Principles of Communication Systems: A Compact First Course

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Prolegomena There are many texts available for the study of communication systems at the undergraduate level. Several of these are given in Appendix 1. They are distinguished by individual styles, the maturity expected of the reader, material covered, and the sequence in which the various topics are presented. In spite of the differences, all can be recommended for supplementary reading and sources of problems.

A common problem of current texts is that they finesse the question of what to teach by presenting an encyclopedic account of the subject that must subsume whatever *anyone* might choose to teach in a first course. This approach sells books, but fails to resolve an important pedagogical question. It is in responding to that question that these notes distinguish themselves – for better or worse. We cover only that which is taught in class.

Appendix 2 gives a brief review of complex numbers and also derives a number of trigonometric identities. Appendix 3 lists important dates and events in the development of communication systems, and Appendix 4 lists keywords and phrases in the field that relate closely to our treatment of the subject.

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1 Signals and systems

1.1 Signal models

For our purposes a signal is a time history of some varying electrical, optical, or other physical quantity. A model is an idealization of reality (e.g., lumped-element circuits) that serves as a useful approximation or to develop useful insights.

Our interest is in *deterministic* signals: given (known) functions of time, e.g., the sinusoidal signal

$$x(t) = A\cos(\omega t + \theta) \tag{1}$$

with A (amplitude), ω (frequency in radians/sec), and θ (phase angle in radians) all given, and *random* signals: (sample) functions of time drawn randomly from some set of possible signals. In the above example, any of A, ω, θ could be random variables. The first part of the course will be restricted to deterministic signals.

Other examples that will recur often are the *unit rectangular pulse* centered at $t = 0$

$$\Pi(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

also denoted by $rect(t)$, and the *triangle function*

$$\Delta(t) = \begin{cases} 0, & |t| > 1/2 \\ 1 - 2|t|, & |t| < 1/2 \end{cases}$$

Note that in (1), $x(t) = x(t+T), \ -\infty < t < \infty$, so (1) is a *periodic* signal, with period $T = 2\pi/\omega$ and frequency $f := \frac{\omega}{2\pi} = 1/T$ in hertz. By choosing a $T > 1$, (2) converts to a periodic function $\Pi^*(t) = \Pi^*(t+T)$ for all t, where $\Pi^*(t) = \Pi(t), \ |t| \leq T/2$. A similar extension can be made to the triangle function.

1.2 Phasors and spectra

1.2.1 Phasors

A *rotating phasor* is a (complex) signal

$$\tilde{x}(t) = Ae^{j(\omega t + \theta)}, \quad \infty < t < \infty$$

defined by a frequency ω, a magnitude A, and a phase θ. The term *phasor* by itself refers only to $Ae^{j\theta}$, the frequency being implicit. The period is $T = 2\pi/\omega$ since this is the smallest number such that

$$e^{j(\omega[t+T]+\theta)} = e^{j(\omega t + \theta)}, \quad -\infty < t < \infty.$$

Alternative geometric interpretations of the real signal

$$x(t) = A\cos(\omega t + \theta)$$

correspond to the expressions

$$x(t) = \Re\{\tilde{x}(t)\}$$

and

$$x(t) = \frac{1}{2}Ae^{j(\omega t+\theta)} + \frac{1}{2}Ae^{-j(\omega t+\theta)} = \frac{\tilde{x}(t) + \tilde{x}^*(t)}{2}$$

1.2.2 Spectra

The amplitude (phase) spectrum of a signal gives the amplitude (phase) of the signal as a function of frequency. For a simple sinusoid like $A\cos(\omega_0 t + \theta_0)$, the (one-sided) amplitude spectrum is

$$\begin{aligned} A(\omega) &= A, \quad \omega = \omega_0, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

and phase spectrum

$$\begin{aligned} \theta(\omega) &= \theta_0, \quad \omega = \omega_0 \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

The corresponding rotating-phasor, two-sided amplitude spectrum is

$$\begin{aligned} \tilde{A}(\omega) &= A/2, \quad \omega = \omega_0 \\ &= A/2, \quad \omega = -\omega_0 \\ &= 0, \quad \text{otherwise} \end{aligned}$$

and the corresponding phase spectrum is

$$\begin{aligned} \tilde{\theta}(\omega) &= +\theta_0, \quad \omega = \omega_0 \\ &= -\theta_0, \quad \omega = -\omega_0 \\ &= 0, \quad \text{otherwise} \end{aligned}$$

More general example: Find the spectra of

$$\begin{aligned} x(t) &= 10\cos(4\pi t + \pi/6) + 3\sin(10\pi t + 2\pi/3) \\ &= 10\cos(4\pi t + \pi/6) + 3\cos(10\pi t + \pi/6) \\ &= 5e^{j(4\pi t+\pi/6)} + 5e^{-j(4\pi t+\pi/6)} + \frac{3}{2}e^{j(10\pi t+\pi/6)} + \frac{3}{2}e^{-j(10\pi t+\pi/6)} \end{aligned}$$

So by inspection the amplitude spectrum is $\tilde{A}(\omega) = 5$ $(\omega = \pm 4\pi)$, $= \frac{3}{2}$ $(\omega = \pm 10\pi)$, $= 0$ (otherwise), and the phase spectrum is $\tilde{\theta}(\omega) = \pm\frac{\pi}{6}$ $(\omega = \pm 4\pi, \pm 10\pi)$, $=$ 0 (otherwise).

1.3 Useful operations on functions

The study and manipulation of signals is often made easier by expressing them in terms of operations on more elementary functions. Three such operations are translation (advance or delay), scaling up or down, and reversal. A function $f(t)$ is a translation of $g(t)$ if, for some constant T, we have $f(t) = g(t - T)$; if $T > 0$, then f is g delayed by T, and if $T < 0$, then f is g advanced by T. Clearly, $f(t) = g(at)$ rescales g by the factor a; if $a > 1$, then f scales g down by a, and if $0 < a < 1$, then f scales g up by $1/a$. Time reversal is embodied in $f(t) = g(-t)$.

Example: Sketch the functions

$$g(t) = \begin{cases} t, & 0 \le t \le 1 \\ 0, & otherwise \end{cases}$$

and

$$f(t) = \begin{cases} 1 - t, & 0 \le t \le 1 \\ \frac{1}{2}(t-1), & 1 \le t \le 2 \\ 0, & otherwise \end{cases}$$

It is easily verified that, in terms of g and the rectangular pulse function, we can express f as

$$f(t) = \Pi(t - \frac{1}{2}) - g(t) + g(\frac{1}{2}(t-1))$$

1.4 Singularity functions

Define $\delta(t)$ by the property

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$$

for all functions x continuous at 0. Call it the *unit impulse* (or the *Dirac delta function*). If we choose $x(t) = 1 \ (t_1 < t < t_2), \ = 0$ (otherwise), we get

$$\begin{aligned} \int_{t_1}^{t_2} \delta(t - t_0)dt &= 1, \ t_0 \in (t_1, t_2) \\ &= 0, \ t_0 \notin (t_1, t_2) \end{aligned}$$

as an alternative definition of δ (either definition follows from the other). The *sifting* or *sampling* property is an immediate consequence:

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0).$$

Other basic properties include:

(i)
$$\delta(at) = \frac{1}{|a|}\delta(t)$$

This property follows from

$$\int_{-\infty}^{\infty} \delta(|a|t)dt = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(z)dz = \frac{1}{|a|}$$

and

$$\int_{-\infty}^{\infty} \delta(-|a|t)dt = -\frac{1}{|a|}\int_{\infty}^{-\infty} \delta(z)dz = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(z)dz = \frac{1}{|a|}$$

We see that $\delta(t) = \delta(-t)$ is a special case of the above argument.

(ii)

$$\begin{aligned} \int_{t_1}^{t_2} x(t)\delta(t-t_0)dt &= x(t_0),\ t_0 \in (t_1, t_2) \\ &= 0,\ t_0 \notin (t_1, t_2) \end{aligned}$$

(iii)
$$\int_{t_1}^{t_2} x(t)\delta^{(n)}(t-t_0)dt = (-1)^n x^{(n)}(t_0),\ t_0 \in (t_1, t_2)$$

For $n = 1$, integrate by parts to obtain for $t_0 \in (t_1, t_2)$,

$$\begin{aligned} \int_{t_1}^{t_2} x(t)\delta^{(1)}(t-t_0)dt &= x(t)\delta(t-t_0)|_{t_1}^{t_2} - \int_{t_1}^{t_2} x^{(1)}(t)\delta(t-t_0)dt \\ &= -\int_{t_1}^{t_2} x^{(1)}(t)\delta(t-t_0)dt = -x^{(1)}(t_0) \end{aligned}$$

For a complete proof of property (iii), use induction on n.

We can approximate δ by any of a large class of unit-area functions whose "widths" are infinitesimally small. For example,

$$\delta_\epsilon(t) := \frac{1}{2\epsilon}\Pi\left(\frac{t}{2\epsilon}\right) = \begin{cases} \frac{1}{2\epsilon} & |t| < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

Indeed, the definition of the singularity function δ can be made via a limit of an approximation, e.g.,

$$\begin{aligned} \int_{-\infty}^{\infty} x(t)\delta(t)dt &:= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \frac{1}{2\epsilon}\Pi\left(\frac{t}{2\epsilon}\right) x(t)dt \\ &= \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} \frac{x(t)}{2\epsilon}dt = x(0), \end{aligned}$$

assuming $x(t)$ is continuous at 0.

The *unit step* is also a useful function, e.g., for generating rectangular signal shapes,

$$u(t) := \int_{-\infty}^{t} \delta(t)dt = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

with $u(0)$ undefined. Then $\frac{du(t)}{dt} = \delta(t)$. Note that

$$\Pi(t) = u(t + 1/2) - u(t - 1/2)$$

where $u(.)$ is defined.

1.5 Signal classifications

The energy in a (possibly complex) signal $x(t)$ over an interval of length T centered at 0 is

$$E := \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

For example, if $e(t)$ is the voltage across a resistance R and $i(t)$ is the current through R, then the instantaneous per-ohm power is $|x(t)|^2 = |i(t)|^2$. The power in $x(t)$ is

$$P := \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

We say that $x(t)$ is an *energy signal* if and only if $0 < E < \infty$ (and hence $P = 0$) and a *power signal* if and only if $0 < P < \infty$ (and hence $E = \infty$). To find a signal that is neither, find one for which $\int_{-T/2}^{T/2} |x(t)|^2 dt$ is increasing in T (so $E = \infty$) but grows slower than linearly in T (so $P = 0$). For example, take $x(t) = |t|^{-1/4}$, so that $x^2(t) = |t|^{-1/2}$ and

$$\int_{-T/2}^{T/2} |t|^{-1/2} dt = 4t^{1/2}|_0^{T/2} = 4\sqrt{T/2}$$

Then

$$E = \lim_{T\to\infty} 4\sqrt{T/2} = \infty$$

whereas

$$P = \lim_{T\to\infty} \frac{1}{T} 4\sqrt{T/2} = 0$$

so $x(t)$ is neither an energy signal nor a power signal.

For our sinusoid example, $x(t) = A\cos(\omega_0 t + \theta)$, with period $T_0 = 2\pi/\omega_0$, no limit actually needs to be taken; we get

$$\begin{aligned} P &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \\ &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} A^2 \cos^2(\omega_0 t + \theta) dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{A^2}{T_0}\int_0^{T_0}[\frac{1}{2}+\frac{1}{2}\cos(2(\omega_0 t+\theta))]dt \\
&= \frac{A^2}{2},
\end{aligned}
$$

which shows that the sinusoid is a power signal. Other classifications distinguish signals according as they are digital or analog, continuous-time or discrete time, periodic or nonperiodic, and deterministic or random.

The power in $x(t)$ is the *mean square* value of $x(t)$; the square root of the power is called the *root mean square* value of $x(t)$

1.6 Complex exponential Fourier series

1.6.1 Definitions

Let $x(t)$ be defined over (t_0, t_0+T_0). Then the complex exponential Fourier series is defined on this interval by a sum of phasors:

$$x(t) = \sum_{-\infty}^{\infty} X_n e^{jnw_0 t}, \quad t_0 \le t \le t_0+T_0 \tag{3}$$

where

$$X_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jn\omega_0 t}dt \tag{4}$$

which is an exact representation at all points t of continuity in (t_0, t_0+T_0). Also, the RHS of (3) is periodic with fundamental period T_0, so if $x(t)$ is periodic with the same period, then (3) holds for all points of continuity and (4) is independent of t_0. As can be seen, X_n and X_{-n} are complex conjugates, if $x(t)$ is real. It can be proved that the sequence $\{X_n\}$ representing $x(t)$ is unique. An application of this result follows.

Example: Find the Fourier series for $x(t) = \cos^3(\omega_0 t)$. We have

$$\cos^2(\omega_0 t) = \frac{1}{2}+\frac{1}{2}\cos(2\omega_0 t)$$

and so

$$
\begin{aligned}
\cos^3(\omega_0 t) &= \frac{1}{2}(e^{j\omega_0 t}+e^{-j\omega_0 t})(\frac{1}{2}+\frac{1}{4}[e^{2j\omega_0 t}+e^{-2j\omega_0 t}]) \\
&= \frac{1}{4}[e^{j\omega_0 t}+\frac{1}{2}e^{3j\omega_0 t}+\frac{1}{2}e^{-j\omega_0 t}+e^{-j\omega_0 t}+\frac{1}{2}e^{-3j\omega_0 t}+\frac{1}{2}e^{j\omega_0 t}]
\end{aligned}
$$

and finally, the nonzero coefficients are

$$X_1 = X_{-1} = \frac{3}{8},\ X_3 = X_{-3} = \frac{1}{8}$$

Uniqueness is being used in the sense that we don't have to evaluate (4) directly to obtain the above result. ■

Note that X_n is a time average of $x(t)e^{-jn\omega_0 t}$. The time average of any given function $v(t)$ is often denoted by

$$\langle v(t) \rangle := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v(t)dt$$

and if $v(t)$ is periodic with period T_0, then

$$\langle v(t) \rangle := \frac{1}{T_0} \int_{T_0} v(t)dt,$$

where $\int_{T_0}$ indicates integration over any period.

We see that $X_0 = \langle x(t) \rangle$ is the dc component of $x(t)$. From

$$X_n = \langle x(t)\cos(n\omega_0 t) \rangle - j\langle x(t)\sin(n\omega_0 t) \rangle = |X_n|e^{j\theta_n}$$

we note that $|X_n| = |X_{-n}|$, assuming, as we normally will, that $x(t)$ is real. Thus, $|X_n|$ is an even function of n. Also, again assuming that $x(t)$ is real, and hence that X_n and X_{-n} are complex conjugates, the phase is an odd function of n, i.e., $\theta_{-n} = -\theta_n$.

1.6.2 Symmetry properties

(i) If $x(t)$ is even, i.e., $x(-t) = x(t)$, then $\langle x(t)\sin(n\omega_0 t) \rangle = 0$ since $\sin\theta$ is an odd function. In this case, X_n is real and even in n.

(ii) If $x(t)$ is odd $(x(-t) = -x(t))$, then $\langle x(t)\cos(n\omega_0 t) \rangle = 0$ must hold, so X_n is imaginary and odd in n.

(iii) *Half-wave symmetry* for periodic $x(t)$ means that $x(t \pm \frac{1}{2}T_0) = -x(t)$. It follows that $X_n = 0$ for $|n|$ even, and

$$X_n = \frac{1}{T_0/2} \int_0^{T_0/2} x(t)e^{-jn\omega_0 t}dt,$$

otherwise. (To prove this, break down the integral into the two regions $(-T_0/2, 0)$ and $(0, T_0/2)$, and in the first substitute $-x(t+T_0/2)$ for $x(t)$.) Note that sinusoids are half-wave symmetric independent of phase.

1.6.3 Trigonometric forms

Assume $x(t)$ is real. Fold the $-n$ part of the series over onto the positive part to get

$$x(t) = X_0 + \sum_{n=1}^{\infty} \left(X_n e^{jn\omega_0 t} + X_{-n} e^{-jn\omega_0 t} \right)$$

Then write X_n and X_{-n} in phasor form to obtain

$$
\begin{aligned}
x(t) &= X_0 + \sum_{n=1}^{\infty} |X_n| \left(e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)} \right) \\
&= X_0 + \sum_{n=1}^{\infty} 2|X_n| \cos(n\omega_0 t + \theta_n) \qquad (5)
\end{aligned}
$$

which is called the *compact trigonometric form.* Using standard trigonometric identities, we could have reached the same result by starting with the classical trigonometric form of the Fourier series representation of $x(t)$

$$
x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} B_n \sin(n\omega_0 t) \qquad (6)
$$

with

$$
\begin{aligned}
A_0 &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) dt \\
A_n &= \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(n\omega_0 t) dt, \ n > 0, \qquad (7) \\
B_n &= \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin(n\omega_0 t) dt, \ n > 0 \qquad (8)
\end{aligned}
$$

Exercise: Work out the details of this approach. Also, prove (7) by multiplying (6) by $\cos(m\omega_0 t)$ and then integrating over a period. Verify (8) similarly. ■

Examples:

(i) The periodic pulse train, $x(t) = A\Pi(t)$, $|t| < 1$, with period $T_0 = 2$ so that $x(t) = x(t+2)$ for all t. We have $\omega_0 = 2\pi/T_0 = \pi$, and hence

$$
\begin{aligned}
X_n &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} A\Pi(t) e^{-jn\omega_0 t} dt \\
&= \frac{1}{2} \int_{-1}^{1} A\Pi(t) e^{-jn\pi t} dt \\
&= \frac{A}{2} \int_{-1/2}^{1/2} e^{-jn\pi t} dt = -\frac{Ae^{-jn\pi t}}{2jn\pi} \Bigg|_{-1/2}^{1/2} \\
&= \frac{A}{2} \left(\frac{e^{jn\pi/2} - e^{-jn\pi/2}}{2 \cdot jn\pi/2} \right) \\
&= \frac{A}{2} \frac{\sin(n\pi/2)}{n\pi/2} = \frac{A}{2} \mathrm{sinc}(n\pi/2)
\end{aligned}
$$

where

$$\mathrm{sinc}(z) := \frac{\sin z}{z}$$

and is called the "sinc" (pronounced "sink") function.

(ii) The half-rectified sine wave, $x(t) = A\sin(\omega_0 t)$, $mT_0 < t < mT_0 + T_0/2$, $m = 0, \pm 1, \pm 2, \dots$ and $x(t) = 0$, elsewhere (just truncate the sine wave beneath the t axis). We find

$$\begin{aligned} X_n &= \frac{1}{T_0}\int_0^{T_0/2} A\sin(\omega_0 t)e^{-jn\omega_0 t}dt \\ &= \frac{A}{T_0}\int_0^{T_0/2} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} e^{-jn\omega_0 t}dt \end{aligned}$$

For $n = 0$ we get

$$X_0 = -\frac{A\omega_0}{2\pi}\frac{\cos(\omega_0 t)}{\omega_0}\bigg|_0^{T_0/2} = -\frac{A}{2\pi}(\cos\pi - 1) = \frac{A}{\pi}.$$

With $n = 1$, we arrive at

$$\begin{aligned} X_1 &= \frac{A}{T_0}\int_0^{T_0/2} \frac{1 - e^{-j2\omega_0 t}}{2j} dt \\ &= \frac{A}{T_0}\frac{t}{2j}\bigg|_0^{T_0/2} \\ &= \frac{A}{4j} = -\frac{jA}{4} \end{aligned}$$

Similarly, for $n = -1$, we compute $X_{-1} = \frac{jA}{4}$. Finally, for $n > 1$, we compute

$$\begin{aligned} X_n &= \frac{A}{T_0}\int_0^{T_0/2} \left(\frac{e^{j\omega_0 t(1-n)}}{2j} - \frac{e^{-j\omega_0 t(1+n)}}{2j}\right) dt \\ &= \frac{A}{4\pi}\left(\frac{(n+1)e^{-j(n-1)\pi} - (n-1)e^{-j(n+1)\pi}}{n^2 - 1} - \frac{2}{n^2 - 1}\right) \end{aligned}$$

If $|n| = 2, 4, 6, \dots$, then since $e^{-j(n-1)\pi} = -1 = e^{j(n+1)\pi}$, we get

$$X_n = \frac{A}{\pi(1 - n^2)}, \ n = \pm 2, \pm 4, \dots$$

On the other hand, if $|n|$ is odd, then $e^{-j(n-1)\pi} = 1 = e^{j(n+1)\pi}$, and so

$$X_n = 0, \ n = \pm 3, \pm 5, \dots$$

1.6.4 Spectra

The complex exponential Fourier series of a signal is a sum of phasors. Each contributes an element of the two-sided discrete amplitude and phase spectra of the signal, $|X_n|$ and θ_n. The one-sided discrete spectra apply to the trigonometric forms of the Fourier series, in which case $2|X_n|,\ n>0$, is the amplitude spectrum, and the phase spectrum is confined to positive frequencies. The d-c component of the signal remains the same.

