Performance Analysis of Root-Music

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***Abstract*—In this paper, we analyze the performance of Root-Music, a variation of the popular MUSIC algorithm, for estimating the direction of arrival (DOA) of plane waves in white noise in the case of a linear equispaced sensor array. The performance of the method is analyzed by examining the perturbation in the roots of the polynomial formed in the intermediate step of Root-Music. In particular, asymptotic results for the mean squared error in the estimates of the direction of arrival are derived. Simple closed-form expressions are derived for the one and two source cases to get further insight. Computer simulations are provided to substantiate the analysis. An important outcome of this analysis is an explanation as to why Root-Music is superior to the popular MUSIC algorithm for the linear equispaced array case.**

I. Introduction

Eigendecomposition based methods have recently been extensively used in estimating the Direction of Arrival (DOA) of plane waves in noise. Most of the eigendecomposition based methods decompose the observed covariance matrix into two orthogonal spaces, commonly referred to as the signal and noise subspaces, and estimate the DOA's from one of these spaces [1]–[5]. These methods, often referred to as subspace based methods, have been shown to perform very well and are capable of resolving closely spaced sources. Recently, we analyzed the performance of three subspace based methods [6], [7]. In particular, we examined ESPRIT [2], [3], the Minimum-Norm method [4], and the Toeplitz Approximation Method (TAM) [5]. In addition to the above three methods, another popular method is MUSIC [1]. MUSIC was the first method that showed the benefits of using a subspace based approach [1]. The MUSIC algorithm computes a spatial spectrum from the noise subspace, and determines the DOA's from the dominant peaks in the spectrum. Another popular variation of MUSIC is Root-Music [8]. Root-Music, as described in more detail later, is similar to MUSIC in many respects except that the DOA's are determined from the roots of a polynomial formed from the noise subspace. Although MUSIC is applicable to known general array configurations, Root-Music is only suitable in the context of a linear equispaced sensor array [8]. Some comparisons of MUSIC with ESPRIT, as well as Root-Music with ESPRIT, based on computer simulations, can be found in [3] and [9]. Some theoretical results comparing MUSIC and the Minimum-Norm method can be found in [10] and [11], wherein a characterization of the methods was done by examining the null spectrum. More recent results on MUSIC can be found in [12]. Our work examines Root-Music, and characterizes the mean squared error in the estimates of the DOA's directly. The analysis provides insight into why Root-Music is superior to the popular (Spectral) MUSIC algorithm in the context of a linear equispaced array.

The organization of the paper is as follows. First, the problem is formulated followed by a brief discussion of Root-Music. Asymptotical results for the mean squared error in the estimates of the DOA are derived. The results are specialized for the one and two source case leading to interesting insights. They are compared to the results for Least Squares ESPRIT. Simulation results are presented to support the analysis.

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II. Problem Formulation

The problem of estimating the direction of arrival of M incoherent plane waves incident on a linear equispaced array of L sensors is considered in this paper. For the kth observation period (snapshot), the spatial samples of the signal plus noise are given by

$$\begin{aligned} Y_k^T &= [y_1^{(k)}, y_2^{(k)}, \cdots, y_L^{(k)}] \\ &= \left[\sum_{i=1}^{M} p_i^{(k)}, \sum_{i=1}^{M} p_i^{(k)} e^{j\omega_i}, \cdots, \sum_{i=1}^{M} p_i^{(k)} e^{j(L-1)\omega_i} \right] \\ &\quad + N_k^T, \end{aligned} \tag{1}$$

where $\omega_i = 2\pi(d/\lambda)\sin\theta_i$, d being the separation between sensors, λ the wavelength of the incident signal, and θ_i the direction of arrival. As in [10] and [11], the noise vector N_k is assumed to be a zero mean, complex white Gaussian random vector, i.e., $\overline{N_k N_l^H} = \sigma_n^2 I \delta_{kl}$. The noise is assumed independent of the complex signal amplitudes $p_i^{(k)}$ which are also modeled as being jointly Gaussian. The covariance matrix P of the amplitudes whose elements are P_{ij}, where $P_{ij} = \overline{[p_i^{(k)} p_j^{*(k)}]}$, is assumed to be of rank M and has distinct eigenvalues. In this paper, the overbar "——" will be used to denote the expectation operator.[1]

