Bin Packing with Discrete Item Sizes
Part II: Average-Case Behavior of FFD and BFD

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ABSTRACT

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1. Introduction

Consider the bin packing problem in which a list L_n of n items, with sizes drawn from the unit interval, is to be partitioned into a smallest collection of blocks such that, for each block, the sum of the sizes of the items in the block is at most 1. With applications in mind, a partitioning algorithm solving this problem is said to *pack* the items of L_n into a minimum number of unit-capacity *bins*. The problem is NP-hard, so a great deal of effort has gone into the analysis of efficient approximation algorithms. The analysis of primary interest here is probabilistic, or average-case analysis (see [5] for a recent book on the subject).

This paper analyzes the expected behavior of what are perhaps the two best known off-line algorithms, First-Fit Decreasing (FFD) and Best-Fit Decreasing (BFD), described as follows. Both FFD and BFD pack the items in order of decreasing size into a sequence B_1, $B_2, \ldots$ of initially empty bins, and both begin by packing the largest item into B_1. Thereafter, FFD places the next item to be packed into the first, i.e., lowest indexed, bin having sufficient space for it, whereas BFD places the item into a bin in which it fits best, i.e., minimizes the space left over. Ties under BFD are resolved in favor of the lowest indexed bin.

In the past, the probabilistic analysis of FFD and BFD [1, 7] has assumed that item sizes are drawn from a continuous distribution on $[0, 1]$, even though models of many, if not most applications require distributions on discrete sets of item sizes. The assumption is that continuous distributions are likely to simplify the analysis while serving as reasonable approximations to discrete distributions on large sets. The goal of our analysis of FFD and BFD is to assess this approximation, to see what properties it hides, and to measure the errors it makes in standard performance metrics, such as expected wasted space.

As in the continuous case [1, 7], we focus on uniform distributions. In the discrete case this refers to the distributions $U\{j;k\}$, $1 \le j \le k$, which denote uniform distributions on the sets $\{i/k;\ 1 \le i \le j\}$ of item sizes. As we will see, the continuous model fails to reveal a variety of intricate (and interesting) aspects of FFD and BFD behavior that appear in the discrete model. Moreover, there are major differences in the average-case results for wasted space in the two models.

To illustrate this last point, let $A(L_n)$ denote the number of occupied bins in a packing by algorithm A, let $s(L_n)$ denote the sum of the sizes of the items in L_n (the occupied area of any packing of L_n), and let $W^A(L_n) = A(L_n) - s(L_n)$ denote the wasted space under algorithm A. In [1] it is shown that, if $U(0,u)$ is the uniform item-size distribution on the real interval $[0,u]$, then $E[W^{\mathrm{FFD}}(L_n)]$ is $\Theta(n^{1/2})$ if $u = 1$, $\Theta(n^{1/3})$ if $1/2 < u < 1$, and $O(1)$ if $u < 1/2$. (A further analysis of the case $u = 1/2$ can be found in [7].) By contrast, we will prove that, if $U\{j;k\}$ is the item-size distribution, then expected waste can even be linear in n. In particular, we find that $E[W^{\mathrm{FFD}}(L_n)]$ is either $\Theta(n)$, $\Theta(n^{1/2})$, or $O(1)$, depending on j,k. Moreover, $\Theta(n^{1/2})$ occurs whenever $j \in \{k-1, k\}$, which corresponds to the $u = 1$ case, but expected wastes $O(1)$ and $\Theta(n)$ both occur for specific pairs j,k with $j/k < 1/2$ and with $j/k > 1/2$.

More detailed versions of these asymptotic results are given in Section 2, after certain key combinatorial properties of FFD and BFD packings are introduced. Section 3 then delves into the problem of classifying pairs j,k in terms of the expected waste under the distributions $U\{j,k\}$. Section 3 also presents a tight upper bound on the expected waste over all pairs j,k. The probabilistic analysis of this paper is confined largely to Section 4, where we prove the asymptotic bounds presented in Section 2. The proof of the upper-bound stated in Section 3 is rather long and is given in Section 5.

This paper is one in a series of papers, all based on the results reported in [2], and all having the common theme of average-case analysis of bin-packing under discrete item-size distributions. Other papers in the series examine the average-case behavior of optimal algorithms and the on-line algorithms First-Fit [3] and Best-Fit [4] (these are the FFD and BFD algorithms but with the order of packing items being the given order of the list L_n.

We conclude this section with some notational conventions. The letters j,k, and n will be reserved throughout for the purposes given in this section, i.e., with j,k defining the set of item sizes (the constraint $j \leq k$ always applies), and with n being the number of items to be packed. We will use $m = n/j$ consistently, when n is divisible by j. Small letters toward the beginning of the Greek and Roman alphabets are used generically to denote constants (their meaning in one place need not be the same as in another).

2. Asymptotic Bounds on Expected Waste

For given j, k and n, with n a multiple of j, let $V_n = V_n(j,k)$ denote the perfectly uniform list consisting of n/j items of each size $1/k, \ldots, j/k$. The asymptotic behavior of FFD and BFD will be expressed in terms of simple combinatorial properties of the FFD and BFD packings of uniform lists. This section first defines these properties and then states and briefly discusses our main asymptotic result.

For each j, k there exists a sequence of values of n such that the FFD packings of V_n take a particularly simple, normal-form structure. A similar statement applies to BFD, but the structures and sequences may differ in the two cases. To describe these structures it is helpful to study an example. Consider the pair $j = 6$, $k = 13$ and suppose that $n = mj$, where m is a multiple of 24. The FFD and BFD packings of V_n are the same and constructed as follows.

Size-6/13 Items: These m items go two per bin into $m/2$ bins, leaving $m/2$ gaps of size $1/13$.

Size-5/13 Items: These m items go two per bin into $m/2$ bins, leaving $m/2$ gaps of size $3/13$.

Size-4/13 Items: These m items go three per bin into $m/3$ bins, leaving $m/3$ more gaps of size $1/13$, for a total of $5m/6$ such gaps.

Size-3/13 Items: The first $m/2$ of these items go into the gaps of size $3/13$ created by size-5/13 items. The remaining $m/2$ go four per bin into $m/8$ bins, leaving $m/8$ additional gaps of size $1/13$, for a total of $23m/24$.