Examples:

(i) Consider the half-rectified sine wave and our earlier results for this signal. We have for even n with $n>1$

$$X_n = -\frac{A}{\pi(n^2-1)} = \frac{A}{\pi(n^2-1)}e^{-j\pi}, \quad X_{-n} = \frac{A}{\pi(n^2-1)}e^{j\pi}$$

and hence the two-sided amplitude spectrum

$$\begin{aligned} |X_n| &= \frac{A}{\pi(n^2-1)}, \ n=\pm 2, \pm 4, \ldots \\ &= 0,\ n=\pm 3, \pm 5, \ldots \end{aligned}$$

The corresponding two-sided phase spectrum is

$$\theta_n = \begin{cases} -\pi, & n=2,4,6,\ldots \\ +\pi, & n=-2,-4,-6,\ldots \\ 0, & n=\pm 3, \pm 5, \pm 7, \ldots \end{cases}$$

When $|n|=1$, we have $X_{\pm 1} = \mp\frac{jA}{4} = \frac{A}{4}e^{\mp j\pi/2}$, and so $|X_{\pm 1}| = A/4$, with $\theta_1 = -\pi/2,\ \theta_{-1} = \pi/2$. Note finally that $|X_0| = A/\pi$ and $\theta_0 = 0$.

(ii) The signal

$$x(t) = \sum_{-\infty}^{\infty} A\Pi\left(\frac{t-nT_0-\tau/2}{\tau}\right)$$

is a train of width-τ pulses with period T_0 and amplitude A. An extension of Example 2.11, p. 57, which we leave as an exercise, gives the coefficients

$$X_n = \frac{A\tau}{T_0}\mathrm{sinc}(n\pi f_0\tau)e^{-j\pi n f_0 \tau}$$

so the amplitude spectrum is

$$|X_n| = \frac{A\tau}{T_0}|\mathrm{sinc}(n\pi f_0\tau)|$$

and the phase spectrum is

$$\theta_n = \begin{cases} -\pi n f_0\tau & \text{if } \mathrm{sinc}(n\pi f_0\tau) > 0 \\ -\pi n f_0 + \pi & \text{if } n>0 \text{ and } \mathrm{sinc}(n\pi f_0\tau) < 0 \\ -\pi n f_0 - \pi & \text{if } n<0 \text{ and } \mathrm{sinc}(n\pi f_0\tau) < 0 \end{cases}$$

The sinc function of $|X_n|$ has a global maximum at the origin and zeros where nf_0 is some multiple of τ^{-1}. When τ is significantly smaller than T_0, the local maxima of $|X_n|$ out beyond τ^{-1} are rather small compared to the maximum of A at the origin, so the (positive) frequencies in which most of the signal energy is concentrated lie between 0 and τ^{-1}. This region is often associated with the *bandwidth* of the signal. *Note particularly that the bandwidth (under any reasonable measure) varies inversely with the signal width.* As the signal width becomes smaller, the number of phasors needed in a good Fourier-series approximation increases.

1.7 Fourier transform

By passing the discrete transform to a continuous limit, we generalize the Fourier series to take into account nonperiodic functions $x(t)$. In the interval $|t| < T_0/2$ we have the Fourier series

$$x(t) = \sum_{-\infty}^{\infty} \left[\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(\xi) e^{-j2\pi n f_0 \xi} d\xi \right] e^{j2\pi n f_0 t}$$

For very large T_0, nf_0 is "approximately" a continuous variable f as it is a multiple of a "differential" interval $1/T_0$. Thus, in the continuous limit, $T_0 \to \infty$, replace nf_0 by f, $1/T_0$ by df, and the sum by an integral to get the representation

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \tag{9}$$

in terms of the *Fourier transform*

$$X(f) = \int_{-\infty}^{\infty} x(\xi) e^{-j2\pi f \xi} d\xi \tag{10}$$

The existence of these integrals is assured, given that $x(t)$ is an energy (square integrable) signal. Note that the two integrals are identical except for the sign of the exponent. We could change the frequency variable to the radian measure $\omega = 2\pi f$, but in so doing (9) would require a factor $(2\pi)^{-1}$ and some symmetry would be lost. Symbolically, we write $X(f) = \mathcal{F}[x(t)]$ for the Fourier transform and $x(t) = \mathcal{F}^{-1}[X(f)]$ for the inverse Fourier transform. As before, we can write $X(f) = |X(f)|e^{j\theta(f)}$ in terms of the amplitude and phase spectra $|X(f)|$ and $\theta(f)$ and then verify easily the symmetries, for real $x(t)$,

$$|X(f)| = |X(-f)|, \qquad \theta(f) = -\theta(-f)$$

If $x(t)$ is real, then $\Re X(f)$ is even and $\Im X(f)$ is odd (by inspection of the cosine and sine functions). The transform of a real and even $x(t)$ is real and even, and the transform of a real and odd $x(t)$ is imaginary, with the imaginary part being odd.

Note that the amplitude spectrum is in fact a *density* in that its units are amplitude units divided by frequency units (observe that $X(f)$ was the limit $T_0 \to \infty$ of $T_0 X_n = X_n/f_0$).

1.7.1 Energy spectral density

In the expression

$$E := \int_{-\infty}^{\infty} |x(t)|^2 dt$$

replace one of the $x(t)$ by its conjugate and the other by (9). Then reverse the order of integration to obtain *Rayleigh's energy theorem* (or Parseval's theorem for Fourier transforms):

$$E := \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \tag{11}$$

1.7.2 Convolutions

The convolution operation, discrete or continuous, is a fundamental operation in the mathematical and engineering sciences. For present purposes, it will be needed in the analysis of linear systems and has a simple relation to the Fourier transform. The convolution of two continuous signals $x_1(t)$ and $x_2(t)$ yields a third as follows:

$$x(t) = x_1(t) * x_2(t) := \int_{-\infty}^{\infty} x_1(\xi) x_2(t-\xi) d\xi \tag{12}$$

It is useful, for the purposes of working out convolutions, to think of the integrand as being constructed in the sequence 1) time reversal to give $x_2(-\xi)$, 2) translation to give $x_2(t-\xi)$, and 3) multiplication.

Example: Consider $x_1(t) = e^{-\alpha t} u(t), \alpha > 0$, and $x_2(t) = \Pi(t)$. Reversal of the square pulse changes nothing, but we must distinguish $t < 1/2$ from $t > 1/2$, for in the former case the pulse overlaps 0 (the integral must be from 0 to $t+1/2$) and in the latter case, the pulse is entirely beyond 0 and the integral must be between $t - 1/2$ and $t + 1/2$.

$$\begin{aligned} x_1(t) * x_2(t) &= 0, \ t < -\frac{1}{2} \\ &= \int_0^{t+1/2} e^{-\alpha\xi} d\xi = \frac{1}{\alpha}\left[1 - e^{-\alpha(t+\frac{1}{2})}\right], \ -\frac{1}{2} \le t \le \frac{1}{2} \\ &= \int_{t-1/2}^{t+1/2} e^{-\alpha\xi} d\xi = \frac{e^{-\alpha t}}{\alpha}\left(e^{1/2} - e^{-1/2}\right), \ t > \frac{1}{2}. \end{aligned}$$

■

1.7.3 Properties of the Fourier transform

In addition to the symmetry properties already mentioned, there are several other properties of the Fourier transform that simplify computational problems. In the formulas below, $X_i(f)$ is the Fourier transform of $x_i(t)$.

(a) **Superposition:** Substitution into (9) shows that

$$\mathcal{F}(x_1(t) + x_2(t)) = X_1(f) + X_2(f)$$

(b) **Time and frequency translation:** Substitution and change of variables gives

$$\mathcal{F}(x(t - t_0)) = X(f)e^{-j2\pi f t_0} \tag{13}$$

and

$$\mathcal{F}\left(x(t)e^{j2\pi f_0 t}\right) = X(f - f_0) \tag{14}$$

(c) **Modulation formula** Writing $\cos(2\pi f_0 t)$ in exponential form and using the above two properties, one obtains

$$\mathcal{F}(x(t)\cos(2\pi f_0 t)) = \frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0) \tag{15}$$

(d) **Scale change:** As in the case of the complex exponential Fourier series, we have

$$\mathcal{F}(x(at)) = \frac{1}{|a|}X\left(\frac{f}{a}\right)$$

(e) **Duality:** A change of variables gives

$$\mathcal{F}(X(t)) = x(-f)$$

so that if $x(\xi) = x(-\xi)$, we have the complete duality in which $x(t)$ and $X(t)$ are the transforms of one another.

(e) **Convolution:** Write

$$\int_{-\infty}^{\infty} x_1(\xi)x_2(t-\xi)d\xi = \int_{-\infty}^{\infty} x_1(\xi)\left[\int_{-\infty}^{\infty} X_2(f)e^{j2\pi f(t-\xi)}df\right]d\xi$$

and then change the order of integration and transform to obtain

$$\mathcal{F}(x_1(t) * x_2(t)) = X_1(f)X_2(f) \tag{16}$$

Analogously,

$$\mathcal{F}(x_1(t)x_2(t)) = X_1(f) * X_2(f) \tag{17}$$

The convolution of $x(t)$ with $\delta(t - t_0)$ performs a translation

$$\delta(t - t_0) * x(t) = \int_{-\infty}^{\infty} \delta(\xi - t_0)x(t - \xi)d\xi = x(t - t_0). \tag{18}$$

(f) **Differentiation and integration:** Integration by parts and induction gives

$$\mathcal{F}\left(\frac{d^n x(t)}{dt^n}\right) = (j2\pi f)^n X(f) \tag{19}$$

The analogous result for integration is

$$\mathcal{F}\left(\int_{-\infty}^{t} x(\xi)d\xi\right) = \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f) \tag{20}$$

To prove this, note that

$$x(t) * u(t) = \int_{-\infty}^{\infty} g(\xi)u(t-\xi)d\xi = \int_{-\infty}^{t} x(\xi)d\xi$$

so the transform of this signal is the product of the transforms of $x(t)$ and $u(t)$. The latter transform is derived in one of the examples below and is given in (23); multiplying that result by $X(f)$ gives (20).

Examples:

(a) Compute the transform of $x(t) = Ae^{\mp j\pi t/a}$, for $|t| \leq a/2$, and $x(t) = 0$ elsewhere.

$$\begin{aligned}
X(f) &= A\int_{-a/2}^{a/2} e^{\mp j\pi t/a}e^{-j2\pi ft}dt \\
&= A\left.\frac{e^{\mp jt(\frac{\pi}{a}\pm 2\pi f)}}{\mp j(\frac{\pi}{a}\pm 2\pi f)}\right|_{-a/2}^{a/2} \\
&= A\frac{e^{\mp j\frac{a}{2}(\frac{\pi}{a}\pm 2\pi f)} - e^{\pm j\frac{a}{2}(\frac{\pi}{a}\pm 2\pi f)}}{\mp j(\frac{\pi}{a}\pm 2\pi f))} \\
&= \frac{2A\sin(\frac{\pi}{2}\pm \pi fa)}{\frac{\pi}{a}\pm 2\pi f} \\
&= \frac{2Aa/\pi}{1\pm 2fa}\cos\pi fa
\end{aligned}$$

With this result, we can easily find the Fourier transform of the sinusoidal pulse $x(t) = A\cos\frac{\pi t}{a}$ for $|t| \leq a/2$ and $x(t) = 0$ elsewhere. From the exponential form of the cosine, we get

$$\begin{aligned}
X(f) &= \frac{Aa}{\pi}\cos\pi fa\left(\frac{1}{1-2fa} + \frac{1}{1+2fa}\right) \\
&= \frac{2Aa/\pi}{1-(2fa)^2}\cos\pi fa
\end{aligned}$$

Exercise: Give the amplitude and phase spectra corresponding to this result.

(b) **Singularity functions:** These signals are not functions (much less energy signals) in the usual sense, but we can extend the Fourier transform to include such functions in an obvious way. The straight and narrow of mathematics would have us express the integrals/functions as limits of suitably designed functions, but we will risk little by continuing as before. For example,

$$\mathcal{F}(A\delta(t-t_0)) = \int_{-\infty}^{\infty} A\delta(t-t_0)e^{-j2\pi ft}dt = Ae^{-j2\pi ft_0}$$

and analogously,

$$\mathcal{F}(Ae^{j2\pi f_0 t}) = A\delta(f-f_0)$$

These relations will be of frequent use in our discussion of modulation.

But there are hazards. For consider the unit-step function. It is tempting to use the equation

$$\Pi(t) = u(t+\frac{1}{2}) - u(t-\frac{1}{2}) \tag{21}$$

as a means of computing the Fourier transform of the unit-step $u(t)$. Proceeding in this (ill-starred) way, one makes use of

$$\mathcal{F}(u(t+\frac{1}{2})) = \mathcal{F}(u(t))e^{j\pi f}, \quad \mathcal{F}(u(t-\frac{1}{2})) = \mathcal{F}(u(t))e^{-j\pi f}$$

by the translation theorem on p. 14. And then one substitutes into (21) to get

$$\begin{aligned}\mathcal{F}(\Pi(t)) &= \mathcal{F}(u(t))\left[e^{j\pi f} - e^{-j\pi f}\right] \\ &= \mathcal{F}(u(t))2j\sin\pi f\end{aligned}$$

and since $\mathcal{F}(\Pi(t)) = \mathrm{sinc}(\pi f) = \frac{\sin\pi f}{\pi f}$ we get

$$\mathcal{F}(u(t)) = \frac{1}{j2\pi f} \tag{22}$$

But we can show for the signum function, $\mathrm{sgn}(t) = -1$ for $t < 0$ and $+1$ for $t > 0$, that

$$\begin{aligned}\mathcal{F}(\mathrm{sgn}(t)) &= \lim_{\alpha\to 0}[\mathcal{F}(u(t)e^{-\alpha t}) + \mathcal{F}(u(-t)e^{\alpha t}] \\ &= \lim_{\alpha\to 0} -\frac{j4\pi f}{\alpha^2 + (2\pi f)^2} \\ &= \frac{1}{j\pi f}\end{aligned}$$

(See p. 84 of Lathi, for example.) Next, use the easily verified relation $u(t) = \frac{1}{2}[1 + \mathrm{sgn}(t)]$ to get

$$\mathcal{F}(u(t)) = \frac{1}{2}[\mathcal{F}(1) + \mathcal{F}(\mathrm{sgn}(t)]$$

and hence from the earlier expression for $\mathcal{F}(\text{sgn}(t))$,

$$\mathcal{F}(u(t)) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f} \tag{23}$$

which is *not* the same as (22). What is wrong with the development of (22)?

As another similar trap, we know that $\delta(t) = \frac{d}{dt}u(t)$, so along with the formula (19) with $n = 1$ we again find that (22) holds, and therefore again go wrong (why?).

(c) The transform of piece-wise linear waveforms can often be computed by just using the Fourier transform properties and the transforms of $\delta(t)$ and its derivatives. For example, consider the waveform

$$\begin{aligned} x(t) &= 1,\ 0 \le t \le 2 \\ &= -(t-3),\ 2 \le t \le 3 \\ &= 0,\ \text{elsewhere} \end{aligned}$$

This can be rewritten as

$$x(t) = u(t) - u(t-2) - (t-3)[u(t-2) - u(t-3)]$$

Then,

$$x'(t) = \delta(t) - \delta(t-2) + \delta(t-2) - [u(t-2) - u(t-3)]$$

and

$$x''(t) = \delta'(t) - \delta(t-2) + \delta(t-3)$$

so by the differentiation and time-shift properties we get

$$\mathcal{F}(x''(t)) = (j2\pi f)^2 X(f) = j2\pi f - e^{-j4\pi f} + e^{-j6\pi f}$$

and after easy manipulations

$$X(f) = \frac{1}{j2\pi f}\left(1 - e^{-j5\pi f}\text{sinc}(2\pi f)\right)$$

(d) The *ideal sampling waveform* is given by

$$y_s(t) := \sum_{-\infty}^{\infty} \delta(t - mT_s)$$

and describes a sequence of unit impulses spaced T_s time units apart. Writing this periodic function in terms of a Fourier series, we get

$$y_s(t) = \sum_{-\infty}^{\infty} Y_n e^{jn2\pi f_s t},$$

where, by integrating over the period bisected by the origin,

$$Y_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)e^{-j2\pi n f_s t}dt = \frac{1}{T_s} = f_s$$

Then transforming term by term gives

$$Y_s(f) = f_s \sum_{-\infty}^{\infty} \mathcal{F}(e^{jn2\pi f_s t}) = f_s \sum_{-\infty}^{\infty} \delta(f - nf_s)$$

Computing the Fourier transform directly and equating it to the above then gives us

$$\sum_{-\infty}^{\infty} e^{j2\pi m T_s f} = f_s \sum_{-\infty}^{\infty} \mathcal{F}(e^{jn2\pi f_s t}) = f_s \sum_{-\infty}^{\infty} \delta(f - nf_s)$$

where we have used the fact that m can be replaced by $-m$ in the first sum. We will return to this signal and its transform when we study sampling theory.

2 Linear systems

Here we study the operations performed by circuits, viewed as single-input, single-output systems. We concentrate first on linear, time-invariant systems. These systems will be composed of circuit elements like resistors, capacitors, ..., but our principal interest will be in the over-all function that the system (circuit) performs on its input. We write for a single-input, single-output system

$$y(t) = \mathcal{H}(x(t))$$

where $\mathcal{H}$ is the operator, $y(t)$ the output, and $x(t)$ the input. Linearity is defined by *superposition*:

$$\begin{aligned} y(t) &= \mathcal{H}(\alpha_1 x_1(t) + \alpha_2 x_2(t)) \\ &= \alpha_1 \mathcal{H}(x_1(t)) + \alpha_2 \mathcal{H}(x_2(t)) \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

and time-invariance is defined by the fact that a translation (in time) of the input gives an equal translation of the output (which is otherwise unchanged). Formally, $y(t-t_0) = \mathcal{H}(x(t-t_0))$.

2.1 System characterization

Define the *impulse response* characterizing a system by

$$h(t) := \mathcal{H}(\delta(t))$$

The response to an impulse applied at any time t_0 is $h(t-t_0)$ so an induction easily establishes that the response to

$$x(t) = \sum_{n=1}^{N} \alpha_n \delta(t-t_n)$$

is

$$y(t) = \sum_{n=1}^{N} \alpha_n h(t-t_n) \tag{24}$$

From the impulse response we can derive the output for any given input by substitution into

$$y(t) = \int_{-\infty}^{\infty} x(\xi) h(t-\xi) d\xi \tag{25}$$

To prove this, approximate the "sifting" integral

$$x(t) = \int_{-\infty}^{\infty} x(\xi) \delta(t-\xi) d\xi$$

by the sum

$$x(t) \approx \sum_{n=N_1}^{N_2} x(n\Delta t)\delta(t-n\Delta t)\Delta t, \quad \Delta t \ll 1$$

where $N_1\Delta t$ and $N_2\Delta t$ are the starting and ending times of the signal. The output $\tilde{y}(t)$ resulting from the approximate signal, after using (24) with $\alpha_n = x(n\Delta t)\Delta t$ and $t_n = n\Delta t$, is

$$\tilde{y}(t) = \sum_{n=N_1}^{N_2} x(n\Delta t)h(t - n\Delta t)\Delta t$$

In the continuous limit, we have $\Delta t \to 0$, with $n\Delta t$ becoming the continuous variable ξ, and the sum becoming an integral. This gives (25), which is called a *superposition integral*, because of the superposition property exploited in the discrete approximation.

A useful notion of stability is that bounded inputs produce bounded outputs. This can be shown to be the case if the impulse response is absolutely integrable.

By the convolution property, the Fourier transform of (25) is

$$Y(f) = H(f)X(f)$$

where X, Y, H are the transforms of the input signal, the output signal, and the impulse response. The transform $H(f)$ is best known as the *transfer function* (TF); in the circuit setting, it is the ratio of the output impedance $Y(f)$ to the input impedance $X(f)$.

In general, of course, $H(f)$ is complex. We write it in the usual form

$$H(f) = |H(f)|e^{j\theta(f)}$$

where $|H(f)|$ is called the system *amplitude response*, and $\theta(f)$ is called the *phase shift*. Note that, since $h(t)$ is real, we have the symmetry properties

$$|H(f)| = |H(-f)| \qquad \theta(f) = -\theta(-f)$$

For responses of linear time-invariant systems to periodic inputs, we need only focus on the response to the signal $Ae^{j2\pi f_0 t}$. By the superposition integral

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\xi)Ae^{j2\pi f_0(t-\xi)}d\xi \\ &= H(f_0)Ae^{j2\pi f_0 t} \end{aligned}$$

By superposition therefore, if $x(t)$ is a general periodic input with Fourier series coefficients X_n and frequency f_0, then

$$y(t) = \sum_{-\infty}^{\infty} X_n H(nf_0)e^{jn2\pi f_0 t}$$

or

$$y(t) = \sum_{-\infty}^{\infty} |X_n||H(nf_0)|e^{j[n2\pi f_0 t + \theta_n + \theta_H(nf_0)]}$$

where θ_n is the phase of the n-th term of the complex exponential Fourier series for $x(t)$ and θ_H is the phase-shift response of the system. What this means about linear time-invariant systems is that, while amplitudes are attenuated by $|H(nf_0)|$ and phases are shifted by $\theta_H(nf_0)$, *frequencies in the output must also be in the input.*

2.2 Filters

Low pass (LP), high pass (HP), and band pass (BP) filters are perhaps the most important linear time-invariant systems for present purposes, so they make perfect examples to illustrate the definitions of section 1.7.1. A simple RC circuit acting as an LP filter has the input across the resistor and capacitor in series and the output across the capacitor (see Example 3.16, Lathi). Thus, if $i(t)$ denotes the current, and $x(t)$ and $y(t)$ are the input and output voltages, we have $x(t) = Ri(t) + y(t)$ and $y(t) = \frac{1}{C}\int^t i(\xi)d\xi$, so the filter is governed by the equation

$$RC\frac{dy}{dt} + y(t) = x(t)$$

Applying the Fourier transform, we get $(j2\pi fRC + 1)Y(f) = X(f)$ and hence

$$\begin{aligned} H(f) &= \frac{Y(f)}{X(f)} = \frac{1}{1 + j(f/f_3)} \\ &= \frac{1}{\sqrt{1 + (f/f_3)^2}} e^{-j\tan^{-1}(f/f_3)} \end{aligned}$$

for the TF, where $f_3 = 1/(2\pi RC)$ is known as the *3-dB frequency*, or *half-power frequency*. (When $f = f_3$, the power spectrum $|H(f)|^2$ is at $1/2$ of its peak value, and the amplitude response is reduced to $1/\sqrt{2}$ of its peak, or $20\log_{10} 1/\sqrt{2} \approx 3$ in decibels.)

