Kelly's LAN Model Revisited

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Introduction. For a given $k \geq 1$, subintervals of a given interval $[0, X]$ arrive at random and are accepted (allocated) so long as they overlap fewer than k subintervals already accepted. Subintervals not accepted are cleared, while accepted subintervals remain allocated for random retention times before they are released and made available to subsequent arrivals. Thus, the system operates as a generalized many-server queue under a loss protocol. We study a discretized version of this model that appears in reference theories for a number of applications; the one of most interest here is linear communication networks, a model originated by Kelly [2]. Other applications include surface adsorption/desorption processes and reservation systems [3, 1].

The interval $[0, X]$, X an integer, is subdivided by the integers into slots of length 1. An *interval* is always composed of consecutive slots, and a configuration $\mathbf{C}$ of intervals is simply a finite set of intervals in $[0, X]$. A configuration $\mathbf{C}$ is *admissible* if every non-integer point in $[0, X]$ is covered by at most k intervals in $\mathbf{C}$. Denote the set of admissible configurations on the interval $[0, X]$ by $\mathcal{C}_X$. Assume that, for any integer point i, intervals of length ℓ with left endpoint i arrive at rate λ_ℓ; the arrivals of intervals at different points and of different lengths are independent. A newly arrived interval is included in the configuration if the resulting configuration is admissible; otherwise the interval is rejected. It is convenient to assume that the arrival rates λ_ℓ vanish for all but a finite number of lengths ℓ, say $\lambda_\ell > 0$, $1 \leq \ell \leq L$, and $\lambda_\ell = 0$ otherwise.

The departure of intervals from configurations has a similar description: the flow of "killing" signals for intervals of length ℓ arrive at each integer i at rate μ_ℓ. If at the time such a signal arrives, there is at least one interval of length ℓ with its left endpoint at i in the configuration, then one of them leaves.

Our primary interest is in steady-state estimates of the vacant space, i.e., the total length of available subintervals $kX - \sum \ell_i$, where the ℓ_i are the lengths of the subintervals currently allocated. We obtain explicit results for $k = 1$ and for general k with all subinterval lengths equal to 2, the classical *dimer* case of chemical applications. Our analysis focuses on the asymptotic regime of large retention times, and brings out an apparently new, broadly useful technique for extracting asymptotic behavior from generating functions in two dimensions.

Our model, as proposed by Kelly [2], arises in a study of one-dimensional communication networks (LAN's). In this application, intervals correspond to the circuits connecting communicating parties and $[0, X]$ represents the bus. Kelly's main results apply to the case $k = 1$ and to the case of general k with interval lengths governed by a geometric law.

The focus here is on space utilization, so the results here add to the earlier theory in three principal ways. First, we give expected vacant space for $k = 1$, with special emphasis on small-μ asymptotics. Behavior in this regime is quite different from that seen in the "jamming" limit (absorbing state) of the pure filling model (all μ's are identically 0). Second, the important dimer case of chemical applications, where all intervals have length 2, is covered. Finally, the approach of the analysis itself appears to be new and to hold promise for the analysis of similar Markov chains. In very broad terms, expected vacant space is expressed in terms of the geometric properties of a certain plane curve defined by a bivariate generating function.

Results. Let $q_\ell = \lambda_\ell/\mu_\ell$, and note immediately that the Markov chain defined on $\mathcal{C}_X$ is reversible. Its stationary probabilities are given by

$$\pi(\mathbf{C}) = Z_X^{-1} \prod_{\ell: q_\ell > 0} q_\ell^{n_\ell(\mathbf{C})}, \quad Z_X(\overline{q}) = \sum_{\mathbf{C} \in \mathcal{C}_X} \prod_{\ell: q_\ell > 0} q_\ell^{n_\ell(\mathbf{C})}$$

where $n_\ell(\mathbf{C})$ is the number of intervals of length ℓ in the configuration $\mathbf{C}$, and Z_X is the partition function. Let $u(\mathbf{C}) = \sum_\ell \ell n_\ell(\mathbf{C})$ be the total used space and $v(\mathbf{C}) = kX - u(\mathbf{C})$ the total vacant space in configuration $\mathbf{C}$, and extend the partition function to the following polynomial in a formal variable x:

$$Z_X(x;\overline{q}) = \sum_{\mathbf{C} \in \mathcal{C}_X} x^{v(\mathbf{C})} \prod_{\ell: q_\ell > 0} q_\ell^{n_\ell(\mathbf{C})}.$$

In words, $Z_X(x,\overline{q})$ is the generating polynomial for the vacant space in admissible configurations on $[0,X]$. One easily finds that the average vacant space over admissible configurations in $[0,X]$ is given by $\langle v\rangle_X = Z_X^{-1} \frac{\partial Z_X}{\partial x}\big|_{x=1}$ where the subscript in $\langle v\rangle_X$ denotes averaging over the stationary probabilities of the Markov chain on $\mathcal{C}_X$. To find the two terms on the right-hand side, one may use the residue method to obtain

Theorem 1 *Assume that $Z_X = P/Q$ is rational, and assume that there is a single root y_m of Q with the least absolute value. Then the ratio $\frac{\partial Z_X}{\partial x}/Z_X$ is given by $X y_m^{-1} \frac{dy_m}{dx}$ up to a $O(X^0)$ term.*

The case $k = 1$ Define the generating function $Z(x,y;\overline{q}) = \sum_X Z_X(x;\overline{q}) y^X$.