Size-2/13 Items: These m items go six per bin, leaving $m/6$ more gaps of size $1/13$, for a total of $9m/8$.

Size-1/13 Items: These items fill m of the $9m/8$ gaps of size $1/13$ in previous bins, leaving $m/8$ gaps unfilled in bins with 6 size-2/13 items. The final wasted space is then $m/8k = n/8jk = n/624$.

Because of our assumption that 24 divides m, "transition" bins do not appear in the final packing. For example, if m had been odd we would have had one bin with a size-6/13 item and a size-5/13 item, and if m had not been divisible by 3 we would have had one bin with both size-4/13 and size-3/13 items. For many values of n, the last bin would also be a transition bin, since it would have a gap at least as large as its smallest item. The bin types created by the above procedure are *repeating* types; the number of bins of each one of these types grows in proportion to m as m increases through multiples of 24.

FFD (BFD) packings of uniform lists that have no transition bins are said to be *normal-form packings under* FFD (BFD). It should be clear, based on the above procedure, how to design an algorithm that takes a pair j, k as input and outputs the smallest m such that $n = mj$ yields a normal-form packing. We call such an algorithm a *fluid* FFD or BFD algorithm, because initially the algorithm treats the number n of items as if it were infinitely divisible. The output of a fluid algorithm can be bounded as a function of j and k, but it seems difficult to find a reasonably simple bound that does not grossly overestimate the output.

It is easily verified that the number of repeating bin types in an FFD or BFD packing is at most j. Correspondingly, in any FFD or BFD packing there are at most j transition bins.

The pairs j, k give three qualitatively different types of normal-form packings.

Type 1. Just before the size-$1/k$ items are packed, the sum of the gap sizes exceeds m/k, so not all gaps can be filled (as was the case for the pair 6,13).

Type 2. Just before the size-$1/k$ items are packed, the sum of the gap sizes is less than m/k, so all gaps will be filled and a nonzero number of bins will be filled with size-$1/k$ items. This happens, for instance, for the pair 6,11.

Type 3. Just before the size-$1/k$ items are packed, the total sum of gap sizes is precisely m/k, so that these become filled with size-$1/k$ items with none left over. This happens, for instance, for the pair 12,13.

Observe that no space is wasted in Type-2 and 3 normal-form packings, but that wasted space grows in proportion to m for Type-1 normal-form packings.

We say that a pair j, k is of Type i under FFD (BFD) if it leads to FFD (BFD) normal-form packings of Type i. The theorem below shows that the type of j, k completely determines the asymptotic expected wasted space under the distribution $U\{j; k\}$.

Theorem 2.1 *Let A denote either FFD or BFD. For any pair j, k let L_n be an n-item list with sizes drawn independently from $U\{j; k\}$. If j, k is of Type 1 under A, then $E[W^A(L_n)] \sim cn$ as $n \to \infty$, where c is the constant of proportionality for the wasted space in normal-form A*

packings. If j,k is of Type 2 under A, then $E[W^A(L_n)] = O(1)$, and if j,k is of Type 3 under A, then $E[W^A(L_n)] = \Theta(n^{1/2})$.

Our proof of this theorem, given in Section 4, shows that $E[W^A(L_n)]$ is estimated up to a $O(n^{1/2})$ term by the waste in normal-form packings, if j,k is of Type 1 or 3 under A, and up to a constant term if j,k is of Type 2 under A. The theorem follows immediately from these estimates. Although the proof does not yield any closed-form formula that might help to explain the behavior in Theorem 2.1, extending the output of the fluid algorithms gives us procedures that, given j,k, determine which of the three options applies, and in the case of linear expected waste, computes the constant of proportionality.

For Type-1 pairs, we can also obtain a formula for the limiting expected ratio $R_\infty^A = \lim_{n\to\infty} R_n^A$, where $R_n^A = E[A(L_n)/s(L_n)]$. In Section 4 we show that, for A either FFD or BFD and j,k of Type 1,

$$R_\infty^A = 1 + \frac{2ck}{j+1} \ , \tag{2.1}$$

with c as defined in Theorem 2.1.

3. Analysis of Pair Types

A computer program implementing fluid FFD and BFD algorithms was written to evaluate pair types and, in the case of Type 1 pairs where linear waste occurs, to determine constants of proportionality. Computer runs not only revealed the intricate behavior of FFD and BFD, but also simplified theoretical results (see Theorem 3.4). This section begins by proving theorems that helped limit the computations that had to be made. We then discuss the results of the computer runs. The section concludes by stating and briefly discussing a theorem that supplies a sharp upper bound on linear expected waste over all pairs j,k.

Normal-form packings were computed for all combinations of j,k with $k \le 1000$. The fluid algorithms were needed only for pairs j,k, $j \le k/2$, as a result of the following easy theorem.

Theorem 3.1

(a) All pairs j,k with $j = k-1$ or k are Type 3 under both FFD and BFD.

(b) A pair j,k with $k/2 < j \le k-2$ has the same type under FFD (BFD) as the pair $k-j-1,k$ has under FFD (BFD).

Proof. When $j = k-1$, FFD and BFD normal-form packings of $n = mj$ items contain m bins for each matched pair $(i/k, 1-i/k)$, $i = j, j-1, \ldots, \lceil k/2 \rceil$, except for the last pair ($i = \lceil k/2 \rceil$) when k is even, in which case there are $m/2$ bins pairing size-1/2 items with each other. The same prescription holds for $j = k$ if we imagine size-1 items being paired with fictional size-0 items. This proves part (a).

If $k/2 < j \leq k-2$ then the FFD and BFD normal-form packings begin as described above, but after packing bins with matched pairs, the items of sizes $1/k, \ldots, 1-(j+1)/k$, m of each, are left over. The remainder of the packing will then be a normal-form packing for the pair $k-j-1, k$. Part (b) follows. ■

Note that for Type-1 pairs j, k with $k/2 < j \leq k-2$, the constants of proportionality in the expected waste under FFD (BFD) are easily computed from those in the expected waste under FFD (BFD) for the corresponding pairs $k-j-1, k$. Interestingly, the computer runs showed that there were no Type-3 pairs with $j \leq k-2$, for any $k \leq 1000$ under either FFD or BFD. The existence of such Type-3 pairs for $k > 1000$ remains an open question.