Now $\alpha/(\alpha + j2\pi f)$ is the transform of the function given by $\alpha e^{-\alpha t}$, for $t \geq 0$, and 0 for $t < 0$, so we have for the impulse response

$$h(t) = \frac{1}{RC}e^{-t/(RC)}, \quad t \geq 0$$

Note that, for $f \ll f_3$, the amplitude and phase responses are

$$|H(f)| \approx 1, \qquad -\tan^{-1}\frac{f}{f_3} \approx -\frac{f}{f_3}$$

so the time delay is approximately $1/f_3$. Thus, for low enough frequencies, the circuit acts as a distortionless filter.

With the LP-filter system thus characterized, let us look at a couple of specific cases.

Examples:

(i) Consider first what happens to an input pulse (energy signal)

$$x(t) = A\Pi\left(\frac{t - T/2}{T}\right)$$

From the transform for $H(f)$ above, and from the transform of a time-shifted gating pulse (extending the result on p. 81, Lathi), we get

$$Y(f) = \frac{Ae^{-j\pi fT}}{1 + j(f/f_3)}\mathrm{sinc}(\pi fT)$$

To find $y(t)$ it seems easier to evaluate the superposition integral than to invert the above transform. For the former approach, we see that

$$h(t-\sigma) = \frac{1}{RC} e^{-(t-\sigma)/RC}, \quad \sigma \le t$$

but is 0 otherwise. Thus,

$$\begin{aligned} y(t) &= \begin{cases} 0, & t < 0, \\ \int_0^t \frac{A}{RC} e^{-(t-\sigma)/RC} d\sigma, & 0 \le t \le T, \\ \int_0^T \frac{A}{RC} e^{-(t-\sigma)/RC} d\sigma, & t > T \end{cases} \\ &= \begin{cases} 0, & t < 0, \\ A(1 - e^{-t/(RC)}), & 0 \le t \le T, \\ A(e^{-(t-T)/(RC)} - e^{-t/(RC)}), & t > T \end{cases} \end{aligned}$$

Taking the first two derivatives will show that the first derivative is positive and decreasing in $(0, T)$ and negative and increasing in (T, ∞). To assure a sharp rise at the origin and fall at T in response to the square pulse, T/RC needs to be large. A standard (far from unique) measure of the frequency band passed by the filter is the 3dB frequency, which in the present case is $f_3 = 1/2\pi RC$. When the ratio $T/(2\pi RC) = f_3/T^{-1}$ of the 3dB frequency to the approximate spectral width T^{-1} of the pulse is large, the pulse is passed largely undistorted; conversely, when this ratio is small the filter output bears little resemblance to the input. The inverse relation between bandwidth and pulse width (as measured by f_3 and T^{-1}, respectively) will be a recurring theme.

(ii) Suppose a filter with the TF

$$H(f) = 3\Pi(f/26)$$

has as input a half-rectified cosine waveform $x_c(t)$ with fundamental frequency 10Hz. Starting with the half-rectified sine wave $x_s(t)$, we recall that the Fourier coefficients are given by (see Section 2.5.4)

$$X_n^s = \begin{cases} \frac{A}{\pi}(1-n^2), & n = 0, \pm 2, \pm 4, \ldots \\ 0, & n = \pm 3, \pm 5, \ldots \\ \frac{jA}{4}, & n = -1, \\ -\frac{jA}{4}, & n = 1 \end{cases}$$

and so

$$\begin{aligned} x_s(t) = \ldots &- \frac{8A}{\pi} e^{-j80\pi t} - \frac{3A}{\pi} e^{-j40\pi t} + \frac{jA}{4} e^{-j20\pi t} + \frac{A}{\pi} \\ &- \frac{jA}{4} e^{j20\pi t} - \frac{3A}{\pi} e^{j40\pi t} - \frac{8A}{\pi} e^{j80\pi t} - \ldots \end{aligned}$$

The corresponding Fourier series for the half-rectified cosine is obtained simply by multiplying the n-th term in the Fourier series for x_s by $e^{-jn\pi/2}$.

The only components passed by the filter cutting off at $f = 13Hz$ (the pulse shaped amplitude response terminates at f/26 = 1/2) are the middle 3, so after phase shifting and amplifying by $H(0) = 3$, we find that

$$\begin{aligned} y(t) &= \frac{3A}{\pi} + \frac{3jA}{4}\left[e^{-j20\pi t} - e^{j20\pi t}\right]e^{-j\pi/2} \\ &= \frac{3A}{\pi}[1 - \frac{\pi}{2}\cos(20\pi t)] \end{aligned}$$

■

Causality: A linear time-invariant system is *causal* if the output at time t is a function only of the input up to and including time t (not of the future beyond t). Thus, we must have $h(\tau) = 0$ for $\tau < 0$, and we can write

$$y(t) = \int_{-\infty}^{\infty} x(\xi)h(t-\xi)d\xi = \int_{-\infty}^{t} x(\xi)h(t-\xi)d\xi$$

By the Paley-Wiener criterion, if h, H are square integrable, then

$$\int_{-\infty}^{\infty} \frac{|\ln|H(f)||}{1+f^2}df < \infty$$

must hold for causality. This integral is infinite if $H(f)$ vanishes over some finite interval (and hence completely rejects all frequencies over some finite interval), so it is clear that ideal filters are physically unrealizable. Indeed, even exponential decay of the amplitude response is disallowed by the criterion.

As another limitation on practical systems, we note that the signal properties of being strictly band limited and strictly time limited are provably incompatible; if one holds the other can not. Informally, a time limited signal lacks the "smoothness" that's needed for finite Fourier series representations. An analogous statement applies to signal amplitude spectra.

2.3 Distortion

A *distortionless system* is one that produces an output that differs from the output by at most a scale factor, say H_0, and a time delay, say t_0 (where $t_0 > 0$ is needed for causality), i.e.,

$$y(t) = H_0 x(t - t_0)$$

and shape is preserved. Transforms then show that

$$Y(f) = H_0 e^{-j2\pi f t_0} X(f)$$

and hence

$$H(f) = H_0 e^{-j2\pi f t_0}$$

is the TF of a distortionless system, one with an amplitude response uniform over frequency and a phase shift linear in frequency (and hence a delay uniform over frequency).

The *group delay* imposed by a given TF is simply the negative of the derivative of the output phase angle with respect to frequency in radians per second:

$$T_g(f) := -\frac{1}{2\pi}\frac{d\theta(f)}{df}$$

If T_g is a constant as in the distortionless case, then every frequency in the "group" of frequencies passed by the system is delayed in the output by the same amount. The term *phase delay* converts the phase shift of a given periodic signal into a delay:

$$T_p = -\frac{\theta(f)}{2\pi f}$$

The nonlinear case: In addition to amplitude and phase distortion, there are usually nonlinearities in a nominally linear system. For example, a simple form of nonlinear distortion is given in the input/output characteristic

$$y(t) = a_1 x(t) + a_2 x^2(t)$$

For an input with a discrete set of frequencies, consider

$$x(t) = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t$$

which yields the output

$$\begin{aligned} y(t) &= a_1(A_1 \cos \omega_1 t + A_2 \cos \omega_2 t) \\ &+ \frac{a_2}{2}(A_1^2 + A_2^2) + \frac{a_2}{2}(A_1^2 \cos 2\omega_1 t + A_2^2 \cos 2\omega_2 t) \\ &+ a_2 A_1 A_2[\cos(\omega_1 + \omega_2) + \cos(\omega_1 - \omega_2)] \end{aligned}$$

where we have used standard trigonometric identities. Whereas linear time-invariant systems can not introduce new frequencies, we see here that nonlinear systems do; here, double frequencies, called *harmonic distortion* terms, arise as well as sum and difference frequencies, called *intermodulation distortion* terms.

Note that with appropriate filtering, systems with harmonic distortion can be used as devices for muliplying frequency by an integer constant.

For signals with frequencies over a continuous frequency spectrum, the distortion is more general. For example, consider the system above, for which

$$Y(f) = a_1 X(f) + a_2 X(f) \star X(f),$$

with the input transform

$$X(f) = A\Pi\left(\frac{f}{2W}\right)$$

Recalling that $\Lambda(t/\tau) := 1 - |t|/\tau$, for $|t| < \tau$, and $= 0$, otherwise, we get for the distortion

$$a_2 X(f) \star X(f) = 2a_2 W A^2 \Lambda\left(\frac{f}{2W}\right)$$

which introduces interference at all frequencies in the desired output. Note that the bandwidth of the distortion term is twice that for the input.

Ideal, distortionless filters: These are summarized as follows:

- Lowpass: The TF is

$$H_{LP}(f) = H_0 \Pi(f/2B) e^{-j2\pi f t_0}$$

and the impulse response is

$$h_{LP}(t) = 2BH_0 \mathrm{sinc}[2\pi B(t - t_0)]$$

where B is the bandwidth of the passband $[0, B]$.

- Highpass: The TF is

$$H_{HP}(f) = H_0 \left[1 - \Pi\left(\frac{f}{2B}\right)\right] e^{-j2\pi f t_0}$$

- Bandpass: The TF is

$$H_{BP} = H_0 \left[\Pi\left(\frac{f - f_0}{B}\right) + \Pi\left(\frac{f + f_0}{B}\right)\right] e^{-j2\pi f t_0}$$

To find the corresponding impulse response, note that the bracketed term is the transform of $2m(t)\cos 2\pi f_0 t$, by the modulation theorem, where

$$m(t) := \int_{-\infty}^{\infty} \Pi(f/B) e^{j2\pi f t} df = B\mathrm{sinc}(\pi B t),$$

by a change of variables. Thus, multiplying by H_0 and using the time-shift theorem, we get

$$h_{BP}(t) = 2BH_0 \mathrm{sinc}(\pi B[t - t_0]) \cos(2\pi f_0 [t - t_0])$$

Note particularly, that the ideal filters are non-causal, since the impulse response is not identically zero for negative argument. In spite of this, they are useful constructs as they give good insights, and in many cases workable approximations.

There are a variety of techniques for approximating ideal filters by real ones. The approximation can focus on any of several properties, but in so doing a sacrifice is generally made in other properties. The flatness of the amplitude response, the sharpness of the cut-off, and the linearity of the phase shift are examples of such properties.

As an illustration, consider the Butterworth filters that improve amplitude response at the expense of the filter time delay. The amplitude response of the n-th order such filter is given by (the RC circuit already studied is a first order filter)

$$|H(f)| = \frac{1}{\sqrt{1+(f/f_3)^{2n}}} \tag{26}$$

which becomes the ideal flat characteristic as $n \to \infty$. (The 3dB frequency f_3 is defined as before.) The larger n, the *order* of the filter, the flatter the response but the worse the time delay, and the more complicated the circuit (the number of reactive elements grows with n). To see how we can achieve (26), consider a circuit with the input across an inductor L in series with a parallel RC circuit; the output $y(t)$ is taken across the parallel RC circuit. Then we get

$$H(f) = \frac{Z_{RC}}{Z_{RC} + j\omega L}$$

where Z_{RC} is the impedance of the parallel RC circuit

$$Z_{RC} = \frac{R}{1 + j\omega RC}$$

Expanding, we find

$$H(f) = \left[1 + j\frac{2\pi L}{R} - (2\pi\sqrt{LC}f)^2\right]^{-1}$$

Now if we choose the parameters so that $R = \sqrt{L/(2C)}$, then $H(f)$ will have the form (26) with $n = 2$.

Bandwidth vs. pulse width and rise time In our coverage of the RC filter, we saw the inverse relation between pulse width and bandwidth, as measured by the half-power frequency. The constant of (inverse) proportionality depends on circumstances. For example, the zero crossings of the sinc pulse are also useful as a measure of the bandwidth requirement of a pulse centered at 0; such a pulse of width τ has an amplitude spectrum with first zero crossings at $f = \pm\tau^{-1}$.

A similar relation appears when comparing *rectangular approximations* of a signal and its Fourier transform. We say that the rectangle extending vertically from 0 to $x(0)$ and horizontally from $-T/2$ to $T/2$ is a rectangular approximation of the signal $x(t)$ if $\int_{-\infty}^{\infty} |x(t)|dt = x(0)T$. A rectangular approximation of $X(f)$ is defined similarly. If the rectangular approximations of $x(t)$ and its Fourier transform $X(f)$ have widths T and W, respectively, then

$$W \geq \frac{1}{T}$$

is an immediate consequence of

$$Tx(0) = \int_{-\infty}^{\infty} |x(t)|dt \geq \int_{-\infty}^{\infty} x(t)dt = X(0)$$

and

$$WX(0) = \int_{-\infty}^{\infty} |X(f)|df \geq \int_{-\infty}^{\infty} X(f)df = x(0)$$

Rise times are a good measure of distortion and have a relation to bandwidth that is the same, within some constant of proportionality, as the relation that pulse width has. A rise time is variously defined as the time T_R taken by the response to the leading edge of a rectangular pulse to go from 10% to 90% of its final value. This *step response* of a filter is given by

$$y_s(t) = \int_{-\infty}^{\infty} h(\sigma)u(t-\sigma)d\sigma = \int_{-\infty}^{t} h(\sigma)d\sigma$$

Examples:

(i) The impulse response of an LP RC filter is

$$h(t) = \frac{1}{RC}e^{-t/(RC)}u(t)$$

so, with $f_3 = 1/(2\pi RC)$, the step response is

$$y_s(t) = (1 - e^{-2\pi f_3 t})u(t)$$

The rise time as defined above works out numerically to be $T_R \approx 1/(3f_3)$ which shows the inverse relation between rise time and bandwidth.

(ii) The step response of an ideal LP filter with impulse response $2B\mathrm{sinc}[2\pi B(t-t_0)]$ is

$$\begin{aligned} y_s(t) &= \int_{-\infty}^{t} 2B\mathrm{sinc}[2\pi B(\sigma - t_0)]d\sigma \\ &= \int_{-\infty}^{t} 2B\frac{\sin[2\pi B(\sigma - t_0)]}{2\pi B(\sigma - t_0)}d\sigma \\ &= \frac{1}{\pi}\int_{-\infty}^{2\pi B(t-t_0)} \frac{\sin\xi}{\xi}d\xi \\ &= \frac{1}{2} + \frac{1}{\pi}\mathrm{Si}[2\pi B(t-t_0)] \end{aligned}$$

where $\mathrm{Si}(x) := \int_0^x \sin\xi/\xi)d\xi = -\mathrm{Si}(-x)$ is the *sine-integral* function. Numerics show that $T_R \approx 0.44/B$, yet another illustration of the inverse relationship between bandwidth and rise time.

3 Linear modulation

3.1 Introduction

For our purposes, the term *carrier* refers to a given periodic signal, usually just a simple sinusoid like

$$x_c(t) := A(t)\cos[\omega_c t + \phi(t)]$$

The definition of *modulate*, i.e., "to alter or adapt according to circumstances" (Webster's unabridged 1996), will mean in our application to modify the characteristics of a carrier according to the information contained in some *message signal* $m(t)$. Thus, $x_c(t)$ "carries" the information in $m(t)$ from a source to a destination by variations in one or more of its characteristics, which include the amplitude $A(t)$ and the phase angle $\omega_c t + \phi(t)$ or its derivative $\omega_c + \phi'(t)$ Modulation usually shifts the spectral location of the message signal to a region that facilitates transmission. For example, in radio, the relatively small voice spectrum is translated up by hundreds of kHz to levels for which antennas are easily built.

Modulation is classified as *continuous wave* or *pulse* modulation; the latter is further classified as *analog* or *digital*. We begin with continuous-wave modulation which is broken down into linear modulation and angle modulation. The remainder of this chapter covers linear modulation while the next covers angle modulation.

With linear modulation the modulated carrier has the simpler form

$$x_c(t) = A(t)\cos\omega_c t$$

where the amplitude $A(t)$ and the transmitted signal is a linear function of the message signal.

3.2 Double-sideband modulation (DSB)

In the simplest case when the amplitude is directly proportional to $m(t)$, we write

$$x_c(t) = A_c m(t)\cos\omega_c t$$

Applying the modulation theorem (and now we see the origin for that term),

$$X_c(f) = \frac{1}{2}A_c M(f+f_c) + \frac{1}{2}A_c M(f-f_c), \quad f_c = \omega/(2\pi)$$

where $M(f)$ is the Fourier transform of $m(t)$. If we consider the amplitude spectrum, we see that the first term above adds in a replica of the message spectrum centered at $-f_c$ while the second term adds in a replica centered at f_c. Normally, the bandwidth of $m(t)$ is much smaller than f_c, so the replicas are essentially non-interfering. Each replica has an *upper sideband* and *lower sideband* above and below the centering frequency.

It is worth noting that virtually any periodic signal can be used to generate the modulated carrier, so long as its period is $1/f_c$ and an appropriate filter is available. For, if we have such a periodic signal, it can be represented by the Fourier series $\sum_{-\infty}^{\infty} c_n e^{j2\pi n f_c t}$, so its product with the message signal can be written

$$\hat{x}_c(t) := \sum_{-\infty}^{\infty} c_n m(t) e^{j2\pi n f_c t}$$

which has the amplitude spectrum

$$\hat{X}_c(f) = \sum_{-\infty}^{\infty} c_n M(f - n f_c)$$

so that, if $\hat{x}_c(t)$ has no dc component and hence $c_0 = 0$ (as for example in the bipolar pulse waveform of Fig. 4.6(b) in Lathi), we need only lowpass-filter $\hat{x}_c(t)$ to get $x_c(t)$ to within an amplification factor. If there is a dc component (as for example in the gating waveform of Fig. 4.4(b) in Lathi), then we need to bandpass-filter $\hat{x}_c(t)$ in order to eliminate the copy of $M(f)$ centered at the origin. Note particularly that the modulated gating waveform can be obtained by merely switching the message signal $m(t)$ on and off at frequency f_c. As noted by Lathi, there are standard circuits for doing this.

The process of *demodulation* attempts to recover an undistorted version of the original signal $m(t)$. A way to do this is to multiply the received signal by a version of the carrier generated at the receiver, introducing a factor of 2 for convenience, to obtain the demodulated signal

$$\begin{aligned} d(t) &= 2x_c(t)\cos\omega_c t = 2A_c m(t)\cos^2\omega_c t \\ &= A_c m(t) + A_c m(t)\cos 2\omega_c t \end{aligned}$$

With the usual assumption that the message signal bandwidth W is small compared to ω_c, we simply pass $d(t)$ through an LP filter, stopping all frequencies above some cut-off between W and $2\omega_c$. (Noise considerations dictate as low a cut-off frequency as possible, but this topic has to be postponed at this point.)

Note that the demodulator output $y_D(t) = A_c m(t)$ can be interpreted as the envelope of $d(t)$.

Of course, the carrier introduced at the receiver must be in phase with the carrier of the received signal, and for this reason this method of demodulation is called *synchronous* or *coherent* modulation. The difficulty of independently generating in-phase signals is a disadvantage of DSB modulation. To see the need for coherence, suppose the demodulation carrier is actually given by $2\cos[\omega_c t + \theta(t)]$ where $\theta(t)$ is a phase error. Then since

$$2\cos x\cos y = \cos(x+y) + \cos(x-y)$$

we find that

$$d(t) = A_c m(t)\cos\theta(t) + A_c m(t)\cos[2\omega_c t + \theta(t)]$$

and hence

$$y_D(t) = A_c m(t) \cos\theta(t)$$

with the distortion factor $\cos\theta(t)$. If $\theta(t)$ is nearly constant in time and not too close to a multiple of π, then we will have no serious distortion or attenuation; but if either assumption doesn't hold, we have a problem.

To avoid the phase error, we could attach the carrier as a discrete component of the transmitted signal and filter it out at the receiver, or we could just square the received signal to get

$$x_r^2(t) = A_c^2 m^2(t) \cos^2 \omega_c t = \frac{1}{2} A_c^2 m^2(t) + \frac{1}{2} A_c^2 m^2(t) \cos 2\omega_c t,$$

so that if $m(t)$ is a power signal ($m^2(t)$ has a nonzero dc value), then $x_r^2(t)$ has a discrete component at $2\omega_c$ which can be filtered out and then halved to yield ω_c. DSB systems that do not transmit the carrier itself are called *suppressed carrier* systems.

