{\vartheta - t}.
\end{aligned}
$$

We derive a similar estimate for I_2 when $y > x$, again by integration by parts. The estimates for I_1 and I_2 yield

$$
\int_{-\vartheta-i\infty}^{-\vartheta+i\infty} \mathcal{V}_y^{(2,1)}(t,u) e^{ux} du \to 0, \text{ as } x \to \infty,
$$

which along with (42) proves (41). ∎

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