The next result shows that the fluid algorithms are needed only for pairs j, k with $k^{1/2} \leq j \leq k - k^{1/2}$.

Theorem 3.2 *For all j, k with $j < k^{1/2}$ or $j > k - k^{1/2}$, the FFD and BFD normal-form packings are of Type 2, and hence $E[W_n^{FFD}]$ and $E[W_n^{BFD}]$ are both $O(1)$ when item sizes are drawn independently from the distributions $U\{j,k\}$ based on these pairs.*

Proof. By Theorem 3.1 we need only consider the case $j < k^{1/2}$. Note that for each i, $1 < i \leq j$, the size-i/k items go at least $k^{1/2}$ per bin, creating at most $m/k^{1/2}$ gaps of some specified size g. A simple induction can thus be used to show that, when we pack items of size g, no gaps of size greater than g remain, and that the total number of gaps of size g itself is at most $(m/k^{1/2})(k^{1/2}-1)$. This means that all of the size-g gaps will be filled perfectly, with some items of size g left over. Since this holds true for $g = 1/k$, the normal-form packings are of Type 2, by definition. ■

A somewhat weaker result of this kind can be proved for Type-1 pairs. However, it gives no further constraints on the pairs that need to be evaluated by the fluid algorithms.

Theorem 3.3 *Under FFD and BFD Type-1 pairs j, k exist for arbitrarily large values of j with j/k both above and below $1/2$.*

Proof. Consider the pairs j, k with $k = 1260t + 1$, $j \in \{420t, 840t\}$ and $t \geq 1$ arbitrary. We first note that if $j = 840t$, then the FFD and BFD normal-form packings pair each of the items of size $1/2 + a/k$, $1/2 \leq a \leq 210t - 1/2$, with an item of size $1/2 - a/k$, thus

Figure 1: Pairs j, k yielding linear waste.

less than $10^{-8}n$ for 311,943 to $0.00160256n$ for 6,13. Table 1 lists the ten pairs j, k that were found to yield the largest rates of wasted space under FFD and BFD. (For these values of j, k the waste under FFD and BFD is the same.) The third column gives the expected waste as a function of n, with the rate rounded to the nearest multiple of 10^{-8}. The fourth column in the table gives the corresponding asymptotic expected ratios $E[R_{\infty}^{A}]$. The first 9 pairs yield the 9 largest values encountered for the expected ratio; the tenth largest value is given by 22,61 rather than 20,41, the last pair listed. Note that for each entry in the table, we have k odd and $j = \lfloor k/2 \rfloor$; these pairs tend to give the highest rates of wasted space.

In view of Theorem 3.3 and the computational results in Table 1, it is natural to ask whether the pair 6,13 gives the worst linear waste for *all* $k \geq 1$. The following theorem answers this question and shows in addition that the maximum rate of expected wasted space tends to 0 as $k \to \infty$.

j	k	$E[W_n^A]$	$E[R_\infty^A]$
6	13	$0.00160256n$	1.00595238
12	25	$0.00122685n$	1.00471866
15	31	$0.00046723n$	1.00181052
18	37	$0.00046694n$	1.00181859
24	49	$0.00043285n$	1.00169676
30	61	$0.00027820n$	1.00109484
36	73	$0.00021314n$	1.00084103
27	55	$0.00020804n$	1.00081730
48	97	$0.00018985n$	1.00075163
20	41	$0.00018770n$	1.00073291

Table 1: Combinations of j, k with the greatest expected waste. (A = FFD or BFD.)

Theorem 3.4 *Let A denote either FFD or BFD, and let item sizes be drawn independently from $U\{j;k\}$. Then for all $j, k,\ k \geq 1$,*

(a) $E[W_n^A] \leq n/624 = (.00160\ldots)n$,
the rate attained by the pair $6, 13$,

(b) $E[W_n^A] \leq (.00614\ldots)n/k^{1/2}$,
the constant being determined by the pair $12, 25$, *and*

(c) $E[R_\infty^A] \leq 1.00595\ldots$,
attained again by the pair $6, 13$.

The constants revealed by this theorem confirm that the linear waste that FFD and BFD can create relative to OPT under $U\{j;k\}$ may well lack significance in practice.

The proof of this theorem, given in Section 5, is based on a worst-case analysis of normal-form packings, and relies on our computational results, which show that a violation of the theorem would have to be for some $k > 1000$. The proof involves two separate arguments, both somewhat complicated. The first, covering the waste in bins that start with items larger than $k^{1/2}$, has something of the flavor of the analysis of FFD in [1] for the continuous case, although edge effects that previously could be ignored must now be taken into account. The argument for bins that start with items smaller than $k^{1/2}$ must confront distinctly new issues.

4. Probabilistic Analysis

Before getting into the asymptotic estimates of this section, we prove a useful combinatorial lemma. With this lemma in hand, we then derive asymptotic bounds on the tails of the distribution of wasted space under FFD and BFD. The section concludes with a proof of the average-case behavior given in Theorem 2.1.

Lemma 4.1 *Let I be a set of items with sizes drawn from $\{i/k;\ 1 \le i \le j\}$, and suppose I' is obtained from I by the deletion of a single item. If A denotes either FFD or BFD, then the A packings of I and I' differ in at most k^j bins, i.e., the packings of I and I' have at least* $\max\{A(I), A(I')\} - k^j$ *bins in common.*

Proof. Assume that I has at least one item of size i/k, and that I' is obtained from I by deleting one such item. Following the A packing of all items of size i/k or more, the packing of I' will leave one bin less full. Now consider the packing of size-$(i-1)/k$ items. Some of these, but certainly no more than $k-1$ might go in the bin that is less full. This means that at most $k-1$ other bins, which would otherwise have received items of this size, will be less full. Thus, following the packing of these items, there are at most k bins whose contents differ from those that would have been obtained by packing I.

The argument continues in this fashion. Suppose we are about to pack items of size $s < i/k$ and there are N bins whose contents differ from what they would have been if we had been packing I. Any such bin that is less full might take up to $k-1$ more items of size s, so there might be up to $k-1$ other bins that receive fewer items of size s than would have been the case if we had been packing I. Any bin that is more full might take up to $k-1$ fewer items of size s, so there might be up to $k-1$ other bins that receive more items of size s than would have been the case if we had been packing I. Consider those bins whose contents start this phase no differently and receive exactly the same number of items of size s as they would have received if we had been packing I; since items of size s are identical, such bins end this phase of the packing holding exactly the same contents as if we had been packing I. Thus at the end of this phase there are no more than kN bins whose contents differ from what they would have been if we had been packing I. It follows that there are no more than k^j bins whose contents differ for packings of I and I'. ■

The above bound can usually be improved for specific j, k. However, all we need from the lemma is that the change in the packing of I is limited to a number of bins, and hence to a change in wasted space, that is independent of the size of I.