3.3 Amplitude modulation

AM is like DSB except that AM introduces a dc bias to the message signal, so that the transmitted signal is $x_c(t) = [A + m(t)]A'_c \cos\omega_c t$ or, using a more convenient form,

$$x_c(t) = A_c[1 + a m_n(t)] \cos\omega_c t$$

where $m_n(t)$ is $m(t)$ normalized so that its minimum value is -1, i.e., $m_n(t) := m(t)/|\min m(t)|$. (We assume for convenience that $m(t)$ has a zero dc value so that the dc component of the signal is measured entirely by A_c.) The parameter $a = |\min m(t)|/A$ is the *modulation index.*

If $a \leq 1$, the message signal is in the envelope defined by the positive maxima (which equals the reflection about the time axis of the envelope defined by the negative maxima) of a sinusoidal carrier with a frequency much larger than the bandwidth of the message signal. Thus, to demodulate the AM signal, we need only an envelope detector – a circuit that responds to the relatively slow variations in the envelope but not to the high frequency variation of the carrier. A simplified (nonlinear) circuit for this purpose is a diode in series with a parallel RC circuit, with the output taken across the RC circuit. The time constant RC is tuned to a value that gives the appropriate response (tracking the low frequencies and insensitive to the high carrier frequency). The simplicity of envelope detection as compared to coherent demodulation is the primary advantage of AM. In the analysis of this detector, one may be able to treat the circuit as a square law device.

Exercise. Show that an AM signal with a large carrier can be demodulated by squaring it and then appropriately filtering it. ■

The primary disadvantage of AM is the poorer power efficiency that results from the fact that the dc component carries no information. To measure this inefficiency, consider the time-average power in the modulator output

$$\langle x_c^2(t)\rangle = \langle A_c^2[1 + am_n(t)]^2 \cos^2 \omega_c t\rangle$$

If $m(t)$ is much more slowly varying than the carrier, then after replacing $\cos^2 \omega_c t$ by $[1 + \cos 2\omega_c t]/2$ and expanding, we can eliminate the zero-average terms containing $\cos 2\omega_c t$ to obtain

$$\begin{aligned}\langle x_c^2(t)\rangle &= \langle \frac{1}{2}A_c^2[1 + 2am_n(t) + a^2 m_n^2(t)]\rangle \\ &= \langle \frac{1}{2}A_c^2[1 + a^2 m_n^2(t)]\rangle\end{aligned}$$

since $\langle m(t)\rangle = 0$ is assumed. Thus, if we define the power efficiency *Eff* to be the ratio of the power in the message signal to that in the transmitted signal, we get

$$Eff = \frac{a^2\langle m_n^2(t)\rangle}{1 + a^2\langle m_n^2(t)\rangle}$$

3.4 Single-sideband modulation (SSB)

The upper and lower sidebands (USB and LSB) are symmetric so all the information can be extracted from just one of them. Eliminating one of the sidebands reduces power but it runs into implementation problems because of the need to generate signals with potentially big discontinuities in the amplitude spectrum, i.e., build ideal filters. Below, we derive a formula for $x_c(t)$ that suggests a different method of performing the SSB modulation.

If we let $H_L(f)$ denote the transfer function of a filter that takes a DSB signal as input and outputs the corresponding LSB signal, then

$$H_L(f) = \frac{1}{2}[\mathrm{sgn}(f + f_c) - \mathrm{sgn}(f - f_c)]$$

where sgn is the signum function $\mathrm{sgn}(x) = 1,\ x > 0,\ = -1,\ x < 0,\ = 0,\ x = 0.$ Multiplying this by the Fourier transform of the DSB signal given earlier and then expanding, we arrive at the transform of the transmitted signal

$$\begin{aligned}X_c(f) &= \frac{1}{4}A_c[M(f + f_c) + M(f - f_c)] \\ &\quad + \frac{1}{4}A_c[M(f + f_c)\mathrm{sgn}(f + f_c) - M(f - f_c)\mathrm{sgn}(f - f_c)]\end{aligned}$$

The modulation theorem tells us that the first term on the RHS is $\mathcal{F}(\frac{1}{2}A_c m(t)\cos\omega_c t)$. For the remaining term, define the function $\hat{m}(t)$ by its Fourier transform $\mathcal{F}(\hat{m}(t)) := -j\mathrm{sgn}(f)M(f)$. The function $\hat{m}(t)$ is called the *Hilbert transform* of $m(t)$. Expanding the definition,

$$\mathcal{F}(\hat{m}(t)) = \begin{cases} -jM(f) = M(f)e^{-j\pi/2}, & f > 0 \\ jM(f) = M(f)e^{j\pi/2}, & f < 0 \end{cases}$$

we see that the Hilbert transform simply rotates the phase angle by $\pi/2$. Now $\mathcal{F}(\hat{m}(t)e^{\pm j2\pi f_c t}) = -jM(f \mp f_c)\text{sgn}(f \mp f_c)$, so $X_c(f)$ is the transform of

$$\begin{aligned} x_c(t) &= \frac{1}{2}A_c m(t)\cos\omega_c t - A_c\frac{1}{4j}\hat{m}(t)e^{-j2\pi f_c t} + A_c\frac{1}{4j}\hat{m}(t)e^{j2\pi f_c t} \\ &= \frac{1}{2}A_c m(t)\cos\omega_c t + \frac{1}{2}A_c\hat{m}(t)\sin\omega_c t \end{aligned}$$

A similar USB analysis leads to

$$x_c(t) = \frac{1}{2}A_c m(t)\cos\omega_c t - \frac{1}{2}A_c\hat{m}(t)\sin\omega_c t$$

Implementing these formulas directly using multipliers, phase shifters, and amplifiers gives the different method of SSB modulation. Clearly, implementing the Hilbert transform also involves implementing the discontinuity of an ideal filter, but this discontinuity is at the origin and easier to handle.

3.5 Frequency mixing

Frequency mixing, also called translation or conversion, simply translates a carrier to another frequency, or more generally, a modulated-carrier signal to another center frequency. For example, multiplying (mixing) $2\cos(\omega_1 \pm \omega_2)t$ with the baseband signal $m(t)\cos\omega_1 t$ produces

$$e(t) = m(t)\cos\omega_2 t + m(t)\cos(2\omega_1 \pm \omega_2)t$$

and hence the desired new signal $m(t)\cos\omega_2 t$ after appropriate filtering.

However, there is a potential problem created by the fact that other input frequencies will produce the same result. In particular, $\cos(\omega_1 \pm 2\omega_2)t$ is also converted to $\cos\omega_2 t$ after filtering. To see this, multiply out $\cos(\omega_1 \pm 2\omega_2)t$ by $\cos(\omega_1 \pm \omega_2)t$. These other frequencies are called *image* frequencies and must be properly filtered out in common receivers that use mixing, such as the superheterodyne receiver shown in Fig. 4.28 in Lathi.

When this receiver, for AM radio say, is tuned to a particular station, the RF filter/amplifier and the local oscillator (LO) will be tuned respectively to the corresponding carrier frequency, ω_c and to $\omega_c - \omega_{IF}$ or $\omega_c + \omega_{IF}$, depending on whether low-side or high-side tuning is adopted by the receiver. Here, ω_{IF} is the intermediate frequency more suitable for further signal processing. The image frequency of the carrier $\omega_c \pm 2\omega_{IF}$ should be filtered out at the RF filter so that it does not distort the signal carried by ω_c at the input to the IF amplifier. In the AM radio application, the carrier range is 540kHz to 1600kHz and the IF is 455kHz. The filtering problem is not difficult since twice the IF is almost 1 MHz.

We note in passing that it is not difficult to show that high-side tuning has an advantage over low-side tuning in that the ratio of highest-to-lowest freqencies of the LO is much smaller, thus implying a more easily built LO.

4 Angle modulation

4.1 Introduction

An angle modulated signal looks like

$$x_c(t) = A_c \cos[\omega_c t + \phi(t)]$$

with $m(t)$ incorporated in $\phi(t)$ by *phase modulation* (PM) in which case the *phase deviation* is

$$\phi(t) = k_p m(t),$$

and the *instantaneous phase* is $\omega_c t + \phi(t)$, or by *frequency modulation* (FM) in which case the *frequency deviation* is

$$\phi'(t) = 2\pi f_d m(t),$$

and the *instantaneous frequency* is $\omega_c + 2\pi f_d m(t)$, where k_p and f_d are given constants called the *phase deviation constant* and *frequency deviation constant.* (Note that k_p and f_d are radian and hertz rates, respectively.) Thus,

$$x_c(t) = \begin{cases} A_c \cos[\omega_c t + k_p m(t)], & PM \\ A_c \cos[\omega_c t + 2\pi f_d \int^t m(\tau) d\tau], & FM \end{cases}$$

Examples:

1. Consider the responses of PM and FM to a unit step at time t_0. The latter will have a discontinuous change in the frequency at 0, from f_c to $f_c + f_d$ at time t_0, but the waveform will retain its continuity at and beyond t_0.

   The PM response will be a discontinuity in phase at t_0 that is accompanied by a discontinuity in the waveform; the phase will advance by k_p radians.

2. Now suppose the input to the FM and PM modulators is a sinusoid. The frequency deviation of the PM system is proportional to the derivative of the phase deviation and hence to the slope of $m(t)$; as the message signal varies from one peak to the next, the instantaneous frequency goes from a minimum to a maximum and back to a minimum. For the FM output, the opposite is true: the instantaneous frequency varies from a maximum to a minimum and back to a maximum. Thus, since a sinusoid and its derivative are distortionless versions of each other, the PM and FM outputs will look identical up to a time shift corresponding to $\pi/2$ radians.

4.2 Narrowband angle modulation

Write

$$\cos[\omega_c t + \phi(t)] = \cos \omega_c t \cos \phi(t) - \sin \omega_c t \sin \phi(t)$$

and expand the functions of ϕ in power series. If $\phi(t)$ is *narrowband*, i.e., $\phi(t) << 1$, then $\cos\phi(t) \approx 1$ and $\sin\phi(t) \approx \phi(t)$, and we get

$$x_c(t) \approx A_c \cos\omega_c t - A_c\phi(t)\sin\omega_c t \tag{27}$$

which bears some resemblance to an AM signal, the main differences being the minus sign and the $\pi/2$ phase-shifted carrier: $\sin\omega_c t$. The form of (27) shows that the Fourier transform will feature sidebands, but the carrier term and the resultant of the sideband terms will not be colinear in a phasor diagram, as in AM; rather they will be in phase quadrature.

As an example to fix ideas for the FM case, let $m(t) = A\cos\omega_m(t)$ so that the phase deviation created in the interval $[0, t]$ is

$$\phi(t) = 2\pi f_d \int_0^t A\cos(\omega_m\tau)d\tau$$

With the *modulation index* f_d/f_m much smaller than one, we have narrowband, and as an approximation to

$$x_c(t) = A_c \cos\left[\omega_c t + \frac{Af_d}{f_m}\sin\omega_m t\right]$$

we can write

$$\begin{aligned} x_c(t) &= A_c\left[\cos\omega_c t - \frac{Af_d}{f_m}\sin\omega_c t\sin\omega_m t\right] \\ &= A_c\left(\cos\omega_c t - \frac{Af_d}{f_m}[\cos(\omega_c+\omega_m)t - \cos(\omega_c-\omega_m)t]\right) \end{aligned}$$

In phasor form, this becomes

$$x_c(t) = A_c\Re\left\{e^{j2\pi f_c t}\left[1 + \frac{Af_d}{2f_m}(e^{j2\pi f_m t} - e^{-j2\pi f_m t})\right]\right\}$$

The equivalent result for an AM signal is

$$x_c(t) = A_c(1 + a\cos 2\pi f_m t)\cos 2\pi f_c t$$

where a is the modulation index as before. In phasor form this is

$$x_c(t) = A_c\Re\left\{e^{2\pi f_c t}\left[1 + \frac{a}{2}(e^{j2\pi f_m t} + e^{-j2\pi f_m t})\right]\right\}$$

which should be compared to the narrowband FM result.

4.3 Spectra

As $m(t)$ enters into the argument of a transcendental function, it is only for the simplest of functions that spectra can be computed. For this reason, considerable effort has been invested in the study of the simple sinusoid; much insight can be gleaned for more general signals from this simple example.

Thus, take

$$\phi(t) = \beta \sin \omega_m t$$

and call β the modulation index. Then,

$$\begin{aligned} x_c(t) &= A_c \cos(\omega_c t + \beta \sin \omega_m t) \\ &= A_c \Re(e^{j\omega_c t} e^{j\beta \sin \omega_m t}) \end{aligned}$$

and from the Fourier series for the periodic function $e^{j\beta \sin \omega_c t}$ we get the coefficient

$$\frac{\omega_m}{2\pi} \int_{-\pi/\omega_m}^{\pi/\omega_m} e^{j\beta \sin \omega_m t} e^{-jn\omega_m t} dt = J_n(\beta) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j(nx - \beta \sin x)} dx$$

called the *Bessel function* of the first kind of order n and argument β. Then,

$$\begin{aligned} x_c(t) &= A_c \Re \left[e^{j\omega_c t} \sum_{-\infty}^{\infty} J_n(\beta) e^{jn\omega_m t} \right] \\ &= A_c \sum_{-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m) t \end{aligned}$$

so the spectrum structure can be determined by inspection with the help of the modulation theorem. In particular, the spectrum has a component at the carrier frequency ω_c and at the pair of sidebands $\omega_c \pm n\omega_m$ for $n = 1, 2, \ldots$, with magnitudes that can be computed numerically, or read from tables. The symmetries

$$J_{-n}(\beta) = \begin{cases} J_n(\beta), & n \text{ even} \\ -J_{-n}(\beta) & n \text{ odd} \end{cases}$$

and the recurrence

$$J_{n+1}(\beta) = \frac{2n}{\beta} J_n(\beta) - J_{n-1}(\beta), \quad n \geq 1,$$

guide the numerical work.

We can observe the following:

1. For $\beta \ll 1$ we have narrowband with $J_0(\beta)$ dominating.

2. The functions $J_n(\beta)$ are oscillating with amplitudes that decrease with increasing β.

3. The maximum amplitudes of the functions are decreasing in n.

4. At the zeros of $J_0(\beta)$, which are called *carrier nulls* the modulator output has no carrier component. More generally, at the zeros of $J_n(\beta)$ the modulator output is missing components at $f_c \pm nf_m$.

5. If $\beta = 0$, then only the carrier is output by the modulator, and this requires that $J_0(0) = 1$ and $J_n(0) = 0$ for all $n > 0$.

6. With $\phi(t) = A\sin\omega_m t$, the modulation index (magnitude of the phase deviation) is $\beta = k_p A$ for a message signal $m(t) = A\sin\omega_c t$ under PM, but under FM it is
$$\beta = \frac{Af_d}{f_m}$$
for a message signal $m(t) = A\cos\omega_m t$, and so is dependent on the modulation frequency, whereas PM is not. As f_m decreases the number of non-negligible frequencies in the FM spectrum increases.

4.4 Power and bandwidth of angle-modulated signals

The power in $x(t)$ is

$$\begin{aligned}\langle x_c^2(t)\rangle &= A_c^2\langle\cos^2[\omega_c t + \phi(t)]\rangle \\ &= \frac{1}{2}A_c^2 + \frac{1}{2}A_c^2\langle\cos 2[\omega_c(t) + \phi(t)]\rangle\end{aligned}$$

so that if, as is common, there is relatively negligible power concentrated in the low frequencies, then the time average in the second term is roughly the time average of $\cos 2\omega_c t$ which is 0.

$$\langle x_c^2(t)\rangle \approx \frac{1}{2}A_c^2$$

and the power is independent of the message signal, an important difference when comparing angle and linear modulation schemes.

A series expansion of the Bessel function shows that, for any fixed modulation index β, we have
$$J_n(\beta) \sim \frac{\beta^n}{n!2^n} \text{ as } n \to \infty$$
and so $J_n(\beta) \to 0$ very fast as $n \to \infty$. A set of frequencies where a "large fraction" of the power is concentrated can be used as an estimate of the bandwidth. Define the power ratio $P_r(k)$ as the ratio to the total power of the power in the carrier and the k components to either side of it,

$$\begin{aligned}P_r(k) &= \frac{\frac{1}{2}A_c^2\sum_{-k}^{k} J_n^2(\beta)}{\frac{1}{2}A_c^2} \\ &= J_0^2(\beta) + 2\sum_1^k J_n^2(\beta)\end{aligned}$$

For a desired power ratio, one can determine numerically the value of k needed to attain it; the bandwidth estimate then becomes $2kf_m$.

For general signals, define the *deviation ratio*

$$D := \frac{f_d}{W}\max|m(t)|$$

as the peak frequency deviation in $x_c(t)$ divided by the bandwidth W of $m(t)$. Then for FM the bandwidth W_c of $x_c(t)$ can be estimated by

$$W_c \approx 2(D+1)W$$

which is called *Carson's rule.* Note that this estimate varies from $2W$ in the narrowband case when $D \ll 1$, to $2DW$ in the wideband case when D is large. The estimate agrees with the FM case when $A\cos\omega_m t$ modulates the instantaneous frequency and where β plays the role of D; the contributions to bandwidth for $n > \beta + 1$ can be neglected in a first-order approximation of bandwidth (see Fig. 5.8 in Lathi). For PM, the estimate above also applies but with $\max|m(t)|$ replaced by $\max|m'(t)|$ and f_d replaced by $k_p/(2\pi)$ in the definition of D.

4.5 Demodulation of angle-modulated signals

For FM demodulation we need a circuit, called a *discriminator* that takes the received angle-modulatied signal $x_r(t)$ as input and produces an output $y_D(t)$ that varies in proportion to the frequency variation in $x_r(t)$. Thus, if

$$x_r(t) = A_c \cos[\omega_c t + \phi(t)] \tag{28}$$

then we want, for FM demodulation,

$$y_D(t) = \frac{1}{2\pi} K_D \frac{d\phi}{dt} \tag{29}$$

for some *discriminator constant* K_D. Note that integration of the discriminator output can convert the discriminator to a PM demodulator.

The discriminator can be composed of a differentiator, which produces from (28) the AM-like signal

$$e(t) = -A_c\left(\omega_c + \frac{d\phi}{dt}\right)\sin[\omega_c t + \phi(t)]$$

followed by an envelope detector which yields

$$y(t) = A_c\left(\omega_c + \frac{d\phi}{dt}\right)$$

Of course, $y(t)$ is always positive if $\omega_c > -\frac{d\phi}{dt}$ for all t, which is normally the case since ω_c is usually much greater than the bandwidth of the baseband signal.

The above set-up is an ideal in that it is usual for the amplitude to be distorted by the channel randomly in time. This is dealt with by putting at the input of the differentiator, a *limiter*, which delivers a signal close to a square wave, followed by a bandpass filter to retrieve the sinusoid of constant amplitude.

A phase-locked loop FM demodulator. The phase-locked loop (PLL) is a feedback system consisting of a phase detector (PD), voltage controlled oscillator (VCO), and a filter/amplifier (FA). The PD has two inputs: the received, modulated signal

$$x_r(t) = A_c \cos[\omega_c t + \phi(t)]$$

and the (quadrature) VCO output

$$e_o(t) = A_v \sin[\omega_c t + \theta(t)]$$

whose instantaneous frequency $\frac{d\theta}{dt}$ varies with the VCO input voltage $e_v(t)$ as

$$\frac{d\theta}{dt} = K_v e_v(t)$$

We consider a simple product PD that filters out the second harmonic term of the product and produces

$$e_d(t) = \frac{1}{2} A_c A_v K_d \sin[\phi(t) - \theta(t)] \tag{30}$$

where K_d is a multiplier characteristic. Note that we have used the identity

$$\cos\beta_1 \sin\beta_2 = \frac{1}{2}[\sin(\beta_2 - \beta_1) + \sin(\beta_2 + \beta_1)]$$

After further filtering and amplification, we get a signal $e_v(t)$ which is the output of the PLL as well as the input to the VCO. When the PLL is operating *in lock*, i.e., tracking the appropriate input characteristic, $|\theta(t) - \phi(t)|$ will be approximately a small constant so that $\frac{d\theta(t)}{dt} \approx \frac{d\phi(t)}{dt}$, that is, the VCO frequency deviation is a good estimate of the input frequency deviation. Thus, $e_v(t)$ will be the demodulated FM output, as it is approximately proportional to the instantaneous frequency $\frac{d\phi}{dt}$ which in turn is proportional to the message signal.

To analyze the dynamics of the PLL, we assume for simplicity that the FA functions only as an amplifier. Then the input to the VCO, $e_v(t)$, differs from (30) by the gain factor of the FA, and so

$$\frac{d\theta(t)}{dt} = K_t \sin[\phi(t) - \theta(t)] \tag{31}$$

where K_t is the loop gain, which multiplies the characteristic K_v of the VCO, the gain of the FA, and the factor in (30). We now demonstrate informally that the phase error tends to drive the PLL into lock, assuming that the PLL is near enough to a stable operating point. First, introduce the phase error $\psi(t) := \phi(t) - \theta(t)$ and rewrite the above as

$$\frac{d\psi}{dt} + K_t \sin\psi(t) = \frac{d\phi(t)}{dt}$$

with the frequency deviation at the input being the forcing function. To graph this relation on a phase-plane plot (the phase error ψ vs. its derivative), we need an initial operating point. For this, suppose the PLL is quiescent with a phase error of ψ_0 when a jump of $\Delta\omega$ is made in the instantaneous input frequency $\frac{d\phi(t)}{dt}$. The phase plane plot is then as shown in Fig. 5.15 in Lathi[1]. The derivative of the phase error is positive in the upper half plane so as time elapses an operating point in the upper half plane must move to the right in the direction of increasing ψ, and hence down towards a stable point on the ψ axis, assuming that the curve crosses the axis. This assumption will be valid if the loop gain K_t is large enough, as can be seen in the figure. Similarly, if the operating point goes below the abscissa, then it must move left and up and hence back towards the intersection with the abscissa. In the steady state, the frequency error will be 0, but the phase error will have some positive value depending on the loop gain. The constant K_t gives the (one-sided) *lock range* of the PLL, since stable operation requires that $\Delta\omega < K_t$ – the maximum frequency deviation in the input must be sufficiently small that the sinusoidal curve intersects the abscissa.