Lemma 1 *We have that $Z(x,y;\overline{q}) = 1/[1 - yx - \sum_\ell q_\ell y^\ell]$.*

A singularity analysis then proves

Theorem 2 *The vacant space rate is given by*

$$\langle v\rangle_X/X = \frac{1}{1 + \sum_\ell \ell q_\ell y_1^{\ell-1}} + O(1/X).$$

Of interest to us is the behavior of this result as the rates $q_i \to \infty$. One cannot expect any reasonable (nondegenerate) limiting behavior without further assumptions. We need:
Let ρ be the unique real root of the polynomial $\sum_\ell q_\ell y^\ell - 1$. Then we assume that $\rho \to 0$ and that the (nonzero) rescaled coefficients q_ℓ/ρ^ℓ converge to nonnegative coefficients $c_\ell = \lim q_\ell \rho^{-\ell}$

We notice that $\sum_\ell c_\ell = 1$ and therefore 1 is the unique real root of the polynomial $\tilde{Q} = 1 - \sum c_\ell y^\ell$. It follows in particular that $y_1/\rho \to 1$. Further, easy calculations yield the following asymptotic result. Let $\overline{v} := \lim_{X\to\infty} \langle v\rangle_X/X$ be the rate of vacant space, i.e., the vacant space per unit length.

Theorem 3 *Under the above assumption, the vacant-space rate scales as ρ. More precisely, as $\rho \to 0$, $\rho^{-1} \lim_{X\to\infty} \langle v\rangle_X/X \to 1/\sum_\ell \ell c_\ell$*

One might interpret the denominator on the right-hand side as the average conditional length in the rescaled interval flow.

Transient behavior for a version of our model has been studied by Talbot, Tarjus, and Viot [4]. Through simulations, they describe convergence to statistical equilibrium starting with $[0,X]$ empty, as the departure rate μ tends to 0^+. The process begins with an initial, essentially pure filling phase in which vacant space reduces at a $O(1/t)$ rate until the $[0,X]$ is filled to a fraction that is approximately equal to Renyi's constant $\alpha = .748\ldots$. Thereafter, equilibrium behavior is approached in a very slow densification phase with vacant space decreasing at a $O(1/\log t)$ rate; as a typical event in this process, awkwardly placed subintervals straddled by gaps summing to greater than 1 eventually depart and are replaced by two subintervals. Note particularly the singular perturbation point at $\mu = 0$: the occupancy approaches 100% as $\mu \to 0^+$, but in the model where $\mu = 0$ the average occupancy in the jamming state is α. A rigorous proof of the details of transient behavior appears to be a challenging open problem.

Dimer packing, k-channels. For general k, the analysis seems to be quite involved, but for the important dimer case, where all subintervals have length 2, an essentially complete analysis is possible, once one discovers the inductive structure of packings. As above, we denote by $Z(x, y; q)$ the partition function. In our current situation we have only one parameter q, the retention rate for the incoming intervals.

Let $f_X^j[n]$ be the number of elements in the set $\mathcal{C}_X^j[n]$ of admissible configurations on $[0, X+1]$ with exactly j dimers straddling X and n units of vacant space in $[0, X]$. The key recurrence is given in the following result.

Lemma 2 *The* $f_X^j[n]$ *satisfy* $f_X^j[n] = \sum_{m=0}^{k-j} f_{X-1}^m[n-(k-m-n)]$. *and so the generating polynomial* $f_X^j(x) = \sum_n f_X^j[n]x^n$ *for vacant space satisfies* $f_X^j = \sum_{m=0}^{k-j} x^{k-m-j} f_{X-1}^m, \quad X > 1$.

A formal solution is easily derived for the generating (vector) function $\overline{f}(x, y) = \sum_X \overline{f}_X y^X$, with $\overline{f}_X(x) = (f_X^0(x), \ldots, f_X^k(x))$. A singularity analysis requires considerable effort, however, and leads to the following asymptotic result.

Theorem 4 *As* $q \to \infty$, *the average vacant space per unit length scales as* $q^{-k/2}$: $\frac{\langle v \rangle_X}{X} q^{k/2} \to c(k)$ *for some* $c(k) > 0$. *The constants* $c(k)$ *can be found explicitly. In particular,* $1 - c(k)$ *scales as* k^{-2}.

References

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