The probability background needed for the results below is elementary; we review it here for convenient later reference. These preliminary results, or the techniques needed to prove them, can be found in [6]. The chapter on basics in [5] (see Chapter 2) can also be recommended as it is oriented specifically towards our needs.

Our chief concern is with sums of bounded i.i.d. random variables. Let $(m_1, \ldots, m_j)$ denote the size distribution of L_n, i.e., m_i, $1 \le i \le j$, is the number of items of size i/k in L_n. Then $m_1, \ldots, m_j$ has a multinomial distribution and each m_i has a binomial distribution with parameters $1/j$ and n. In the usual way, we can represent m_i as the sum of n i.i.d. random variables X_r, $1 \le r \le n$, where $X_r = 1$ if the r^{th} item of L_n has size i/k and $X_r = 0$ otherwise. Then since $E[X_r] = 1/j$ and $\sigma^2(X_r) = (1/j)(1 - 1/j)$ are the mean and variance, we have $E[m_i] = m = n/j$ and $\sigma^2(m_i) = n(j-1)/j^2 = \Theta(n)$ for fixed j, k. Classical normal limit laws apply to m_i. Then for fixed j, k and constants $\alpha, \beta > 0$, standard estimates for the tails of the normal distribution show that

$$Pr\{|m_i - m| > \alpha n^{1/2} \log^{\beta} n\} = O(1/n^{2\beta}\ .) \tag{4.1}$$

and

$$E|m_i - m| = \Theta(n^{1/2})\ . \tag{4.2}$$

As we are fixing j, k and suppressing multiplicative constants, we can extend (4.1) and (4.2) to sums of the $|m_i - m|$. For, if $\sum_{i=1}^{j}\ |m_i - m| > \alpha n^{1/2} \log^{\beta} n$, then for one or more values of i, $1 \le i \le j$, we must have $|m_i - m| > (\alpha/j) n^{1/2} \log^{\beta} n$. Then

$$Pr\left\{ \sum_{i=1}^{j} |m_i - m| > \alpha n^{1/2} \log^{\beta n} \right\} \le j\, Pr\{|m_i - m| > (\alpha/j) n^{1/2} \log^{\beta} n\}$$

so we can conclude that

$$Pr\left\{ \sum_{i=1}^{j} |m_i - m| > \alpha n^{1/2} \log^{\beta} n \right\} = O(1/n^{2\beta})\ . \tag{4.3}$$

Chernoff bounds also apply to the m_i (e.g. see [5]). Simple versions of these bounds state that, for fixed j, k and any constant $\alpha > 0$, there exists another constant $\beta > 0$ such that

$$\begin{aligned} Pr\{m_i > (1+\alpha)m\} &\le e^{-\beta n} \\ Pr\{m_i < (1-\alpha)m\} &\le e^{-\beta n} \end{aligned} \tag{4.4}$$

and hence $Pr\{|m_i - m| > \alpha m\} \le 2e^{-\beta n}$.

The limit law for m_i can be extended to more general functions of n. For example, if n is replaced by $n + \alpha\sqrt{n}$ or $n - \alpha\sqrt{n}$ for some fixed constant $\alpha > 0$, then the asymptotics in (4.1) and (4.2) still hold (only hidden constants change). To apply this fact, suppose that for some $l < j$ and (positive or negative) constants α_i, $1 \le i < l$, we have $m_i = m + \alpha_i\sqrt{n}$, $1 \le i < l$. Then as direct calculations would show, the conditional distribution of m_l given the m_i, $i < l$, is a binomial distribution with the asymptotics in (4.1) and (4.2). It follows easily that if $e_1, \ldots, e_j$ are events of the form $m_i - m \lesseqgtr \alpha_i\sqrt{n}$ (or $|m_i - m| \lesseqgtr \alpha_i\sqrt{n}$) for given constants α_i, $1 \le i \le j$, then there is a strictly positive constant c such that

$$Pr\{e_1, \ldots, e_j\} \to c \quad \text{as} \quad n \to \infty \ . \tag{4.5}$$

Estimates of c are available from the normal distribution, but only the assertion above is needed for our lower bound arguments.

Finally, we note that $s(L_n)$ is also a sum of i.i.d. bounded random variables, in this case with

$$E[s(L_n)] = \frac{n(j+1)}{2k} \ , \quad \sigma^2(s(L_n)) = \frac{n(j^2-1)}{12k^2} \ . \tag{4.6}$$

Then (4.1) and (4.2) also apply with m_i and m replaced by $s(L_n)$ and $E[s(L_n)]$, respectively.

We now use these results in a proof of the following probability bounds on wasted space.

Theorem 4.1 *Let A denote either FFD or BFD and let the sizes of the items in L_n be drawn independently from $U\{j;k\}$. Then with probability $1 - O(1/n)$, we have $W^A(L_n) = O(n^{1/2}\log^{1/2} n)$ if j,k is of Type 3, and $W^A(L_n) = cn + O(n^{1/2}\log^{1/2} n)$ if j,k is of Type 1, where cn denotes the linear waste in normal-form A packings of $V_n(j,k)$. If j,k is of Type 2 then there exists a $\beta > 0$ such that $W(L_n) = O(1)$ with probability $1 - O(e^{-\beta n})$.*

Proof. Consider the A packings of items with sizes drawn from $\{i/k;\ 1 \le i \le j\}$. It is easily verified that if, for fixed j,k, we can prove the theorem for n limited to multiples of some constant, then the theorem holds for general n. Thus, as a convenience we restrict n in the remainder of the proof to those values for which the sets V_n have normal-form packings.