Let us assume that $\phi(t) - \theta(t)$ is small enough to justify the linear approximation $\sin[\phi(t) - \theta(t)] \approx \phi(t) - \theta(t)$. Then from (31),

$$\frac{d\theta(t)}{dt} + K_t\theta(t) = K_t\phi(t)$$

The loop transfer function is (using Laplace transforms given by capital letters)

$$H(s) = \frac{\Theta(s)}{\Phi(s)} = \frac{K_t}{s + K_t}$$

Clearly, as $K_t \to \infty$ the impulse response (the inverse transform of H(s)) is $\delta(t)$, which means that, for large loop gain, we indeed have $\theta(t) \approx \phi(t)$ and hence a justification of our linear approximation.

5 Interference

5.1 Complex envelope

It will be useful to describe first a generalization of product modulation, or heterodyning, as represented by $m(t)\cos\omega_c t$. To this end, we refer to *baseband* signals as real signals with spectra around $f = 0$, and to *bandpass* signals as the same signals after a translation of their spectra to higher frequencies usually centered around some frequency $f = f_c$, where f_c exceeds (usually far exceeds) the bandwidth of the baseband signal.

The generalization of $m(t)\cos\omega_c t$ is the so-called *analytic* signal $g(t)e^{j\omega_c t}$ where $g(t)$ is complex (as well as $e^{j\omega_c t}$); observe that the complex exponential introduces a quadrature component explicitly. Any real bandpass signal $v(t)$ can be expressed as

$$v(t) = \Re g(t)e^{j\omega_c t} \tag{32}$$

[1] Some obvious notational changes need to be made: θ_e is ψ, $\omega_o - \omega_c$ is $\Delta\omega$, $A = 1$ and K is K_t.

for an appropriate center frequency ω_c and a (complex) function $g(t) := x(t) + jy(t)$ called a *complex envelope*. But

$$g(t)e^{j\omega_c t} = x(t)\cos\omega_c t - y(t)\sin\omega_c t + j[x(t)\sin\omega_c t + y(t)\cos\omega_c t]$$

so (32) is equivalent to

$$v(t) = x(t)\cos\omega_c t - y(t)\sin\omega_c t \tag{33}$$

We can write from $g(t) = x(t) + jy(t)$

$$g(t) = R(t)e^{j\theta(t)} \tag{34}$$

where

$$\theta(t) = \tan^{-1}\frac{y(t)}{x(t)}, \quad R(t) = \sqrt{x^2(t) + y^2(t)}$$

so then, since $x(t) = R(t)\cos\theta(t),\ y(t) = R(t)\sin\theta(t)$, we have

$$v(t) = R(t)[\cos\omega_c t\cos\theta(t) - \sin\omega_c t\sin\theta(t)]$$

and thus, by a standard trigonometric identity,

$$v(t) = R(t)\cos(\omega_c t + \theta(t)) \tag{35}$$

We stress our original assumptions that the signal $v(t)$ is real and bandpass (with no power in frequencies in some neighborhood of the origin).

5.2 Linear modulation

Interference consists of unwanted signals that corrupt received signals. As an elementary illustration, suppose we have at the input to a receiver a bandpass signal composed of a carrier component at frequency ω_c, a pair of sidebands at frequencies $\omega_c \pm \omega_m$ representing a sinusoidal message signal, and an interfering tone of frequency $\omega_c + \omega_i$:

$$x_r(t) = A_c\cos\omega_c t + A_i\cos(\omega_c + \omega_i) + A_m\cos\omega_m t\cos\omega_c t \tag{36}$$

Using coherent demodulation, we multiply by $2\cos\omega_c t$ and LP-filter to obtain the demodulator output

$$y_D(t) = A_m\cos\omega_m t + A_i\cos\omega_i t \tag{37}$$

where the filter cut-off is above ω_i and the dc component has been eliminated. Here, we see that interference has been additive; coherent detection creates a linear demodulator.

Envelope detection can be quite a different matter, as it is nonlinear. Rewrite (36) as

$$\begin{aligned} x_r(t) &= A_c\cos\omega_c t + A_m\cos\omega_m t\cos\omega_c t \\ &\quad + A_i[\cos\omega_c t\cos\omega_i t - \sin\omega_c t\sin\omega_i t] \end{aligned}$$

or

$$x_r(t) = [A_c + A_m \cos\omega_m t + A_i \cos\omega_i t]\cos\omega_c t - A_i \sin\omega_c t \sin\omega_i t$$

In the case of most interest, $A_c \gg A_i$, and the first term dominates, so that

$$y_D(t) \approx A_m \cos\omega_m t + A_i \cos\omega_i t,$$

which exhibits the linearity we saw before.

However, if $A_i \gg A_c$, we will effectively lose the message signal, which can be seen as follows. Render (36) in the form

$$\begin{aligned} x_r(t) &= A_c \cos(\omega_c + \omega_i - \omega_i)t + A_i \cos(\omega_c + \omega_i)t \\ &\quad + A_m \cos(\omega_m + \omega_i - \omega_i)t \end{aligned}$$

Then after applying the proper trigonometric identities and rearranging, we find that

$$\begin{aligned} x_r(t) &= [A_i + A_c \cos\omega_i t + A_m \cos\omega_m t \cos\omega_i t]\cos(\omega_c + \omega_i)t \\ &\quad + [A_c \sin\omega_i t + A_m \cos\omega_m t \sin\omega_i t]\sin(\omega_c + \omega_i)t \end{aligned}$$

which is dominated by the first term when $A_i \gg A_c$. Thus, the process of envelope detection applied to the first term yields

$$y_D(t) \approx A_c \cos\omega_i t + A_m \cos\omega_m t \cos\omega_i t$$

We still have an AM signal, but the carrier has changed from ω_c to ω_i!

5.3 Angle modulation

Analysis remains difficult in general, since angle modulation is nonlinear. To get some idea of the effects of interference, consider the demodulator input to be an unmodulated carrier at frequency ω_c plus an interfering tone at frequency $\omega_c + \omega_i$. Precisely,

$$x_r(t) = A_c \cos\omega_c t + A_i \cos(\omega_c + \omega_i)t$$

which expands to

$$x_r(t) = A_c \cos\omega_c t + A_i \cos\omega_i t \cos\omega_c t - A_i \sin\omega_i t \cos\omega_c t$$

Now referring to (33), we can rewrite the above as

$$x_r(t) = R(t)\cos(\omega_c t + \psi(t)), \qquad (38)$$

where

$$R(t) = \sqrt{(A_c + A_i \cos\omega_i t)^2 + (A_i \sin\omega_i t)^2}$$

and

$$\psi(t) = \tan^{-1} \frac{A_i \sin\omega_i t}{A_c + A_i \cos\omega_i t}$$

If $A_c \gg A_i$, then we have the approximations

$$R(t) \approx A_c + A_i \cos \omega_i t$$
$$\psi(t) \approx \frac{A_i}{A_c} \sin \omega_i t \tag{39}$$

so (38) becomes

$$x_r(t) \approx A_c \left(1 + \frac{A_i}{A_c} \cos \omega_i t\right) \cos \left(\omega_c t + \frac{A_i}{A_c} \sin \omega_i t\right)$$

The phase deviation in (39) shows that for PM

$$y_D(t) = K_D \frac{A_i}{A_c} \sin \omega_i t$$

and by (29) we have for FM

$$y_D(t) = K_D \frac{A_i}{A_c} f_i \sin 2\pi f_i t$$

We see that, in both cases, the deviation is a sinusoid. But the amplitude for PM is independent of f_i whereas for FM it is proportional to f_i. For f_i beyond the bandwidth of a message signal, the interfering tone can be filtered out; otherwise, more complicated filtering is needed to reduce the effects of the interference.

6 Pulse modulation

6.1 Sampling theory

An analysis of pulse modulation must begin with sampling theory, which plays an essential role in the mathematical foundations of communication systems, especially digital communication systems.

An analog, continuous, baseband signal over any given time interval can be reconstructed from a finite number of samples of the signal during this interval, so long as they are sufficient in number, in particular, so long as they are taken at a frequency of at least $2W$ per second, where W is the bandwidth of the baseband signal. This result underpins discrete pulse modulated systems for the communication of analog signals; samples are converted to modulated pulses at the transducer/modulator, transmitted over the channel, and converted back to analog waveforms at the receiver/transducer.

Theorem: (Uniform sampling theorem for baseband signals) *If the spectrum of a signal $x(t)$ is identically zero for all frequencies beyond some W Hz, then for all t, $x(t)$ can be expressed as a function only of samples uniformly spaced in time and taken at a frequency of at least $2W$ samples/sec.* ■

Remarks: Generalizations of this result exist in which the uniform spacing is not required. The frequency $2W$ is called the *Nyquist frequency.* Over a time interval T_0, we need $N = 2WT_0$ samples to reconstruct a signal bandlimited to W Hz; N can be viewed as the "dimensionality" of the given signal over the T_0-sec interval. ■

To prove the theorem, it is convenient to work with the *ideal instantaneous sampled waveform* (see Example (d) of Section 2.7):

$$x_\delta(t) = \sum_{-\infty}^{\infty} x(nT_s)\delta(t - nT_s),$$

where T_s is the sampling interval. It is enough to prove that $x(t)$ can be reconstructed from $x_\delta(t)$, since $x_\delta(t)$ is a function only of the samples of $x(t)$. Indeed, we will verify that $x(t)$ is obtained from $x_\delta(t)$ by passing it through an ideal LP filter.

Write $x_\delta(t) = x(t)\sum_{-\infty}^{\infty}\delta(t - nT_s)$ and transform to get the convolution

$$X_\delta(f) = X(f) \star [f_s \sum_{-\infty}^{\infty} \delta(f - nf_s)]$$

where we have used the Fourier transform of the ideal sampling waveform derived in Example (d) in Sect. 2.7.3. Interchanging summation and convolution and then applying the sifting property of $\delta(.)$ gives

$$X_\delta(f) = f_s \sum_{-\infty}^{\infty} X(f - nf_s)$$

We are assuming that $f_s > 2W$ so a study of this formula shows that $X_\delta(f)$ is a periodic sequence of functions $X(f)$ centered at $0, \pm f_s, \pm 2f_s, \ldots$ and multiplied by f_s, in which successive instances of $X(f)$ do not overlap. It then follows that, if we pass $x_\delta(t)$ through an ideal LP filter with a TF given by $H_0\Pi(f/2B)e^{-j2\pi f t_0}$, where B satisfies $W < B < f_s - W$, then the output will have the spectrum

$$Y(f) = f_s H_0 X(f) e^{-j2\pi f t_0}$$

and hence be the signal

$$y(t) = f_s H_0 x(t - t_0),$$

i.e., a scaled and delayed (undistorted) version of $x(t)$. Thus, we have verified that $x(t)$ is completely determined by its samples $x(nT_s)$. To express $x(t)$ as a function of its samples, and hence complete the proof of the theorem, write

$$y(t) = \sum_{-\infty}^{\infty} x(nT_s)h(t - nT_s),$$

where $h(t) = 2Bf_s\text{sinc}[2\pi B(t - t_0)]$, $W < B < f_s - W$, is the impulse response of the ideal LP filter. Then we obtain the *interpolation formula*

$$y(t) = 2Bf_s \sum_{-\infty}^{\infty} x(nT_s)\text{sinc}[2\pi B(t - t_0 - nT_s)] \tag{40}$$

which completes the proof. Note the structural similarity between (40) and the Fourier series: Here the basis functions are the sinc functions and the coefficients are the sample values.

The bandwidth conditions of the theorem can not be expected to hold in real life, and so distortion, called *aliasing* is present. Aliasing is reduced by increasing the sampling rate, or by an initial LP filtering of the input $x(t)$. A second type of error results from the fact that ideal LP filters are an abstraction as well. This error can be reduced by sharpening the "roll-off" characteristics of the TF (e.g., see our discussion of Butterworth filters in Section 2.8.3), or again, by increasing the sampling rate.

Useful implications of the sampling theorem are found from the perspective of noise-free bandlimited channels. For, if a channel is bandlimited to B Hz, then a signal can be sent error free over the channel only if it is also bandlimited to B Hz, and this can be done (by the sampling theorem) only if at least 2 samples/sec are carried over the channel. Regarding the samples as symbols and representing each by a pulse, we see that: *A channel bandlimited to B Hz can carry at most $2B$ pulses per second.* Of course, the capacity of the channel to transmit information is not specified here, since we have not constrained the alphabet of symbols, e.g., the number amplitude levels in a discretized PAM signal. Practical constraints on information transmission arise from noise, distortion, interference, etc. and will be discussed more fully later. The sinc pulse plays a unique role in achieving the theoretical upper bound. Recall that $A\text{sinc}2\pi Bt$ vanishes at (nonzero) multiples of $1/(2B)$; it follows that any such pulse centered at $t \pm k/(2B),\ k > 0$, vanishes at the midpoint of a pulse centered at t and hence does not interfere with the latter pulse.

We now discuss pulse modulation, covering the analog case first, and then the case of pulse code modulation (PCM).

6.2 Analog pulse modulation

In these systems, a pulse train carries information by modulating pulses in one of three ways: varying the pulse amplitude, width (i.e., duration), or position relative to some given clock time. We cover only pulse amplitude modulation (PAM), because of its relevance to communication systems. PAM is produced in two ways: natural sampling (gating) and instantaneous sampling (*flat-top* PAM). In the latter, the sampled signal (which we've seen before) with period T_s between samples, is

$$m_\delta(t) = \sum_{-\infty}^{\infty} m(nT_s)\delta(t - nT_s)$$

and is the input to a *holding circuit* having the impulse response

$$h(t) = \Pi\left(\frac{t - \tau/2}{\tau}\right)$$

and transfer function $H(f) = \tau \mathrm{sinc}(\pi f \tau) e^{-j\pi f \tau}$ which gives the flat-topped waveform

$$m_c(t) = \sum_{-\infty}^{\infty} m(nT_s) \Pi \left(\frac{t - (nT_s + \tau/2)}{\tau} \right).$$

The holding network's failure to have a constant amplitude response can be compensated by reducing τ or by inserting an *equalizer* with transfer function $1/|H(f)|$ over the bandwidth of the signal.

The PAM signal that uses natural gating is given by

$$m_c(t) = m(t)s(t)$$

in terms of the *gating waveform*

$$s(t) = \sum_{-\infty}^{\infty} \Pi \left(\frac{t - nT_s}{\tau} \right)$$

The spectrum of $m_c(t)$ is easily shown to be

$$M_c(f) = \frac{\tau}{T_s} \sum_{-\infty}^{\infty} \mathrm{sinc}(n\pi\tau/T_s) M(f - nf_s)$$

where M is the spectrum of the baseband signal. (Expand the periodic square wave in a Fourier series before computing the transform, as we did in Section 3.2.) We leave as an exercise proofs that product demodulators can be designed for each form of PAM.

6.3 Pulse code modulation

The principal advantages of digital communications that have been responsible for its widespread adoption include the following.

1. the relative simplicity of digital circuits (the ease of applying integrated-circuit techniques)
2. the increasing use and availability of digital signal processing systems
3. the fact that digital signals are the language of digital computers.
4. the ease of coding of digital signals for combatting noise.
5. the use of regenerative repeaters to repair and reshape signals that have been attenuated or otherwise corrupted by noise.

The three principal components yielding PCM output from a baseband analog signal $m(t)$ are the sampler, the quantizer, and the encoder, in that order. Each sample of $m(t)$ is quantized into one of L levels, where $L = 2^n$ is taken as a power of 2 for obvious efficiency reasons. The discretized sample is encoded into an n-bit code(word), and is then passed along to further modulation and multiplexing stages in preparation for transmission (these latter stages will be discussed shortly).

If W denotes the bandwidth of $m(t)$, then the sampling rate is at least $2W$ samples per second, and the PCM output is at least $2Wn = 2W \log_2 L$ bits per second. The bandwidth required by each pulse is inversely proportional to the pulse width, which is at most $1/(2Wn)$, so if K denotes the constant of proportionality, then

$$B \geq 2WKn = 2WK \log_2 L$$

Note that quantization error is inherent and irrecoverable; it can be reduced by increasing L and hence the bandwidth: an error-bandwidth communications trade-off. For a closer analysis of quantization error, suppose we adopt the standard technique of discretizing a sample by replacing it with the midpoint of the quantization interval in which it falls. Let Δv be the quantization interval size, and let $m_p := \max_t |m(t)|$ be the peak signal magnitude. Then we choose $L = 2m_p/\Delta v$ for the number of quantization levels spanning the interval $[-m_p, m_p]$.

Let $m(t)$ be the original signal and let $\hat{m}(t)$ denote the signal recovered by the interpolation formula (40) applied to the quantized samples. Then the quantization error is

$$q(t) = \hat{m}(t) - m(t)$$

and we can write, in analogy with (40),

$$q(t) = \sum_k q(kT_s) \mathrm{sinc}(2\pi Bt - k\pi) \tag{41}$$

In the customary way, we measure the effects of quantization error by computing a signal-to-noise power ratio. For the quantization "noise," we need to find $\langle q^2(t) \rangle$. We make the standard assumption that successive quantization errors are independent and uniformly distributed over the error range $[-\Delta v/2, \Delta v/2]$. Then

$$\langle q^2(t) \rangle = \int_{-\Delta v/2}^{\Delta v/2} \frac{q^2 dq}{\Delta v} = \frac{(\Delta v)^2}{12} = \frac{(2m_p/L)^2}{12}$$

so the SNR is

$$\frac{\langle m^2(t) \rangle}{(\Delta v)^2/12} = 3L^2 \frac{\langle m^2(t) \rangle}{m_p^2}$$

which verifies, for one thing, that the SNR is boosted a great deal (quadratically) by increases in the number of quantization levels. Of course, as L increases and Δv decreases other sources of noise begin to predominate. But the formula also shows an independence of signal power and quantization noise power, which bodes ill for weaker signals when the ratio of m_p to rms signal power, the *crest factor*, is large, as in voice communications. We can improve on the SNR during periods when the signal is weak by letting Δv be a function of signal level, in particular, by narrowing the quantization levels in the region of weak signals at the expense of the levels in the region of strong signals. In practice, this is accomplished indirectly by compressing the high end of the range, applying uniform quantization levels, and then expanding the high end at the receiver back to where it was originally; the process is called *companding.* The most common characteristics defining the compression are logarithmic.

For passband communications, elementary forms of digital modulation are

1. On-off keying (OOK), also called amplitude shift keying (ASK), in which a carrier is switched on and off according as a bit is 1 or 0

2. Phase shift keying (PSK) in which 180 deg phase shifts distinguish 1 and 0 bits

3. Frequency shift keying (FSK), in which two distinct frequencies represent 1's and 0's.

6.4 Delta modulation (DM)

Low bandwidth signals $m(t)$ that change relatively slowly (or are sampled at a suitably high rate) can be tracked by recording just regularly spaced discrete increments/decrements with respect to the last value recorded. Figure 6.20 of Lathi illustrates the concept. (We will take $E = 1$ for simplicity.) The output of the delta modulator is a sequence of impulse functions, with polarities representing increments or decrements. In general, weights represent the sizes of the changes.

The modulator first forms the difference of $m(t)$ and $m_s(t)$, the latter being the estimate of the former up to time t. The DM then hard limits $d(t)$ producing the function $D(t)$ which is either a $+1$ or -1 according as the sign of $d(t)$ is positive or negative, respectively. Multiplying $D(t)$ by the output

$$\delta_s(t) = \sum_{-\infty}^{\infty} \delta(t - nT_s)$$

of a pulse generator then gives the DM output

$$x_c(t) = D(t) \sum_{-\infty}^{\infty} \delta(t - nT_s)$$

In a feedback loop, $x_c(t)$ is integrated to form the updated staircase estimate

$$m_s(t) = \sum_{-\infty}^{\infty} D(nT_s) \int^{t} \delta(\xi - nT_s) d\xi$$

Demodulation of DM integrates $x_r(t)$ to form $m_s(t)$, which can then be LP-filtered to suppress discrete jumps in $m_s(t)$. (Note that the LP filter can also be used for the integration in principle.)

There are two potential problems that must be faced:

(i) *granular noise*, which arises from the discrete approximation, and, like quantization noise, is unavoidable.

(ii) *slope overload*, which occurs when the signal changes faster than the maximum rate at which the modulator can change $m_s(t)$.

As an example of slope overload, suppose

$$m(t) = A \sin 2\pi f_1 t$$

and the delta functions of the modulator have weights δ_0 (steps in $m_s(t)$ are $\pm\delta_0$). The maximum slope the DM will allow without overload is then $S_m = \delta_0/T_s$. The slope (derivative) of $m(t)$ is

$$m'(t) = 2\pi A f_1 \cos 2\pi f_1 t$$

so slope overload occurs when

$$\frac{\delta_0}{T_s} < 2\pi A f_1,$$

which shows that, as expected, $m(t)$ must have a bandwidth constraint to avoid slope overload for given δ_0 and T_s.

Slope overload is reduced in more complicated DM's that adapt increment/decrement sizes to the rate of change of the baseband signal. *Differential* PCM is such a generalization. In the telephone application, 4 values, $\pm\delta_0, \pm 2\delta_0$, have been adopted, and have resulted in a halving of the voice channel bandwidth.

6.5 Multiplexing

It is desirable in many applications (e.g., voice, data) to combine the data created by many sources at some central switching point, and transmit the composite signal over a single channel. Frequency division multiplexing (FDM) and time division multiplexing (TDM) are two principal ways of doing this, the first being parallel use of sub-bands within a given spectrum, and the second being a simple time sharing of the channel. Quadrature multiplexing (QM) is yet another, but more limited technique that we discuss briefly.

6.5.1 Frequency division multiplexing

In this technique, some number, say N, of different baseband signals are heterodyned to disjoint spectral regions using *subcarriers*, and then summed to produce a new baseband signal which can then be further modulated, e.g., shifted to an appropriate RF band. The original baseband signals need not be all of the same format (DSB, SSB, and angle modulated signals could be present in the N baseband waveforms). Moreover, this process can be repeated recursively. As an example, consider the following 3 levels of the North American FDM hierarchy that uses FDM to combine 600 SSB-modulated voice channels over a common transmission channel. At the first level, 12 4kHz voice channels are multiplexed to form a *group* covering the range 60–108kHz (12 subcarriers spaced 4kHz apart), 5 groups at the 2nd level combine to form a *supergroup* covering the range 312–552kHz, and finally 10 supergroups combine to form a *mastergroup*.

6.5.2 Time division multiplexing

The simplest implementation of TDM, say with N identical PCM sources, assigns to the i-th source the i-th *time slot*, $1 \leq i \leq N$, in every sequence (cycle) of N consecutive time slots starting at multiples of N. If τ is the time slot duration, then the rate at which each source transmits is at most $1/(N\tau)$ bps, and by the sampling theorem, this rate must be high enough to provide for $2W$ samples/sec, where W is the bandwidth of the individual sources.

The cycle or *frame* usually accommodates varying numbers of slots per source, when the source bandwidths vary. Thus, in general, a frame will consist of k_i slots for source i, $1 \leq i \leq N$, the assumption being that, for each i, k_i slots out of every frame, each frame of length $\sum_i k_i$ is enough to provide the minimum rate, $2W_i$ samples/sec required by source i, where W_i is the bandwidth of source i. If the composite signal has bandwidth B, then in any large T sec interval there must be $2BT$ samples, and since this must also be the sum $\sum_{i=1}^{N} 2W_iT$, we have $B = \sum_{i=1}^{N} W_i$.

Example At the lowest level of the North American TDM hierarchy, a 64kbps channel comprises 8,000 samples/sec, each sample of 7 bits being augmented by a signaling bit. (This sampling rate handles a 4kHz-bandwidth voice signal.) Twenty four of these channels make up a T1 channel, which puts 24(8)+1=193 bits/frame, the extra bit being used for synchronization. With 8000 frames/sec, this gives 1.544Mbps for the T1 carrier. Four T1's make up a T2, 7 T2's make up a T3, and 6 T3's make up a T4 carrier. Not only are synchronization bits added to frames, occasional "bit stuffing" takes place in a synchronous system (when the bit rate falls below the clock rate, bits are inserted artificially). The system is not tied to voice communications; other digital sources may be assumed so long as they adhere to the rate hierarchy.

6.5.3 QM

In this scheme, the carrier $\cos\omega_c t$ and its quadrature $\sin\omega_c t$ are modulated separately by different baseband signals, $m_1(t), m_2(t)$, then summed and transmitted. At the receiver, the signal

$$x_c(t) = A_c[m_1(t)\cos\omega_c t + m_2(t)\sin\omega_c t]$$

is product demodulated coherently with both $\cos\omega_c t$ and $\sin\omega_c t$ to isolate the original message signals. For example, if there is a phase error θ (in the coherence), one obtains from multiplication by $2\cos(\omega_c t + \theta)$ and LP filtering

$$y_D(t) = A_c[m_1(t)\cos\theta - m_2(t)\sin\theta]$$

where $\theta = 0$ gives the desired output. This shows that the phase error creates not only attenuation but crosstalk from the quadrature signal as well. Note that QM specializes to both DSB and SSB if $m_2(t)$ is chosen properly. QM can be used in an FDM system that needs only half as many subcarriers as messages and that devotes to each baseband signal only its bandwidth (not twice its bandwidth as in DSB), where guardbands are being excluded from this discussion.

6.6 Pulse shaping

In the ideal Nyquist channel of bandwidth B Hz, $2B$ pulses per second can be transmitted only if we use the sinc pulse in a noise-free environment. In reality, jitter, interference, etc. make it impossible to maintain tolerable error rates. A principle problem is intersymbol interference (ISI) in which pulses in the same general neighborhood (in time) interfere with each other. For example, ideally the centers of the sinc pulses should be spaced at multiples of the basic pulse interval $T_b = 1/2B$. Recall that these pulses satisfy the *Nyquist criterion* whereby all pulses not centered at a given multiple of T_b have the value 0 there. If the centers become even slightly displaced relative to some pulse, the cumulative effect on that pulse can be quite large. A sharper pulse is needed to combat ISI and keep these cumulative effects small, but the cost will be the greater bandwidth requirement.

To design such a pulse, $p(t)$, we would like to keep the Nyquist criterion. Thus, normalizing the peak magnitude to 1, we want

$$p(t) = \begin{cases} 1, & t = 0 \\ 0, & t = \pm nT_b, n > 0 \end{cases}$$

For convenience, let us also require $p(t)$ to have a real, even-symmetric spectrum. To see what these properties imply, consider the pulse sequence evaluated at multiples of the basic period. By the Nyquist criterion

$$\sum_{-\infty}^{\infty} p(t)\delta(t - nT_b) = \delta(t)$$

so the transform gives the spectrum

$$\frac{1}{T_b}\sum_{-\infty}^{\infty} P(\omega - n\omega_b) = 1$$

Focusing on the interval $0 < \omega < \omega_b$, we see that the sum reduces to the terms for $n = 0$ and $n = 1$ (see Figure 7.11 in Lathi)

$$P(\omega) + P(\omega - \omega_b) = T_b,$$

or, by a change of variables,

$$P\left(\frac{\omega_b}{2} + x\right) + P\left(\frac{\omega_b}{2} - x\right) = T_b$$

where we have used the even symmetry of $P(\omega)$. This equation shows that the spectrum of $p(t)$ must be odd symmetric about the coordinates axes translated to $(\omega_b/2, T_b/2)$ (see Figure 7.12 in Lathi).

The *full cosine roll-off* characteristic gives us such a signal; the spectrum is

$$P(\omega) = \frac{1}{2}\left(1 + \cos\frac{\omega}{2R_b}\right)\Pi\left(\frac{\omega}{4\pi R_b}\right)$$

where $R_b := 1/T_b$. (The calculation of $p(t)$ as the inverse transform is left as an exercise.) Notice that the bandwidth ω_b of this pulse doubles that of the sinc pulse. For intermediate shapes, we can use any signal of the form

$$P(\omega) = \begin{cases} \frac{1}{2}\left[1 - \sin\left(\frac{\pi[\omega - \omega_b/2]}{2\omega_x}\right)\right] & |\omega - \omega_b/2| < \omega_x \\ 0 & |\omega| > \omega_x + \omega_b/2 \\ 1 & |\omega| > \omega_x - \omega_b/2 \end{cases}$$

where the bandwidth is $\omega_b/2 + \omega_x \leq \omega$. The *roll-off factor* is a convenient shape index and is defined as $r := \frac{\omega_x}{\omega_b/2}$. It follows that the bandwidth of $p(t)$ is $\frac{(1+r)R_b}{2}$. We have the lower limit $r = 0$ with the sinc pulse and the upper limit $r = 1$ for the raised cosine pulse.

7 Elements of Information Theory

7.1 Introduction

Consider a discrete source that puts out symbols (e.g., characters in the English alphabet, or the integers 0–9, or bits, etc.) at some given rate. For simplicity, suppose a symbol is emitted every time unit and let $X_j,\ j = 1, 2, \ldots$ denote the symbol emitted at time j. Let the symbol values be taken from the set $\{x_1, x_2, \ldots, x_n\}$ and suppose that for all j, $Pr\{X_j = x_i\} = p_i$ *independently* of all prior symbols $X_j, X_{j-1}, \ldots$. The source is said to be a *discrete, memoryless source* (DMS) emitting symbols from an alphabet of size n.

Now suppose the distribution $\{p_i\}$ is known to you, and you have just observed that x_i was the j-th symbol emitted from the source. By any reasonable measure of "information," how much information did you receive? Trivially, if $p_i = 1$ you received 0 information; you knew in advance that the symbol would be x_i so you learned nothing when you actually observed that fact. Thus, if $I(.)$ is your measure of information and x_i has probability 1 of occurring, then you want $I(x_i) = 0$. And in addition, you would want your measure to increase as the probability of occurrence decreases, i.e., as the surprise factor increases. As an extreme case, suppose the symbol you observed has probability 10^{-6} of occurring, only one chance in a million. Then you must have been astounded when it occurred, and you should have gained a great deal of information by your measure.

Finally, suppose you want to measure the information in the composite symbol represented by two consecutive outputs X_{j-1}, X_j. The outputs are independent so your measure of the composite symbol should add the information in the individual symbols. In particular, let $Y_j = X_{2j-1}X_{2j}$ take values from the composite alphabet of n^2 symbols $\{y_1, \ldots, y_{n^2}\}$ with probabilities $\{q_i\}$. If the composite symbol is $y_i = x_k x_\ell$ then the probability of y_i is given by the product $q_i = p_k p_\ell$, but we want the information in y_i to be the sum $I(y_i) = I(x_k) + I(x_\ell)$.

The exponential function converts sums to products, and its inverse, the logarithmic function, converts products to sums, the latter being just what we want. Thus we take as our measure of information

$$I(x_i) := \log 1/p_i$$

The base of logarithms has yet to be specified, and the properties we described above do not require any particular base. However, it is a distinct advantage to have base-2 logs and to take the *bit* as the unit of information (although this latter assumption creates an ambiguity, since we've been using bits to mean binary digits). Hereafter, base 2 is implicit in the notation $\log x$; we write $\ln x$ to denote the natural logarithm.

Example: A DMS outputs symbols from the alphabet $\{A, B, C, D\}$ with probabilities $\{1/2, 1/4, 1/8, 1/8\}$. The information in A, B is $\log \frac{1}{1/2} = 1$ bit, and $\log \frac{1}{1/4} = 2$ bits, respectively, and the information in both C and D is $\log \frac{1}{1/8} = 3$ bits.

The average information in one source symbol is obtained by averaging over the source probabilities $\{p_i\}$ and is denoted by

$$H(X) := \sum_{i=1}^{n} p_i \log 1/p_i,$$

where X is a random symbol. It is called the source *entropy.* We can also refer to it as the *information rate* of the source in bits per symbol. The information rate in bits/sec is obtained by multiplying the entropy by the rate in symbols/sec that the source emits symbols.

Examples:

1. The average information per symbol in the previous example is

$$H(X) = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + 2 \cdot \frac{1}{8} \cdot 3 = 1.75 \text{ bits} \tag{42}$$

2. An important special case is the binary case $n = 2$ with the probability of a 1 being denoted by p,

$$H(X) = p \log 1/p + (1-p) \log 1/(1-p)$$

   A graph of this function shows that, since $\lim_{z \to 0} z \log 1/z = 0$, we have $H = 0$ if either $p = 0$ or $p = 1$. The function is shaped like an inverted cup with a maximum when $p = 1/2$, in which case $H = 1$ bit (which is equivalent to saying that there is one bit of information in the toss of a fair coin).

It is easy to verify that $0 \leq p_i \leq 1$, for all i, implies that $H(X) \geq 0$, and that $H(X) = 0$ (the no-information case) if and only if $p_i = 1$ for some i. We can generalize the second of the preceding two examples and also prove without difficulty that, for an n symbol alphabet,

$$H(X) \leq \log n \tag{43}$$

To prove this upper bound, we first note that

$$\left.\frac{d^k}{dx^k} \ln x\right|_{x=1} = (-1)^{k-1} k!, \quad k \geq 1$$

so from Taylor's series for $\ln x$ in the neighborhood of $x = 1$ (see p. 5), we have

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

and hence

$$\ln x \leq x - 1, \quad x > 0$$

which is easily checked by differentiation.

Now consider two distributions $\{p_i\}$ and $\{q_i\}$ on an n symbol alphabet for a DMS. Changing to the natural logarithm, we can write

$$\sum_{i=1}^{n} p_i \log q_i/p_i = \frac{1}{\ln 2} \sum_{i=1}^{n} p_i \ln q_i/p_i,$$

and so from $\ln x \le x - 1$ we get

$$\sum_{i=1}^{n} p_i \log q_i/p_i \le \frac{1}{\ln 2} \sum_{i=1}^{n} p_i \left(\frac{q_i}{p_i} - 1 \right) = 0$$

On substituting $q_i = 1/n$, we find that

$$\begin{aligned} \sum_{i=1}^{n} p_i \log 1/(np_i) &= \sum_{i=1}^{n} p_i \log 1/p_i + \sum_{i=1}^{n} p_i \log 1/n \\ &= \sum_{i=1}^{n} p_i \log 1/p_i - \log n \le 0 \end{aligned}$$

and (43) follows.

7.2 Source encoding

In digital systems, symbols are encoded into bit strings (code words) for transmission. These codes must be chosen carefully if one is to minimize the channel capacity allocated to the source. If the n symbols are equally likely then about $\log n$ bits per symbol will be needed, so the channel capacity needed by the source will be $r \log n$ bps, where r is the source symbol rate. However, we can reduce this requirement when we can exploit cases where some symbols are output more frequently than others; in such cases, we can encode the more frequent symbols using fewer bits at the expense of longer codes for the less frequent symbols.

We limit the present treatment to *prefix* codes: no code word can be a prefix (the first k bits for some $k \ge 1$) of any other code word. For example, suppose we have symbols A, B, C, D with source probabilities $1/2, 1/4, 1/8, 1/8$, and we adopt the respective codes 0, 10, 110, 111. Replacing the letters in any sequence on the source alphabet $\{A, B, C, D\}$ by their corresponding codes gives a uniquely decodable bit string. For example, because of the prefix property of the above code, the sequence

11001001110110111110

scanned left to right can be the encoding of only the one sequence CABADACDC. These codes are also called *instantaneous* codes, since you know the source symbol the instant you scan the last bit of the codeword.

In the above example, we see that the average number of bits per symbol is

$$E(L) = .5 \cdot 1 + .25 \cdot 2 + 2 \cdot .125 \cdot 3 = 1.75 \text{ bits}$$

The *efficiency* of a code is defined to be $\eta = \overline{L}_{min}/\overline{L}$, where $\overline{L}_{min}$ is the minimum achievable average number of bits per symbol.

Shannon's *noiseless coding theorem* states that $\overline{L}_{min} = H(X)$, so that

$$\eta = H(X)/\overline{L}$$

gives the efficiency of a code. Note that the code of the previous example is 100% efficient, since both H and $\overline{L}$ are 1.75 bits (see (42)). However, this good fortune was a result of the inverse-power-of-two probabilities. In general, one can not expect to achieve 100% efficiency with such simple codes. Later, we will briefly discuss techniques for encoding that give efficiencies as close as desired to 100%.

Huffman codes. A classic, efficient code for DMS's is the Huffman code. Constructing a Huffman code is equivalent to the following process of building a binary tree on the source probabilities $p_1, p_2, \ldots, p_n$. The process begins with (a forest of) n one-node trees, each node being labeled with a distinct p_i, and it ends as a single tree whose leaves are the n initial nodes. At each step of the process, there will be some forest of binary trees; each non-leaf node in the forest will have a label equal to the sum of the labels of its two children. At an arbitrary step, two root nodes with minimal labels are made the children of the root of a new tree, the new root being given a label equal to the sum of the labels of its children. One of the edges from the new root is labeled with a 0, the other with a 1.

The codewords generated by this process are read off the final labeled tree top down; the code for symbol s_k (with probability p_k) is the sequence of edge labels (bits) along the path from the root to the leaf labeled p_k.

It can be shown that, for a given source entropy, the Huffman code is a prefix code with minimum average length. It can also be shown that the average length satisfies

$$H(X) \leq \overline{L} \leq H(X) + 1 \tag{44}$$

Shannon-Fano codes. These codes are also represented by a tree building process. However, in contrast to Huffman codes, these codes are contructed top-down. At each step of the process we have a tree with nodes corresponding to disjoint ordered subsets of the set of symbols. The process begins with a single node containing the probabilites of all symbols in decreasing or increasing order, and ends with a tree having leaves containing all of the original probabilities, one to a leaf.

At an arbitrary step of the process, a leaf node containing an ordered set of more than one probability is selected and divided into two ordered subsets with total probabilities as nearly equal as possible. (The probabilities in one of the ordered subsets are all at least as large as all of the probabilities in the other.) The two ordered subsets become the contents of children of the original node and the edges to the children are labeled so that one carries a 0 and the other a 1. When the process is complete, the leaf nodes all contain exactly one probability; the code for symbol s_k is read off the edges top down along the path from the root to the leaf containing p_k.

Source extension. More efficient codes can often be obtained by encoding the source symbols taken n at a time for some given extension parameter n. As an example, let us consider a source that generates symbols A and B with probabilities .7 and .3, respectively. Encodings for this example are trivial with a 0 and 1 encoding A and B, or vice versa. The efficiency of this code is

$$\frac{H(X)}{\overline{L}} = \frac{.7\log 1/.7 + .3\log 1/.3}{1}$$

which comes to 88.16%. Now let us take the symbols $n = 2$ at a time, so that we have four composite source symbols AA, AB, BA, and BB with probabilities $.49, .21, .21, .09$. As demonstrated below, it is not difficult to show that the entropy of the n-th source extension is just n times the entropy of the original source. For the current example, we leave as an exercise the proof that a Huffman code for the extended source is AA:1, AB:01, BA:001, BB:000 with respective symbol probabilities $.49, .21, .21, .09$. Then the new code has an average length

$$\overline{L}_2 = 1(.49) + 2(.21) + 3(.3) = 1.81$$

and the improved efficiency $2(.8816)/1.81 = 97.41\%$.

The process can be continued for improved efficiency. If $\overline{L}_n$ denotes the average code length for the n-th extension, then by (44) and $H(X^n) = nH(X)$, we can write

$$\frac{nH(X)}{nH(X)+1} \leq \frac{nH(X)}{\overline{L}_n} \leq 1$$

and so the efficiency approaches 100% as n approaches infinity, which realizes the promise of Shannon's noiseless coding theorem.

We conclude with a proof that the entropy $H(X^n)$ of the n-th source extension is n times the entropy $H(X)$ of the original source. We confine ourselves to the binary case, but the method extends to arbitrary alphabet sizes. If $i_1, \ldots, i_n$ represents an output symbol of the n-th order extension, then

$$\begin{aligned} H(X^n) &= \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} p_{i_1} \cdots p_{i_n} \log \frac{1}{p_{i_1} \cdots p_{i_n}} i \\ &= \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} p_{i_1} \cdots p_{i_n} \left(\log \frac{1}{p_1} + \cdots + \log \frac{1}{p_n} \right) \qquad (45) \end{aligned}$$

But

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} p_{i_1} \cdots p_{i_n} \log \frac{1}{p_j} = \sum_{i_j=1}^{2} p_j \log \frac{1}{p_j}$$

and so (45) reduces to

$$H(X^n) = \sum_{j=1}^{n} \sum_{i_j=1}^{2} p_j \log \frac{1}{p_j} = \sum_{j=1}^{n} H(X) = nH(X).$$

7.3 Reliable transmission in discrete channels

To extend our information-theoretic analysis of sources to an analysis of noisy channels, we introduce a fundamental probability model of channels called the *discrete memoryless channel* (DMC). The DMC takes an input X and delivers an output Y to be thought of as a noisy version of X. The alphabets from which X and Y are drawn are finite; this accounts for the term *discrete*. The alphabets are usually the same for our purposes, but the theory makes no such restriction. The system is *memoryless* in the sense that the i-th output symbol depends only on the i-th input symbol, and not on prior input symbols.

The simplest nontrivial examples are *binary* channels, where the inputs and outputs are drawn from $\{0, 1\}$. The conditional distribution of Y given X is called the the *channel transition probability distribution.* Let

$$p_{ij} := Pr\{Y = y_j | X = x_i\}$$

be the probability that an input x_j results in an output y_i. The *channel transition* matrix is

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

so

$$\begin{aligned} p(y_1) &= p_{11}p(x_1) + p_{21}p(x_2) \\ p(y_2) &= p_{12}p(x_1) + p_{22}p(x_2) \end{aligned}$$

can be written in matrix form as

$$[p(y_1), p(y_2)] = [p(x_1), p(x_2)] \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

For the *binary symmetric channel,* $p(y_2|x_1) = p(y_1|x_2) = p$ and the matrix is

$$\begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

We take the BSC as a convenient vehicle for discussing reliable transmission in the presence of noise. We assume a binary DMS with equi-probable symbols A and B, such as would be the approximate output of a Huffman source encoding.

An obvious technique for coping with errors is redundancy: send codes over the channel with sufficiently more bits than are needed to represent source symbols so as to be able to figure out at the receiver the symbol sent, even when bit errors occur, so long as they are sufficiently small in number.