Recall that $(m_1, \ldots, m_j)$ denotes the size distribution of L_n. Then the total number of items that need to be added or deleted to obtain V_n from L_n is $\sum_{i=1}^{j} |m_i - m|$. Thus, repeated applications of Lemma 4.1 show that the difference δ_n in the wasted space of the A packings of L_n and V_n is bounded by

$$(4.7)\qquad \delta_n \le k^j \sum_{i=1}^{j} |m_i - m| \ .$$

By definition of δ_n, we have for a Type-1 pair

$$(4.8)\qquad |W^A(L_n) - cn| = \delta_n \le k^j \sum_{i=1}^{j} |m_i - m| \ .$$

Type-3 normal-form packings waste no space, so for a Type-3 pair we have

$$(4.9)\qquad W^A(L_n) = \delta_n \le k^j \sum_{i=1}^{j} |m_i - m| \ .$$

Combining (4.8) and (4.9) with the estimate in (4.3), we obtain the Type-1 and Type-3 results of the theorem.

Now suppose j,k is of Type 2, and let δ_n^* denote the difference in the wasted space of the A packings of L_n and V_n just before the size-$1/k$ items are packed. By the same arguments leading to (4.7), we have

$$(4.10)\qquad \delta_n^* \le k^{j-1} \sum_{i=2}^{j} |m_i - m| \ .$$

Next, we observe that, in the A packing of V_n, the number of bins with only size-$1/k$ items is bm for some b, $0 < b < 1$. Then $W^A(L_n) > 1$ implies that $\delta_n^* > bm/2k$ or $m_1 < bm/2$. For if neither inequality held, then by definition of δ_n^* the packing of L_n before the size-$1/k$ items were added would have a total gap size of at most $bm/2k$ and L_n would have at least $bm/2$ size-$1/k$ items with which to fill the gaps; so if there were any wasted space in the packing of L_n, it would be confined to the last bin, and hence be less than 1. Thus

$$Pr\{W^A(L_n) > 1\} \le Pr\{\delta_n^* > bm/2k\} + Pr\{m_1 < bm/2\}$$

and

$$(4.11)\qquad Pr\{W^A(L_n) \le 1\} \ge 1 - Pr\{\delta_n^* > bm/2k\} - Pr\{m_1 < bm/2\} \ .$$

For the second probability on the right of (4.11), the Chernoff bound in (4.4) shows that there exists a $\beta_1 > 0$ such that

$$(4.12)\qquad Pr\{m_1 < bm/2\} = O(e^{-\beta_1 n}) \ .$$

By (4.10), we see that $\delta_n^* > bm/2k$ implies that, for one or more values of i, $2 \le i \le j$, we have $|m_i - m| > b'm$ for some $b' \ge b/[2(j-1)k^j]$. Then

$$Pr\{\delta_n^* < bm/2k\} \le (j-1) Pr\{|m_1 - m| > b'm\} \ ,$$

so we can again use (4.4) to conclude that there exists a $\beta_2 > 0$ such that

$$Pr\{\delta_n^* > bm/2k\} = O(e^{-\beta_2 n}) \ . \tag{4.13}$$

If β denotes the minimum of β_1 and β_2, then (4.12) and (4.13) yield $Pr\{W^A(L_n) \le 1\} = 1 - O(e^{-\beta n})$. The Type-2 result of the theorem follows. ∎

Except for the lower bound for Type-3 pairs, the average-case results in Theorem 2.1 fall out as an easy corollary to Theorem 4.1.

Proof of Theorem 2.1. As before, we restrict n to those values for which V_n has normal-form A packings.

Observe that n is a trivial upper bound to the wasted space in any packing of L_n. Then by Theorem 4.1 for Type-1 pairs

$$E[W^A(L_n)] = [cn + O(n^{1/2} \log^{1/2} n)][1 - O(1/n)] + n \cdot O(1/n)$$

and hence $E[W^A(L_n)] \sim cn$ as $n \to \infty$. Similar arguments show that $E[W^A(L_n)] = O(1)$ if j,k is of Type 2. From (4.2) and (4.9), we have $E[W^A(L_n)] = O(n^{1/2})$ if j,k is of Type 3.

To prove the lower bound for Type-3 pairs consider the event

$$|m_i - m| \le \sqrt{n}/(j-1), \ \ 2 \le i \le j, \qquad m_1 \le m - \frac{1}{2} k^j \sqrt{n} \ . \tag{4.14}$$

The wasted space in the A packing of V_n just before the size-$1/k$ items are added is exactly m/k. Thus, if (4.14) holds, the wasted space in the packing of L_n just before the size-$1/k$ items are added is at least

$$m/k - \delta_n^* \ge m/k - k^{j-1} \sum_{i=2}^{j} |m_1 - m| \ge m/k - k^{j-1}\sqrt{n} \ .$$

But $m_1 \le m - \frac{k^j}{2}\sqrt{n}$, so the wasted space *after* the size-$1/k$ items are packed satisfies

$$W^A(L_n) \ge m/k - k^{j-1}\sqrt{n} - \left(m - \frac{k^j}{2}\sqrt{n} \right) /k = \frac{1}{2} k^{j-1} \sqrt{n} \ . \tag{4.15}$$

The probability that the inequalities in (4.14) hold jointly for all $1 \le i \le j$ is strictly positive for all n sufficiently large, as in (4.5). We conclude from (4.15) that $W^A(L_n) = \Omega(n^{1/2})$ with positive probability. The lower bound $E[W^A(L_n)] = \Omega(n^{1/2})$ follows. ∎

We also obtain formula (2.1) for Type 1 pairs as a corollary. We have

$$R_n^A = \frac{A(L_n)}{s(L_n)} = 1 + \frac{W^A(L_n)}{s(L_n)} \ ,$$

and from Theorem 4.1, $W^A(L_n) = cn + O(n^{1/2}\log^{1/2} n)$ with probability $1 - O(1/n)$. For the sum of item sizes we apply (4.1) and (4.2) with m_i replaced by $s(L_n)$ to obtain a similar estimate: $s(L_n) = n(j+1)/2k + O(n^{1/2}\log^{1/2} n)$ with probability $1 - O(1/n)$. Now R_n^A is bounded by a constant for all $n \geq 1$ and all problem instances L_n, so we can conclude that, as $n \to \infty$,

$$E[R_n^A(L_n)] = \frac{2ck}{j+1} + o(1) ,$$

which gives formula (2.1).