A simplistic example is to send out a k-bit code ($k > 1$) for A and another such code for B at the channel input. We make the codes (codewords) as different as possible to maximize insensitivity to errors. In this case, we take two codes that maximize the *Hamming distance* between them, defined to be the number of bit positions in which they differ. For example, with $k = 5$, we could choose the pair of codes (11111,00000) or the pair (10101,01010) since in each case the Hamming distance between the codewords is maximum at 5, i.e., they are maximally different in the sense of Hamming distance. If in sending either code in this example, there are at most 2 bit errors, the receiver will guess the correct source symbol sent just by deciding on the codeword at the smallest Hamming distance from the k-bit symbol (possibly damaged codeword) received. Of course, any codeword received which is not either of the two correct codewords signals that at least one error has occurred. (The only bad transmission that goes undetected is one where all k bits are received in error.) In some systems (data communications) the codeword must be resent but in others (telephony) it may be ignored.

With the so-called *repetition* codes we have been considering, we may drive down the probability of detection errors as far as we like. But a painful price is exacted in the reduced rate of transmitting information. The information contained in a channel binary digit decreases like $1/k$ as the codeword length k increases; a fraction $\frac{k-1}{k}$ of the channel binary digits is reserved for error control. One might well ask whether we can drive down the error without sacrificing so much in the number of channel symbols per source symbol. Indeed, without much difficulty we could come up with codes that perform much better than repetition codes, but a code we are likely to design would just imply a function larger (a rate of decrease smaller) than $1/k$; increasing code lengths would still be needed to drive down errors and the question just posed would remain for the new code.

A famous result of Shannon given in the next section states in effect that, so long as the information rate at the channel input is beneath a certain channel-capacity threshold (a threshold to be given in the next section), and so long as we are willing to put up with a positive error rate $\epsilon > 0$, then the fraction of channel binary digits needed for error control is in fact bounded away from 1 independently of ϵ. However, there is no free lunch; the channel encoding process will entail longer time delays as ϵ is made smaller or as the source rate approaches more closely the channel capacity threshold. A primary goal of the next section is to make these ideas more precise.

7.4 Mutual information

With a probability model of the channel in hand, our first objective is a useful model of transmitted information. This will be based on the formal notion of mutual information developed below. First, we need to introduce conditional entropies, often called *equivocations*, as follows. Let Y denote the output of a given channel and let X denote its input. Then we define

$$H(Y|X) := \sum_{i,j} p(x_i, y_j) \log \frac{1}{p(y_j|x_i)} \tag{46}$$

in units of bits per symbol, as the average uncertainty remaining in Y given the input X that produced it. (The sum over i, j is a double sum over the symbol alphabets at the input and output, respectively.) We define $H(X|Y)$ similarly ($p(y_j|x_i)$ is replaced by $p(x_i|y_j)$) which is the average uncertainty about the transmitted symbol that remains after observing the output. Since $H(X)$ is the average uncertainty at the input, the *mutual information*

$$I(X,Y) := H(X) - H(X|Y)$$

is the uncertainty *resolved* by seeing the results of the transmission; it's the information in bits per symbol actually received when $H(X)$ is transmitted and $H(X|Y)$ is lost due to noise. If the channel generates its output independently of its input, then the output tells nothing about the input, $H(X|Y) = H(X)$, and the mutual information is 0 (there is no communication going on in such a channel). On the other hand, if the channel output is precisely its input, then there is no uncertainty about the input after viewing the output. In that case, $H(X|Y) = 0$ and the mutual information is $H(X)$ (the channel implements perfect communication).

Mutual information has the following important properties.

1. Symmetry: $I(X,Y) = I(Y,X)$ so $H(X) - H(X|Y) = H(Y) - H(Y|X)$
2. Nonnegative: $I(X,Y) \geq 0$
3. $I(X,Y) = H(X) + H(Y) - H(X,Y)$, where the joint entropy $H(X,Y)$ is defined as in (46) with the conditional probability replaced by the joint probability.

To prove property 1, we write from the definitions of $H(X)$ and $H(X|Y)$

$$\begin{aligned} I(X,Y) &= \sum_{i,j} p(x_i, y_j) \left[\log \frac{1}{p(x_i)} - \log \frac{1}{p(x_i|y_j)} \right] \\ &= \sum_{i,j} p(x_i, y_j) \log \frac{p(x_i|y_j)}{p(x_i)} \end{aligned} \tag{47}$$

Reversing the roles of x_i and y_j, we get

$$I(Y,X) = \sum_{i,j} p(x_i, y_j) \log \frac{p(y_j|x_i)}{p(y_j)}$$

But

$$\frac{p(x_i|y_j)}{p(x_i)} = \frac{p(y_j|x_i)}{p(y_j)}$$

since $p(x_i, y_j) = p(y_j|x_i)p(x_i) = p(x_i|y_j)p(y_j)$ Thus, the expressions for $I(X,Y)$ and $I(Y,X)$ are equal and we are done.

To prove property 2, multiply and divide the argument of the log in (47) by $p(y_j)$ and obtain

$$\begin{aligned} I(X,Y) &= \sum_{i,j} p(x_i, y_j) \log \frac{p(x_i, y_j)}{p(x_i)p(y_j)} \qquad (48) \\ &= -\sum_{i,j} p(x_i, y_j) \log \frac{p(x_i)p(y_j)}{p(x_i, y_j)} \\ &\geq \frac{1}{\ln 2} \sum_{i,j} p(x_i, y_j) \left[1 - \frac{p(x_i)p(y_j)}{p(x_i, y_j)}\right] = \frac{1}{\ln 2}(1-1) = 0 \end{aligned}$$

where we used in the last line, $\ln x \leq x - 1,\ x \geq 0$ and hence $-\log x \geq \frac{1}{\ln 2}(1-x)$.

To prove property 3, just expand (48).

We now define the *capacity* of a channel as the maximum of the mutual information over all distributions on the input symbols

$$C := \max_{\{p(x_i)\}} I(X,Y)$$

Clearly, C is a function only of the channel transition probabilities and its units are bits per input symbol.

To illustrate the definition, consider the BSC with parameter p. It is useful in this case to write the mutual information as the *destination entropy* less the *noise entropy*, as follows

$$I(X,Y) = H(Y) - H(Y|X)$$

with

$$\begin{aligned} H(Y|X) &= \sum_{i,j} p(x_i, y_j) \log \frac{1}{p(y_j|x_i)} \\ &= \sum_i p(x_i) \sum_j p(y_j|x_i) \log \frac{1}{p(y_j|x_i)} \end{aligned}$$

By symmetry, we can let $y_1 = 0$ and $y_2 = 1$ or vice versa and similarly for x_1 and x_2. In either case,

$$\sum_j p(y_j|x_i) \log \frac{1}{p(y_j|x_i)} = (1-p) \log \frac{1}{1-p} + p \log \frac{1}{p}$$

which is a constant independent of the probability distributions at the input or output. This constant is usually called the *entropy function* and denoted by H_p.

Now maximizing $H(Y)$ is done by taking $p(y_1) = p(y_2) = 1/2$, so

$$C = \max_{\{p(x_i)\}} I(X,Y) = 2\left(\frac{1}{2}\log\frac{1}{1/2}\right) - H_p = 1 - H_p.$$

In general, the problem of computing C is difficult.

With the concepts of this section, we are ready to return to Shannon's fundamental theorem for a noisy channel:

> *Fix $\epsilon > 0$ as a desired detection error probability. Let a source have information rate R and suppose its output is to be transmitted over a channel of capacity C. If $R < C$, then there exists a coding of the source output such that the transmission of the coded output can take place with a detection error of at most ϵ. Conversely, if $R > C$, then there exists an $\epsilon' > 0$ such that the detection error will exceed ϵ' independently of the coding scheme.*

For the case $R < C$, the complexity of the code (e.g., codeword size) will grow, and hence time delays will increase, as ϵ or the difference $C - R$ is made smaller. From a practical point of view the theorem is "existential" only, useful codes that realize the promise of the theorem are not part of the result. However, there are excellent codes that come close to the optimal performance; the structure and theory of such codes is beyond the treatment here.

Note that our digital system entails two coding processes: First, redundancy is coded out of the source (by a Huffman code for example), and second, redundancy is coded back in for transmission over the channel. The nature of the redundancy is usually quite different in the two cases (the second does not simply undo the first); the first is a feature of the application and the second is designed to deal with a noisy channel.

8 Random signals and noise

We focus on random signal and noise processes defined on the entire real line; the state spaces (voltage or current values for example) of the processes are continuous as are the time parameters. Recall that a random process $X(t)$ is determined by its n-th order joint probability distributions $\Pr\{X(t_1) \le x_1, \ldots, X(t_n) \le x_n\}$, and that it is *strictly stationary* if these probabilities are independent of the origin for all n and for all choices of the t_i and x_i. That is, the above joint probabilities do not change if we add the same (positive or negative) constant to each of the t_i's. The same assertion applies to the n-th order joint probability density functions. We will always assume the existence of the mean

$$\mathbb{E}[X(t)] = \mu_X \tag{49}$$

and the autocorrelation

$$R_X(\tau) = R_X(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)] \tag{50}$$

or its centered version, the autocovariance

$$\begin{aligned} \mathrm{Cov}_X(\tau) &= \mathbb{E}[(X(t_1) - \mu_X)(X(t_2) - \mu_X))] \\ &= \mathbb{E}[X(t_1)X(t_2)] - \mu_X^2 \\ &= R_X(t_1, t_2) - \mu_X^2 \end{aligned}$$

where the notation reflects the fact that the autocovariance and autocorrelation evaluated at points t_1 and t_2 depend only on the difference $\tau := t_2 - t_1$. Random processes with just the stationarity implicit in (49) and (50) are said to be *wide-sense stationary*, or just *stationary*.

We will be especially interested in Gaussian random processes in which

$$\Pr\{X(t) \le x\} = \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{x} e^{-(z-\mu_X)^2/(2\sigma_X^2)} dz$$

for all x, t, where $\sigma_X^2 := \mathrm{Cov}_X(0)$ is the variance of the distribution. The n-th order joint distributions will be joint Gaussian distributions, but in the special case of most interest to us, when $X(t_1)$ and $X(t_2)$ are independent for all $t_1 \ne t_2$, the joint distribution will have product form.

Note that the autocorrelation has the properties $R_X(0) = \mathbb{E}[X^2(t)]$, symmetry ($R_X(\tau) = R_X(-\tau)$), and $|R_X(\tau)| \le R_X(0)$. The third property can be verified by expanding

$$\mathbb{E}[(X(t+\tau) \pm X(t))^2] \ge 0$$

to obtain $-R_X(0) \le R_X(\tau) \le R_X(0)$. Note that $R_X(\tau)$ measures the dependence between points spaced τ time units apart; one can also say that it measures the influence $X(t)$ has on $X(t+\tau)$.

A further useful property of $R(\tau)$ is given by

$$R(\infty) = \lim_{|\tau| \to \infty} R(\tau) = \lim_{|\tau| \to \infty} \mathbb{E}[X(t)X(t+\tau)] = \mathbb{E}[X(t)]^2$$

The values of a process at times $t, t+\tau$ become asymptotically uncorrelated as $\tau \to \infty$.

Example: Let $g(t)$ be a stationary random bipolar binary pulse train with bit period T_b. A pulse has duration $T_b/2$ and is $+1$ when representing a 1 and -1 when representing a 0. The pulse train is random in the sense that the polarities are equally probable. To find the autocorrelation for $\tau < T_b/2$, note that each pulse overlaps with its shifted version by an amount $T_b/2 - \tau$, and the product $g(t)g(t+\tau) = 1$ throughout the interval of overlap. The fraction of a period given to the overlap is

$$\frac{T_b/2 - \tau}{T_b} = \frac{1}{2}\left(1 - \frac{2\tau}{T_b}\right)$$

so symmetry of the autocorrelation function yields

$$R(\tau) = \frac{1}{2}\left(1 - \frac{2|\tau|}{T_b}\right), \quad \tau < T_b/2$$

For any $\tau > T_b/2$, both $g(t)$ and $g(t+\tau)$ are independently, equally likely to be $+1$ and -1. It follows easily that $R(\tau) = 0, \quad \tau \geq T_b/2$. ■

Ergodic processes are stationary processes for which ensemble averages are independent of time and equal to time averages, e.g., if $x(t)$ is some sample function of $X(t)$, then

$$\mu_X = \mathbb{E}[X(t)] = \langle x(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)dt$$

and

$$R_X(\tau) = \mathbb{E}[X(t)X(t+\tau)] = \langle x(t)x(t+\tau) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)dt$$

Intuitively, the (ensemble) averages *across* the set of all sample functions are the same as the time averages *along* any given sample function.

For signals and noise, the mean or time average of $X(t)$ is simply its dc component, and the total power $\mathbb{E}[X^2(t)]$ is the sum of the ac power σ_X^2 and the dc power $(\mathbb{E}[X(t)])^2$.

The *power spectral density* of a process $n(t)$ is defined as

$$S_n(f) := \lim_{T\to\infty} \frac{\mathbb{E}[|N_T(f)|^2]}{T}$$

where $N_T(f)$ is the Fourier transform of $n_T(t)$, with $n_T(t) = n(t)$ for $|t| \leq T/2$ and 0 elsewhere. The Wiener-Khinchine theorem states that the power spectral density and the autocorrelation function are transform pairs

$$S_n(f) = \int_{-\infty}^{\infty} R_n(\tau)e^{-j2\pi f\tau}d\tau, \quad R_n(\tau) = \int_{-\infty}^{\infty} S_n(f)e^{j2\pi\tau f}df$$

To prove this result, write

$$\begin{aligned} |N_T(f)|^2 &= \left| \int_{-T/2}^{T/2} n(t)e^{-j\omega t}dt \right|^2 \\ &= \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} n(t)n(s)e^{-j\omega(t-s)}dtds \end{aligned}$$

and then take averages to get, after reversing the order of expectation and integration,

$$\begin{aligned}
\mathbb{E}|N_T(t)|^2 &= \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} \mathbb{E}[n(t)n(s)]e^{-j\omega(t-s)}dtds \\
&= \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} R_n(t-s)e^{-j\omega(t-s)}dtds
\end{aligned}$$

This expression begs for a change of variables $u = t - s,\ v = t$. The extreme values $s = \pm T/2$ become the boundaries $v = u + T/2,\ v = u - T/2$. Thus, the boundary of the region of integration has changed from the square $[-T/2, T/2]^2$ in the s, t plane to the trapezoid with vertices (in clockwise order) $[-T, -T/2], [0, T/2], [T, T/2], [0, -T/2]$ in the u, v plane. The shape of the new region changes from one triangular shape to another at $u = 0$, so we separate out the cases $u < 0$ and $u > 0$ to obtain

$$\begin{aligned}
\mathbb{E}|N_T(t)|^2 &= \int_{u=-T}^{0} R_n(u)e^{-j\omega u}\left(\int_{-T/2}^{u+T/2} dv\right)du + \int_{u=0}^{T} R_n(u)e^{-j\omega u}\left(\int_{u-T/2}^{T/2} dv\right)du \\
&= \int_{-T}^{0}(T+u)R_n(u)e^{-j\omega u}du + \int_{0}^{T}(T-u)R_n(u)e^{-j\omega u}du \\
&= T\int_{-T}^{T}\left(1 - \frac{|u|}{T}\right)R_n(u)e^{-j\omega u}du
\end{aligned}$$

Thus,

$$\begin{aligned}
S_n(f) &= \lim_{T\to\infty}\frac{\mathbb{E}|N_T(t)|^2}{T} \\
&= \lim_{T\to\infty}\int_{-T}^{T}\left(1 - \frac{|u|}{T}\right)R_n(u)e^{-j\omega u}du \\
&= \int_{-\infty}^{\infty} R_n(u)e^{-j\omega u}du
\end{aligned}$$

which is what we set out to prove.

Properties of the power spectral density include: It is always real, nonnegative, and symmetric, the integral $\int_{-\infty}^{\infty} S_x(f)df = P$ gives the total (per ohm) power, and $S_x(0) = \int_{-\infty}^{\infty} R_x(\tau)d\tau$.

The random processes of principal interest to us are *white noise* and linear transformations of it. White noise is defined by its constant power spectral density, which is universally denoted by the *two-sided* spectral density $N_0/2$, where N_0 is the *one-sided* power spectral density. White noise distributes energy equally over all frequencies, and the infinite total power that it implies makes it physically unrealizable. The term "white" stems from the analogous chromatic property of white light. From the inverse Fourier transform of a constant, we see that $R(\tau) = \delta(\tau)$ for white noise; the values of the process at times t_1, t_2

are uncorrelated for all $t_1 \neq t_2$. For a *Gaussian* white noise process this means that the values of the process at times t_1, t_2 are statistically independent for all $t_1 \neq t_2$. For a given bandwidth B, *bandlimited* white noise is a process having a power spectral density $S(f) = N_0/2$ for $|f| \leq B$ and 0 otherwise. Note that the total power of bandlimited white noise is $2B \times N_0/2 = BN_0$, and that the above correlation property no longer holds. (What is the autocorrelation function of bandlimited white noise?)

There are two fundamental input-output relationships for linear time-invariant systems with random processes as inputs. First, if the input is a Gaussian process, then so also is the output. The correlation structure will of course change in general. Second, if $x(t)$ and $y(t)$ are the input and output of a linear time-invariant system with transfer function $H(f)$, then

$$S_y(f) = |H(f)|^2 S_x(f) \tag{51}$$

To prove this result, we work in terms of R_y and then apply the Fourier transform. Let $h(t)$ be the impulse response of the linear time-invariant system and write

$$\begin{aligned}
R_y(\tau) &= \mathbb{E}[y(t)y(t+\tau)] \\
&= \mathbb{E}\left[\int_{-\infty}^{\infty} h(\xi_1)x(t-\xi_1)d\xi_1 \int_{-\infty}^{\infty} h(\xi_2)x(t+\tau-\xi_2)d\xi_2\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi_1)h(\xi_2)\mathbb{E}[x(t-\xi_1)x(t+\tau-\xi_2)]d\xi_1 d\xi_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi_1)h(\xi_2)R_x(\tau-\xi_2+\xi_1)d\xi_1 d\xi_2
\end{aligned}$$

Easy manipulations show that the two integrals can be rendered as a cascade of two convolutions

$$R_y(\tau) = h(-\tau) \star h(\tau) \star R_x(\tau)$$

whereupon the Fourier transform gives the product

$$S_y(f) = H^*(f)H(f)S_x(f) = |H(f)|^2 S_x(f)$$

as desired.

9 Receiver structure in digital communications

This section studies the problem of receiver design in digital (binary data) communication systems. The performance metric will be the probability $\Pr\{\mathcal{E}\}$ of a bit error in the presence of additive white Gaussian noise. Our goal is the characterization and implementation of *matched filters.* The receiver's decision as to whether a 0 or 1 was sent is, as one might expect, based on a threshold rule. We determine where that threshold should be set and what property (computation on the received signal) should enter into the threshold decision. We begin with the problem posed on baseband signals.

9.1 Baseband signals

Let 0's and 1's be represented by $+A$ and $-A$ in an NRZ (non return to zero format); the bit signaling interval duration is denoted by T. We shall ignore synchronization problems; the receiver knows the starting and ending times of the bit intervals. A naive threshold rule would examine the signal in the middle of the bit interval to see whether it is positive or negative; if the former, a 1 is decided and if the latter, a 0 is decided. But one readily sees that the threshold decision applied to the *area* of the signal over the bit interval makes for a less error prone rule. In this case, the receiver structure is simply an integrator followed by a threshold device. If $s(t)$ denotes the signal and $n(t)$ the noise, then the output of the integrator is

$$V := \int_{t_0}^{t_0+T} [s(t) + n(t)]dt$$

which we can shorten to

$$V = \begin{cases} AT + N \text{ if a 1 was sent} \\ -AT + N \text{ if a 0 was sent} \end{cases}$$

where

$$N := \int_{t_0}^{t_0+T} n(t)dt$$

We assume that $n(t)$ is stationary so we can put $t_0 = 0$, and we assume that $n(t)$ is a zero-mean white Gaussian process, so since integration is a linear operation, N is also Gaussian with mean

$$\mathbb{E}N = \mathbb{E}\int_0^T n(t)dt = \int_0^T \mathbb{E}n(t)dt = 0$$

and variance

$$\sigma_N^2 \quad = \quad \mathbb{E}N^2 = \mathbb{E}\left[\int_0^T n(t)dt\right]^2$$

$$
\begin{aligned}
&= \int_0^T \int_0^T \mathbb{E}[n(t)n(u)]dtdu \\
&= \int_0^T \int_0^T \frac{1}{2}N_0\delta(u-t)dtdu \\
&= \frac{1}{2}N_0 T
\end{aligned}
$$

Then the density function is

$$
\phi_N(x) = \frac{e^{-x^2/(N_0T)}}{\sqrt{\pi N_0 T}}
$$

By our threshold rule, an error occurs if $AT + N < 0$ or if $-AT + N > 0$, so we have the conditional error probability given that a 0 was sent,

$$
\Pr\{\mathcal{E}|0\} = \int_{AT}^{\infty} \phi_N(x)dx
$$

which by a change of variables $u = x/\sqrt{N_0T/2}$ can be written

$$
\begin{aligned}
\Pr\{\mathcal{E}|0\} &= \int_{AT}^{\infty} \phi_N(x)dx = \frac{1}{\sqrt{2\pi}} \int_{AT/\sqrt{N_0T/2}}^{\infty} e^{-u^2/2}du \\
&= 1 - \Phi\left(\sqrt{\frac{2A^2T}{N_0}}\right) = \Phi_c\left(\sqrt{\frac{2A^2T}{N_0}}\right)
\end{aligned}
$$

where $\Phi(\xi)$ is the normalized Gaussian distribution with mean 0 and unit variance, and $\Phi_c(\xi) = 1 - \Phi(\xi)$ is the tail of the distribution. The ratio A^2T/N_0 can be interpreted directly as the signal energy, $E_b := A^2T$, divided by the noise power spectral density. But it can also be interpreted as the signal-to-noise ratio $A^2/(N_0B_p)$ where the *bit-rate bandwidth* $B_p := 1/T$ is the estimate of the bandwidth of pulse signals that we have seen before.