5. Proof of Theorem 3.4

A look at Table 1 shows that a violation of the theorem would require $k > 1000$. But for $k > 1000$ we can prove the following stronger results:

(5.1) $$E[W_n^A] < 0.00558n/k^{1/2} < 0.000177n$$

(5.2) $$E[R_\infty^A] < 1.00271 .$$

We prove these bounds for a variant of FFD in which no bin is allowed to have more than one "fallback" item. In an FFD packing, the items that go into bin B_i before bin B_{i+1} is started are called *regular* items. Those that are placed in bin B_i after bin B_{i+1} is started are called *fallback* items. In this modified FFD (MFFD), a bin that has received a fallback item is declared "closed" and receives no further items. In [1] it is proved that for all lists L, FFD(L) $\leq$ MFFD(L). For the analogously modified BFD (MBFD), similar proofs show that for all L, BFD(L) $\leq$ MBFD(L) $=$ MFFD(L). Thus it suffices to prove the theorem for MFFD.

Our proof considers normal-form MFFD packings, defined in analogy with those of FFD and BFD, and bounds the maximum cumulative gap as a function of $n = mj$. Note that Theorem 3.2 also applies to MFFD, and that by it, we can assume $j < k/2$. We will discuss two kinds of deficits. The *initial gap* in a bin is the space left after it has received its last regular item. The *residual gap* is the space left after the bin has received its one allowed fallback item (or simply the initial gap, if no fallback item is ever received). In order to obtain a bound on $E[W_n^{\mathrm{FFD}}]$, we need to bound the sum W of the residual gaps, divided by $n = mj$. To obtain a bound on $E[R_\infty^{\mathrm{FFD}}]$, we bound W/s, where $s = s(V_n) = mj(j+1)/2k$ is the sum of the item sizes.

In what follows, we classify the bins in normal-form MFFD packings in two different ways. The "S-class" of a bin (S for size) is the size of the biggest item it contains. The "M-class" of a bin (M for multiplicity) is the number of copies of its largest item. For instance, when $j = 6$ and $k = 13$ the first bin created contains two items of size $6/k$ and one of $1/k$. Its S-class is thus $6/13$ and its M-class is 2. We call a bin an (h/k)-bin if its S-class is h/k, and an i-bin if its M-class is i. This overloads our notation slightly, but need cause no confusion since all S-class values are less than 1 and, by our assumption that $j < k/2$, all M-class values are greater than 1.

Two bins with the same M-class are said to have different *types* if they belong to different S-classes. The number of types of bins with i as their M-class is simply the number of integers $h \le j$ such that $\lfloor k/(i+1) \rfloor < h \le \lfloor k/j \rfloor$. Note that for $\lfloor k/j \rfloor \le i \le k^{1/2} - 1$ there must be at least one type of i-bin created by MFFD, since $k/(k^{1/2} - 1) - k/k^{1/2} = k^{1/2}/(k^{1/2} - 1) > 1$. Let W_A denote the total residual waste in i-bins with $i \le k^{1/2} - 1$ under MFFD, and let W_B denote the total residual waste in i-bins with $i > k^{1/2} - 1$. We bound W_A (and W_A/s) first.

no further than the length M_i of the longest sequence of sizes such that the total initial billing to those sizes, as of the time the i-bins send in their bills, exceeds m times the length of the sequence. Thus M_i/k bounds the residual gap in the i-bins under MFFD.

Consider how large M_i can be. The total billing to a sequence of M item sizes can be bounded as follows: The sizes billed by i-bins increase by i/k as we go from one type to the next, since $(k - i(h-1))/k - (k - ih)/k = i/k$. Thus the total number of sizes in a sequence of M bins that can be billed by i-bins is at most $\lceil M/i \rceil$, with each being billed for at most m/i. To begin the analysis, let us assume that $j > k/3$, so that 2-bins exist. In this case, as we have already observed, we cannot have $M_i \geq 1$ until $i \geq 4$. Moreover, note that although the 4-bins can cause the bill to an individual item size to reach $((5/6) + (1/4))m > m$, at least $(1/6)(1/4) = 2/3$ of that bill will be paid, so that only $1/3$ of those bins will actually have residual gaps. Those residual gaps will be of size $1/k$. This is because $\lceil 2/t \rceil = 1$ for all $t \geq 2$, so we cannot have $M_i \geq 2$ until $\sum_{i=2}^{i}(1/t) \geq 2$, which does not occur until $i = 11$.

Now consider a sequence of three item sizes. These can be billed from two different types of 2-bin, but can be billed by at most one type of t-bin, $t > 2$. Thus we cannot have $M_i \geq 3$ until $1 + \sum_{i=3}^{i}(1/t) \geq 3$, which does not occur until $i = 19$. Thus when $j > k/3$ we can bound W_A/mj by the following expression (in which k' denotes $\lfloor k^{1/2} - 1 \rfloor$):

$$\frac{W_A}{mj} < \frac{1}{kj}\left(\frac{k}{3 \cdot 4^2 \cdot 5} + \frac{1}{3 \cdot 4} + \sum_{i=5}^{k'}\left(\frac{k}{i^2(i+1)} + \frac{1}{i}\right) + \sum_{i=11}^{k'}\left(\frac{k}{i^2(i+1)} + \frac{1}{i}\right)\right)$$
$$+ \frac{1}{kj}\sum_{i=19}^{k'}\left(\left(\frac{k}{i^2(i+1)} + \frac{1}{i}\right)(M_i - 2)\right) .$$

Noting that $i^3 < i^2(i+1)$ and that $\sum_{i=2}^{x}(1/i) < \ln x$, this can be simplified to

$$\frac{W_A}{mj} < \frac{1}{j}\left(\frac{51}{12,000} + \sum_{i=5}^{\infty}\frac{1}{i^2(i+1)} + \sum_{i=11}^{\infty}\frac{1}{i^2(i+1)} + \sum_{i=19}^{\infty}\frac{M_i - 2}{i^3}\right)$$
$$+ \frac{1}{j}\left(\frac{\ln k}{k} + \sum_{i=19}^{k'}\frac{M_i - 2}{ik}\right) . \tag{5.3}$$

To bound the $M_i - 2$, $i \geq 19$, note that each M_i must satisfy

$$M_i < \sum_{i=2}^{i}\frac{1}{t}\left\lceil\frac{M_i}{t}\right\rceil \leq \sum_{i=2}^{i}\frac{M_i + t - 1}{t^2} < (M_i - 1)\sum_{t=2}^{\infty}\frac{1}{t^2} + \sum_{t=2}^{i}\frac{1}{t} .$$

Observing that $\sum_{i=2}^{\infty}(1/t^2) < .645$, this implies

$$M_i - 2 < \frac{\sum_{i=2}^{i}\frac{1}{t} - .645 - .710}{.355} < 2.817\sum_{i=2}^{i}\frac{1}{t} < 2.817 \ln i . \tag{5.4}$$

Substituting $2.817 \ln i$ for the first $M_i - 2$ in (5.3), we find that

$$\sum_{i=19}^{\infty} \frac{M_i - 2}{i^3} < \sum_{i=19}^{\infty} \frac{2.817 \ln i}{i^3} < \int_{i=18}^{\infty} \frac{2.817 \ln i \; di}{i^3} = \left[-\frac{2.817}{2i^2} \left(\ln i + \frac{1}{4} \right) \right]_{18}^{\infty} < .01474 \; .$$

Next, we substitute for the second $M_i - 2$ and use the relations $k' = \lfloor k^{1/2} - 1 \rfloor$ and $k > 1000$ to arrive at

$$\begin{aligned} \sum_{i=19}^{k'} \frac{M_i - 2}{ik} &< \int_{i=18}^{k'} \frac{2.817 \ln i \; di}{ik} < \left[-\frac{2.817}{2k} (\ln i)^2 \right]_{18}^{k'} \\ &< \frac{1.409}{k} \left((\ln(k^{1/2} - 1))^2 - (\ln 18)^2 \right) < \frac{1.409}{1000} (11.930 - 8.354) < .00473 \; . \end{aligned}$$

Substituting into (5.3), noting that $\ln k / k < .00691$ for $k > 1000$, and bounding the remaining infinite sums, we then obtain

$$\begin{aligned} \frac{W_A}{mj} &< \frac{.00425 + .02133 + .00426 + .01474 + .00691 + .00473}{j} \\ &< \frac{.05622}{j} < \frac{.00534}{k^{1/2}} \; , \end{aligned} \tag{5.5}$$

since, by assumption, $j > k/3$ and $k > 1000$, so $1/j < (3/1000^{1/2})(1/k^{1/2}) < .09487 k^{1/2}$.

For this case $(j > k/3)$, bounding W_A / s is straightforward:

$$\frac{W_A}{s} = \frac{W_A}{mj(j+1)/2k} < \frac{2k}{j+1} \frac{.00534}{k^{1/2}} < \frac{(6)(.00534)}{k^{1/2}} = \frac{.03204}{k^{1/2}} < .00102 \; . \tag{5.5a}$$

Let us now consider the situation where $\lfloor((k/(k-1))(k^{1/2}+1) < j \le k/3$. Let $d \ge 0$ be the integer such that $k/(4+d) < j \le k/(3+d)$. We argue much as in the case for $j > k/3$, deriving a bound on W_A / mj that is maximized when $d = 0$.

First, consider the case $d = 0$. By looking at the sums $\sum_{i=3}^{i} (1/t)$ for various $i \ge 3$, we can verify that there will be no residual gap in any i bins for $i = 3, 4, 5, 6$ (since $1/3 + 1/4 + 1/5 + 1/6 < 1$). Similarly, the gap will be at most $1/k$ for $7 \le i \le 18$, and will be at most $2/k$ for $19 \le i \le 50$. (Note that with $j \le k/3$, each string of 3 item sizes can receive bills from at most one type of i-bin for each i.)

When $d > 0$, it is not difficult to see that there is no residual gap in an i-bin until $i \ge 7 + 2d$. This is because $\sum_{i=3+d}^{7+2d} (1/t) < 1$, which can be proved by an induction based on the inequality $1/(3+d) \ge 2/(7+2d+1)$. Similarly, the gap is at most $1/k$ until $i \ge 19 + 6d$, and cannot exceed $2/k$ until $i \ge 51 + 17d$.

The analogue of the bound (5.3) can now be derived. Here, we get a further simplification by bounding $\lceil k/(i^2(i+1)) \rceil$ by $2k/i^3$ rather than $k/(i^2(i+1)) + 1$. We also do not make a special case out of the first i with a residual gap. We immediately bound all the summations by the appropriate integrals and obtain

$$\begin{aligned}
\frac{W_A}{mj} &< \frac{1}{j}\left(\int_{i=6+2d}^{\infty}\frac{2di}{i^3}+\int_{i=18+6d}^{\infty}\frac{2di}{i^3}+\int_{i=50+17d}^{\infty}\frac{5.634\ln i\; di}{i^3}\right)\\
&< \frac{4+d}{k}\left(\frac{1}{(6+2d)^2}+\frac{1}{(18+6d)^2}+\frac{.70425+1.409\ln(50+17d)}{(50+17d)^2}\right)\ ,
\end{aligned}$$

where we make use of our assumptions that $j > k/(4+d)$. But note that this sum is clearly maximized for $d=0$, in which case we have

$$\frac{W_A}{mj} < \frac{1}{31.62k^{1/2}}\left(\frac{1}{9}+\frac{1}{81}+\frac{.70425}{625}+\frac{(1.409)(3.9121)}{625}\right)$$

$$(5.6)\qquad < \frac{.11111+.01235+.00113+.00882}{31.62k^{1/2}} < \frac{.00422}{k^{1/2}}\ .$$

This is seen to be less than the bound (5.5) computed for the case $j > k/3$, so the former provides a universal bound on W_A (when $k > 1000$).