By symmetry, $\Pr\{\mathcal{E}|1\} = \Pr\{\mathcal{E}|0\}$, so the unconditional error probability is

$$
\Pr\{\mathcal{E}\} = \Pr\{\mathcal{E}|1\}\Pr\{1\} + \Pr\{\mathcal{E}|0\}\Pr\{0\} = \Phi_c\left(\sqrt{2E_b/N_0}\right) \tag{52}
$$

Note the useful approximation for large x

$$
\Phi_c(x) \approx \frac{e^{-x^2/2}}{x\sqrt{2\pi}}
$$

and hence

$$
\Pr\{\mathcal{E}\} \approx \frac{e^{-E_b/N_0}}{2\sqrt{\pi E_b/N_0}}
$$

9.2 Modulated signals

Consider now the equally likely pulse signals $s_1(t)$ and $s_2(t)$ with T time-unit durations; they represent 0's and 1's, respectively. We first derive results for general signals; later we will choose these signals so as to apply our results to ASK, PSK, and FSK modulation. The energies of the signals are denoted by

$$E_i := \int_{t_0}^{t_0+T} s_i^2(t)dt, \quad i = 1, 2,$$

and are assumed to be finite. We again assume that the receiver is synchronized to the transmitter. The received signal plus noise is

$$y(t) = s(t) + n(t), \quad t_0 \leq t \leq t_0 + T,$$

where $s(t) = s_1(t)$ or $s_2(t)$, and $n(t)$ is white Gaussian noise with the power spectral density $N_0/2$. As before, we can assume that $t_0 = 0$ without loss of generality.

The receiver structure consists of a filter with transfer function $H(f)$ followed by a threshold device. The input to the filter is $y(t)$ and the output is denoted by

$$v(t) = s_{oi}(t) + n_o(t), \quad i = 1, 2$$

where $s_{oi}(t)$ and $n_o(t)$ are the outputs corresponding to $s_i(t)$, $i = 1, 2$, and $n(t)$. At time T, the threshold device is applied to $v(T)$ with the threshold k. If $v(T) > k$ the decision is that a 1, $s_2(t)$, was sent; otherwise, the decision is that a 0, $s_1(t)$, was sent.

The filter is constrained to be linear, so $n_o(t)$ is a stationary Gaussian process with mean 0 and power spectral density

$$S_{n_o}(f) = \frac{1}{2} N_0 |H(f)|^2$$

and hence variance

$$\sigma_o^2 = \int_{-\infty}^{\infty} \frac{1}{2} N_0 |H(f)|^2 df$$

Further, $N := n_o(T)$ has the density

$$\phi_N(x) = \frac{e^{-x^2/(2\sigma_o^2)}}{\sqrt{2\pi\sigma_o^2}}$$

Then the inputs to the threshold device are

$$V := v(T) = s_{oi}(T) + N$$

with $i = 1$ or 2. The $s_{oi}(T)$ are deterministic and since the above is a linear operation, V must also be Gaussian with mean $s_{o1}(T)$ or $s_{o2}(T)$, and with variance σ_o^2 in either case.

The conditional probabilities of error are

$$\Pr\{\mathcal{E}|s_1(t)\} = \int_k^\infty \frac{e^{-(x-s_{o1}(T))^2/(2\sigma_o^2)}}{\sqrt{2\pi\sigma_o^2}}dx$$

and

$$\Pr\{\mathcal{E}|s_2(t)\} = \int_{-\infty}^k \frac{e^{-(x-s_{o2}(T))^2/(2\sigma_o^2)}}{\sqrt{2\pi\sigma_o^2}}dx$$

and since 0's and 1's are equiprobable,

$$\Pr\{\mathcal{E}\} = \frac{1}{2}\left[\Phi_c\left(\frac{k-s_{o1}(T)}{\sigma_o}\right) + \Phi\left(\frac{k-s_{o2}(T)}{\sigma_o}\right)\right] \tag{53}$$

9.2.1 Optimization

The objective now is to minimize (53) by appropriate selection of k and the impulse response $h(t)$ of the filter. Formally, k_{opt} can be found by applying Leibniz's rule for differentiating integrals, which in the simplified case at hand, is

$$\frac{d}{du}\int_{a(u)}^{b(u)} f(z)dz = f(b(u))\frac{db(u)}{du} - f(a(u))\frac{da(u)}{du}$$

With $f(z)$ the Gaussian density function, routine calculations show that the optimum value of k is

$$k_{opt} = \frac{1}{2}[s_{o1}(T) + s_{o2}(T)]$$

i.e., the point of intersection of the conditional probability distribution functions, which we could have predicted initially, by an appeal to symmetry.

Substituting into (53), we find that, since by symmetry $\Phi(-\xi) = \Phi_c(\xi)$,

$$\begin{aligned}\Pr\{\mathcal{E}\} &= \frac{1}{2}\left[\Phi_c\left(\frac{s_{o2}(T)-s_{o1}(T)}{2\sigma_o}\right) + \Phi\left(-\frac{s_{o2}(T)-s_{o1}(T)}{2\sigma_o}\right)\right] \\ &= \Phi_c\left(\frac{s_{o2}(T)-s_{o1}(T)}{2\sigma_o}\right) \qquad (54)\end{aligned}$$

Letting

$$g(t) := s_2(t) - s_1(t), \quad g_o(t) := s_{o2}(t) - s_{o1}(t)$$

we now address the problem of finding an $H(f)$ (or $h(t)$) that maximizes $g_o(t)/\sigma_o$ and hence minimizes the error probability. It is more convenient to find an $H(f)$ maximizing $g_o^2(t)/\sigma_o^2$, as follows.

Since $n_o(t)$ is a stationary and mean-zero process, we have that

$$\sigma_o^2 = \mathbb{E}n_o^2(t) = \frac{1}{2}N_0\int_{-\infty}^{\infty}|H(f)|^2 df$$

If we let $G(f)$ denote the Fourier transform of $g(t)$, then from the superposition integral for linear time-invariant systems,

$$g_o(t) = \mathcal{F}^{-1}[G(f)H(f)] = \int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi ft}df$$

and so

$$\frac{g_o^2(T)}{\sigma_o^2} = \frac{\left|\int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi fT}df\right|^2}{\frac{1}{2}N_0\int_{-\infty}^{\infty}|H(f)|^2df}$$

But by Schwarz's inequality, for any two complex functions X and Y of a real variable, we have

$$\left|\int_{-\infty}^{\infty} X(f)Y(f)df\right|^2 \leq \int_{-\infty}^{\infty}|X(f)|^2df\int_{-\infty}^{\infty}|Y(f)|^2df \tag{55}$$

with equality if and only if $X = \alpha Y^*$ for some constant α. So if we put $X(f) = H(f)$ and $Y(f) = G(f)e^{j2\pi fT}$, then after substitution into (55) we find that

$$\frac{g_o^2(T)}{\sigma_o^2} \leq \frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)|^2df = \frac{2E_g}{N_0} \tag{56}$$

where E_g is the energy in $g(t)$ by Rayleigh's energy theorem. Equality holds if

$$H(f) = \alpha Y^*(f) = \alpha G^*(f)e^{-j2\pi fT}$$

for some constant α. Since α only reflects the gain of the linear system, we set it to one for convenience, and write for the transfer function of the optimal filter

$$H_o(f) = G^*(f)e^{-j2\pi fT}$$

The corresponding impulse response is given by

$$\begin{aligned} h_o(t) &= \int_{-\infty}^{\infty}[G^*(f)e^{-j2\pi fT}]e^{j2\pi ft}df \\ &= \int_{-\infty}^{\infty} G(-f)e^{-j2\pi f(T-t)}df \\ &= \int_{-\infty}^{\infty} G(f)e^{j2\pi f(T-t)}df \end{aligned}$$

and so

$$h_o(t) = g(T-t) = s_2(T-t) - s_1(T-t)$$

which is the impulse response of the *matched filter*, i.e., the filter "matched" to $g(t)$.

9.2.2 The matched filter

We give below the main properties of the matched filter, beginning with the error probability.

Error probability. From (54) and the definition of $g_o(t)$, we can write

$$\Pr\{\mathcal{E}\} = \Phi_c\left(\frac{g_o(T)}{2\sigma_o}\right)$$

which we minimize by choosing (see (56))

$$\zeta := \max \frac{g_o(T)}{\sigma_o} = \left[\frac{2}{N_0}\int_{-\infty}^{\infty} |G(f)|^2 df\right]^{1/2}$$

To put this in terms of the original signal characteristics, we use Rayleigh's energy theorem relating $G(f)$ and $g(t) = s_2(t) - s_1(t)$ to obtain

$$\begin{aligned}
\zeta^2 &= \frac{2}{N_0}\int_{-\infty}^{\infty}[s_2(t) - s_1(t)]^2 dt \\
&= \frac{2}{N_0}\left\{\int_{-\infty}^{\infty} s_2^2(t)dt + \int_{-\infty}^{\infty} s_1^2(t)dt - 2\int_{-\infty}^{\infty} s_1(t)s_2(t)dt\right\} \\
&= \frac{2}{N_0}(E_1 + E_2 - 2\sqrt{E_1E_2}\rho_{12})
\end{aligned}$$

where we have introduced the correlation coefficient (for deterministic signals in this case)

$$\rho_{12} := \frac{1}{\sqrt{E_1E_2}}\int_{-\infty}^{\infty} s_1(t)s_2(t)dt$$

which satisfies $-1 \leq \rho_{12} \leq 1$. Thus,

$$\Pr\{\mathcal{E}\} = \Phi_c\left(\frac{\zeta}{2}\right) = \Phi_c\left(\left[\frac{E_1 + E_2 - 2\sqrt{E_1E_2}\rho_{12}}{2N_0}\right]^{1/2}\right)$$

The signals for 1's and 0's are equally likely so the average pulse energy is $E_b = (E_1 + E_2)/2$, and we can write

$$\Pr\{\mathcal{E}\} = \Phi_c\left(\left[\frac{E_b - \sqrt{E_1E_2}\rho_{12}}{N_0}\right]^{1/2}\right) \tag{57}$$

Correlation detection. The direct implementation of the matched filter consists of two filters, one for each $s_i(t)$, in the following arrangement: the received signal $y(t) = s(t) + n(t)$ is input to a filter with impulse response $h(t) = s(T - t)$, $0 \leq t \leq T$ and output $v(t)$ sampled at time T to give $v(T)$, a comparator input. In an alternative implementation, the input $y(t)$ is mixed with $s(t)$ before being input to an integrator with output $v_*(t)$. The output $v(t)$ is sampled at time T to give $v_*(T)$, a comparator input. To verify that $v(T) = v_*(T)$, observe that

$$v(t) = h(t) \star y(t) = \int_0^T s(T - \tau)y(t - \tau)d\tau$$

After setting $t = T$, a change of variables gives

$$v(T) = \int_0^T s(\xi)y(\xi)d\xi$$

which by definition is the input $v_*(T)$ to the comparator in the correlator detector.

Optimum threshold. From the superposition integral and the matched filter impulse response, we get that

$$\begin{aligned} s_{o1}(T) &= \int_{-\infty}^{\infty} h(t)s_1(T-t)dt \\ &= \int_{-\infty}^{\infty} [s_2(T-t) - s_1(T-t)]s_1(T-t)dt \\ &= \int_{-\infty}^{\infty} s_2(u)s_1(u)du - \int_{-\infty}^{\infty} s_1^2(u)du \\ &= \sqrt{E_1E_2}\rho_{12} - E_1 \end{aligned}$$

A similar calculation shows that

$$s_{o2}(T) = E_2 - \sqrt{E_1E_2}\rho_{12}$$

and so from $k_{opt} = \frac{1}{2}[s_{o2}(t) + s_{o1}(t)]$ we arrive at

$$k_{opt} = \frac{1}{2}(E_2 - E_1)$$

as a function only of the signal energies. As one should expect, $k_{opt} = 0$ when the signals have equal energies.

9.2.3 $\Pr\{\mathcal{E}\}$ for binary signaling

Amplitude shift keying (ASK) is defined by $s_1(t) = 0$ and $s_2(t) = A\cos\omega_c t,\ 0 \leq t \leq T$, so

$$k_{opt} = \frac{E_2}{2} = \frac{1}{4}A^2T$$

is the threshold against which $\int_0^T y(t)\cos\omega_c t dt$ is compared in a correlator detector. We see that $s_1(t) = 0$ implies that $\rho_{12} = 0$, so (57) gives

$$\Pr\{\mathcal{E}\} = \Phi_c(\sqrt{E_b/N_0})$$

Compared to antipodal baseband signaling (see (52)), a factor of $\sqrt{2}$ is missing and hence ASK is 3dB worse.

For binary phase shift keying (BPSK), we take $s_1(t) = A\cos\omega_c t$ and $s_2(t) = A\cos(\omega_c t + \pi) = -A\cos\omega_c t$, where we assume for convenience that $\omega_c = 2\pi n/T$ for some positive integer n. Then $E_1 = E_2 = \frac{1}{2}A^2 T$ and $k_{opt} = 0$. We have

$$\begin{aligned}\sqrt{E_1 E_2}\rho_{12} &= -\int_0^T A^2 \cos^2\omega_c t dt \\ &= -\frac{A^2}{2}\int_0^T [1 + \cos 2\omega_c t]dt \\ &= -\frac{A^2 T}{2}\end{aligned}$$

and so $\rho_{12} = -1$. Then

$$\Pr\{\mathcal{E}\} = \Phi_c(\sqrt{2E_b/N_0})$$

which is the same as the result we found for bipolar baseband signaling.

Finally, for frequency shift keying (FSK), we have

$$s_1(t) = A\cos\omega_c t, \quad s_2(t) = A\cos(\omega_c + \Delta\omega)t, \quad 0 \le t \le T$$

where we assume for simplicity that $\omega_c = 2\pi n/T$ and $\Delta\omega = 2\pi m/T$ for distinct integers m, n. Then

$$\begin{aligned}\sqrt{E_1 E_2}\rho_{12} &= \int_0^T A^2 \cos\omega_c t \cos(\omega_c + \Delta\omega)t dt \\ &= \frac{A^2}{2}\int_0^T \cos\Delta\omega t + \cos(2\omega_c + \Delta\omega)t dt \\ &= 0\end{aligned}$$

Thus,

$$\Pr\{\mathcal{E}\} = \Phi_c(\sqrt{E_b/N_0})$$

which is the same as the result for ASK.

Appendix 1

Recent edition, introductory texts on communication systems

1. Carlson, A. B., Crilly, P. B., and Rutledge, J. C., **Communication Systems**, 4th Edition, 2002, McGraw-Hill.

2. Couch, L. W., II, **Digital and Analog Communication Systems**, Sixth Edition, 2001, Prentice-Hall.

3. Haykin, S., **Communication Systems**, 4th Edition, 2001, Wiley.

4. Lathi, B. P., **Modern Digital and Analog Communication Systems**, 3rd Edition, 1998, Oxford University Press.

5. Proakis, J. G. and Salehi, M., **Communication Systems Engineering,** 2nd Edition, 2002, Prentice-Hall.

Appendix 2

Review of complex numbers

Two basic forms of complex numbers are

$$x = a + jb = ce^{j\theta}, \quad j := \sqrt{-1}$$

which implies, by Euler's identity

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta,$$

that

$$a = c\cos\theta, \quad b = c\sin\theta, \quad \theta = \tan^{-1}\frac{b}{a}.$$

Euler's identity also gives

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2}$$

Note that $e^{j\theta}e^{-j\theta} = 1$ implies

$$\cos^2\theta + \sin^2\theta = 1.$$

From the *complex conjugate*

$$x^* := a - jb = ce^{-j\theta} = c(\cos\theta - j\sin\theta)$$

we get that

$$|x|^2 := xx^* = c^2(\cos^2\theta + \sin^2\theta) = c^2 = a^2 + b^2,$$

and hence the *magnitude* of the complex number x

$$|x| = c = \sqrt{a^2 + b^2}.$$

It is easily verified that $z = xy$ implies that $z^* = x^*y^*$, $|z| = |x||y|$, and z has a phase equal to the sum of the phases of x and y. Further,

$$|x_1| - |x_2| \le |x_1 + x_2| \le |x_1| + |x_2|$$

Note also from Euler's identity that

$$e^{j\pi} + 1 = 0,$$

in which we see (arguably) the 5 most basic constants in mathematics. We have

$$\begin{aligned} j^m &= -1, \ m \text{ a positive multiple of 2 but not 4,} \\ &= +1, \ m \text{ a nonnegative multiple of 4,} \\ &= +j, \ m-1 \text{ a nonnegative multiple of 4,} \\ &= -j, \ m-1 \text{ a positive multiple of 2 but not 4.} \end{aligned}$$

so since $e^x = 1 + x + \frac{x^2}{2!} + \cdots$, we can write

$$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{j\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots$$

Then Euler's identity gives the cosine and sine series

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

and

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

Letting the superscript (n) denote the n-th derivative, we can turn the argument around and start with Taylor's expansion for $f(x)$ in a neighborhood of x_0

$$f(x) = \sum_{n \geq 0} f^{(n)}(x_0) \frac{(x - x_0)^n}{n!}$$

let $x_0 = 0$, and substitute $f(x) = e^{jx}$, $\sin x$, and $\cos x$ to obtain Euler's identity.

Define the standard notation $\Re\{x\} := a$, $\Im\{x\} := b$, when $x = a + jb$.

Trigonometric identities. Many basic identities can be produced by proper choices for θ_1 and θ_2 in

$$e^{j(\theta_1+\theta_2)} = e^{j\theta_1} e^{j\theta_2}$$

For, by Euler's identity, the above gives

$$\begin{aligned}\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2) &= (\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2) \\ &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + j(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\end{aligned}$$

So equating real and imaginary parts, we get

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1$$

For example, if $\theta_1 = \theta_2 = \theta$, then

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$

and

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

Also, combining with the expressions where θ_2 is replaced by $-\theta_2$, we find

$$\sin\theta_1\sin\theta_2 = \frac{1}{2}[\cos(\theta_1-\theta_2) - \cos(\theta_1+\theta_2)]$$

$$\cos\theta_1\cos\theta_2 = \frac{1}{2}[\cos(\theta_1-\theta_2) + \cos(\theta_1+\theta_2)]$$

and

$$\sin\theta_1 \cos\theta_2 = \frac{1}{2}[\sin(\theta_1 - \theta_2) + \sin(\theta_1 + \theta_2)]$$

Using identities like those derived above, it is easy to show that the integral of $\cos n\omega_0 t \sin m\omega_0 t$ over a period $T_0 = 2\pi/\omega_0$ is 0 for all integers m, n. Similar identities can be used to prove that the integral of $\cos n\omega_0 t \cos m\omega_0 t$ over a period T_0 is 0 if $n \neq m$ and is equal to $T_0/2$ if $m = n \neq 0$. One obtains exactly the same result if the cosine is replaced by the sine.

Appendix 3

Historical Highlights of Communication Systems

1838 Samuel Morse demonstrates the telegraph

1876 Alexander Graham Bell patents the telephone

1897 Marconi patents a complete wireless telegraph system

1915 Bell System completes first transcontinental phone line

1918 B. H. Armstrong perfects the superheterodyne radio receiver

1920 J. R. Carson applies sampling to communications

1937 Alec Reeves conceives pulse-code modulation

1938 Television broadcasting begins

WWII Radar and microwave systems developed; statistical methods of signal detection implemented

1948 Claude Shannon's Mathematical Theory of Communications is published

1950 Time division multiplexing applied to telephony

1956 First transoceanic telephone cable

1960–1970
- First telecommunications satellite, Telstar I, launched.
- Live TV coverage of the moon exploration
- Experimental pulse-code-modulation (PCM) systems
- Experimental laser communications
- Digital signal processing becomes prevalent
- Color TV
- Time-shared computing and computer networks invented

1970–1980
- Commercial relay satellite communications (voice and digital)
- Gigabit signalling rates, IC communication circuits
- low-loss optical fibers, off-the-shelf optical communication systems
- packet switched digital data systems
- interplanetary grand tour launched (1977)

1980–1990
- mobile, cellular telephone systems
- programmable digital signal processors
- digitally tuned receivers with autoscan
- single-chip encoders and decoders

– Ethernet developed
– Bell System disbands; competitors are born

1990– – Global positioning system developed
– high definition TV
– global satellite–based cellular phones
– Integrated services networks
– Personal communication systems come of age

Appendix 4

Keywords and phrases

Aliasing
Amplitude
  -modulation (AM)
  -response
  -spectrum
Analog
  -communications
  -pulse modulation
    -pulse amplitude (PAM)
    -pulse position
    -pulse width
Analog-to-digital conversion
Angle modulation
Armstrong indirect method
Autocorrelation
Band limited
Bandpass
  -filter
  -limiter
Bandwidth
Baseband signal
Beating
Binary symmetric channel
Broadcasting
Butterworth filters
Capacity
Carrier
  -acquisition
  -sensing
  -suppressed
Causality
Channel
  -coding theorem
  -capacity
  -binary symmetric
Codes
  -convolutional
  -linear block
  -prefix
Coherent
  -detection
Companding