To get the corresponding bound for W_A/s, we first note that if j exceeds $k/(4+d)$, then so does $j+1$, and hence $2k/(j+1) < 2(4+d)$. Thus,

$$\frac{W_A}{s} < \frac{2(4+d)^2}{k}\left(\frac{1}{(6+2d)^2}+\frac{1}{(18+6d)^2}+\frac{.70425}{(50+17d)^2}+\frac{1.409\ln(50+17d)}{(50+17d)^2}\right)\ .$$

Observe that the first three terms are again maximized for $d=0$, but the fourth and final term is maximized for d as large as possible, which in this case is $k^{1/2}-5$. Thus we obtain

$$\frac{W_A}{s} < \frac{1}{31.62k^{1/2}}\left(\frac{8}{9}+\frac{8}{81}+\frac{5.634}{625}\right)+\frac{11.272}{625k^{1/2}}\left(\frac{1.409\ln(17k^{1/2})}{k^{1/2}}\right)\ .$$

The last term is strictly decreasing for $k \geq 1000$. Thus its value when $k > 1000$ is dominated by the value at $k=1000$, and we have

$$(5.6a)\qquad \frac{W_A}{s} < \frac{.03153}{k^{1/2}}+\frac{.01804}{k^{1/2}}\frac{8.859}{31.62} < \frac{.03153+.00506}{k^{1/2}} = \frac{.03659}{k^{1/2}}\ ,$$

in this case slightly worse than the bound (5.5a) obtained when $j > k/3$.

Let us now turn to bounding the waste W_B in i-bins with $i > k^{1/2}-1$. As argued earlier, there is at most one bin of each type, and indeed there are at most $k^{1/2}+2$ such types present over all. This is because the regular item size in such i-bins can be at most $1/(k^{1/2}-1) < (k^{1/2}+2)/k$, so there are at most $k^{1/2}+2$ item sizes available. This means that the only bin types we can have are (h/k)-bins, $1 \le h \le k^{1/2}+2$. We bound the residual waste in each of these types of bins.

First, note that the M-class of an (h/k)-bin is $\lfloor k/h \rfloor \ge (k-h+1)/h$, so the total number of (h/k)-bins is at most $m/\lfloor k/h \rfloor \le mh/(k-h+1)$. If G is an upper bound on the maximum residual gap in these bins and r is the minimum of j and $k^{1/2}+2$, we then have

$$\begin{aligned} \frac{W_B}{rj} \;&<\; \frac{g}{j}\sum_{h=2}^{m} \frac{h}{k-h+1} \;<\; \frac{1.0327gr(r+1)}{2kj} \;<\; \frac{1.0327g(r+1)}{2k} \\ &<\; \frac{.5164g}{k^{1/2}}\frac{k^{1/2}+2}{k^{1/2}} \;<\; \frac{.5491g}{k^{1/2}}\,, \qquad k \;>\; 1000\ . \end{aligned}$$

An easy bound on g follows from $g < M_k/k$. Applying (5.4) with $i = k$, we conclude that no gap is bigger than $(2.817\ln k + 2)/k$. That, however, is an overestimate, as it assumes that all i-bins, $1 \le i \le k$, are present. A more detailed analysis takes into account the missing i's, and concludes that, for $k > 1000$,

$$\begin{aligned} M_k \;&<\; 2 + 2.817\left(\sum_{i=2}^{k^{1/2}-1}\frac{1}{t} + \sum_{h=1}^{m}\frac{h}{k-h+1}\right) \\ &<\; 2 + 2.817\left(\ln\left(k^{1/2}-1\right) + 1.0327\left(\frac{k+5k^{1/2}+6}{2k}\right)\right) \\ &<\; 2 + 2.817\left(\frac{\ln k}{2} + .6012\right) \;<\; 1.409\ln k + 3.6936\ . \end{aligned}$$

Hence $g \le M_k/k < .01343$ for $k > 1000$. Thus,

$$\frac{W_B}{mj} < \frac{(.5491)(.01343)}{k^{1/2}} < \frac{.00738}{k^{1/2}} < .000233\ . \tag{5.7}$$

Combining (5.7) and (5.5), we can conclude that when $k > 1000$, the total residual waste is at most

$$W_A + W_B < \frac{.00558}{k^{1/2}} < .000177\ ,$$

as claimed by the theorem.

In analyzing W_B/s, we must be a bit more subtle. By the first inequality we displayed for W_B, we know that

$$\frac{W_B}{s} < \frac{2k}{j+1}\frac{1.0327gr(r+1)}{2kj} = 1.0327g\left(\frac{r}{j}\right)\left(\frac{r+1}{j+1}\right)$$

where $g < .01343$ is the maximum gap and $r = \min\{j, k^{1/2}+2\}$. Suppose $j \geq 3k^{1/2}+8$. Then we would have both $j > 3r$ and $j+1 > 3(r+1)$, so we would have

$$\text{(5.7a)} \qquad \frac{W_B}{s} < \frac{(1.0327)(.01343)}{9} < .00155\ .$$

On the other hand, suppose $j < 3k^{1/2}+8$. Then the smallest i for which an i-bin exists is greater than $\lfloor k/(2k^{1/2}+5)\rfloor > \lfloor k/2.1582k^{1/2}\rfloor > .46335k^{1/2}-1$. From this we conclude that M_k must satisfy

$$\begin{aligned}
M_k &< \sum_{t=.46335k^{1/2}-1}^{k^{1/2}-1} \frac{M_k t - 1}{t^2} + \sum_{h=1}^{r} \frac{h}{k-h+1} \\
&< (M_k - 1) \sum_{t=.46335k^{1/2}-1}^{\infty} \frac{1}{t^2} + \sum_{t=.46335k^{1/2}-1}^{k^{1/2}-1} \frac{1}{t} + .6012 \\
&< \frac{M_k - 1}{.46335k^{1/2}-2} + (\ln(.4002k^{1/2}) + .6012 \\
&< \frac{2.499(M_k-1)}{k^{-1/2}} + .9158 + .6012 = \frac{2.499(M_k-1)}{k^{-1/2}} + 1.5170
\end{aligned}$$

Thus we obtain

$$M_k < \frac{1.5170 - 2.499k^{-1/2}}{1-2.499k^{-1/2}} < \frac{1.5170}{1-(2.499)/(31.62)} < \frac{1.5170}{.9209} < 1.6474\ .$$

But since M_k must be an integer, this means that $M_k \leq 1$, so the maximum gap satisfies $g \leq 1/k$. We conclude that

$$\text{(5.8)} \qquad \frac{W_B}{s} < 1.0327g\left(\frac{r}{j}\right)\left(\frac{r+1}{j+1}\right) < \frac{1.0327}{k} < .00104\ .$$

Thus (5.7a) is the dominant bound. Combining (5.7a) and (5.5a), we conclude that

$$\frac{W}{s} = \frac{W_A}{s} + \frac{W_B}{s} < .00116 + .00155 < .00271\ ,$$

as claimed. This completes the proof of the theorem. ∎

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