Root-Music estimates $z_i = e^{j\omega_i}$, $i = 1, \cdots, M$, the signal zeros, from which ω_i, the signal frequencies, and then the DOA's θ_i are determined. It utilizes the eigendecomposition of the covariance matrix of the observation

[1] T is used to denote transpose, + to denote the pseudoinverse, * to denote complex conjugate, and H to denote complex conjugate transpose. Also $\hat{}$ is used to denote estimates, and subscripts s and n denote parameters associated with the signal and noise, respectively.

vector Y_k, i.e.,

$$R = \overline{Y_k Y_k^H} = \sum_{l=1}^{L} \lambda_l S_l S_l^H = E \Lambda E^H = E_s \Lambda_s E_s^H + \sigma_n^2 I, \tag{2}$$

where

$$E = [S_1, S_2, \cdots, S_L], \quad E_s = [S_1, S_2, \cdots, S_M] \tag{3}$$

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_L),$$

$$\text{and } \Lambda_s = \text{diag}(\lambda_1^s, \lambda_2^s, \cdots, \lambda_M^s). \tag{4}$$

Also,

$$\lambda_1 = \lambda_1^s + \sigma_n^2 > \lambda_2 = \lambda_2^s + \sigma_n^2 > \cdots > \lambda_M = \lambda_M^s + \sigma_n^2$$
$$> \lambda_{M+1} = \cdots = \lambda_L = \sigma_n^2.$$

λ_i are the eigenvalues of R, and S_i are the corresponding orthonormal eigenvectors. This paper considers the effect of using an estimated covariance matrix. Usually, an estimate of the covariance matrix is obtained by (time) averaging N independent snapshots, i.e.,

$$\hat{R} = \frac{1}{N} \sum_{k=1}^{N} Y_k Y_k^H = \hat{E} \hat{\Lambda} \hat{E}^H, \tag{5a}$$

where

$$\hat{E} = [\hat{S}_1, \hat{S}_2, \cdots, \hat{S}_L], \quad \hat{E}_s = [\hat{S}_1, \hat{S}_2, \cdots, \hat{S}_M],$$
$$\hat{\Lambda} = \text{diag}(\hat{\lambda}_k). \tag{5b}$$

Let $\hat{S}_k = S_k + \eta_k$ and $\hat{\lambda}_k = \lambda_k + \beta_k$. The analysis makes use of the asymptotic properties of the errors η_k derived in [13]. Since the results in [13] are derived with the assumption of distinct eigenvalues, the results are applicable to the errors in the eigenvectors corresponding to the distinct eigenvalues, i.e., eigenvectors corresponding to the signal subspace. The relevant first- and second-order properties of the errors in the eigenvectors corresponding to the signal subspace are summarized below.

$$\overline{\eta_k \eta_l^H} = \frac{\lambda_k}{N} \sum_{\substack{r=1 \\ r \neq k}}^{L} \frac{\lambda_r}{(\lambda_k - \lambda_r)^2} S_r S_r^H \delta_{kl} + o(N^{-1}),$$
$$1 \le k, l \le M, \tag{6}$$

$$\overline{\eta_k \eta_l^T} = \frac{\lambda_l \lambda_k}{N(\lambda_k - \lambda_l)^2} S_l S_k^T (1 - \delta_{kl}) + o(N^{-1}),$$
$$1 \le k, l \le M, \tag{7}$$

where δ_{kl} is the Kronecker delta. In [10] and [11], it was shown that

$$\overline{\eta_k} = -\frac{\lambda_k}{2N} \sum_{\substack{l=1 \\ l \neq k}}^{L} \frac{\lambda_l}{(\lambda_k - \lambda_l)^2} S_k + o(N^{-1})$$
$$= a_k S_k + o(N^{-1}), \quad 1 \le k \le M, \tag{8}$$

III. Spectral MUSIC and Root-Music

The MUSIC algorithm has been widely used and studied. Comparatively, Root-Music, although used by many researchers, has never been extensively discussed. Root-Music, first suggested in [8], is a variation of MUSIC, and is useful in the context of linear equispaced sensor arrays. Here we first present a brief discussion of MUSIC, followed by a discussion of Root-Music.

Spectral Music: The DOA's in the MUSIC method are given by the locations in the peaks of $S(e^{j\omega})$, referred to here as the spatial spectrum, where

$$S(e^{j\omega}) = \frac{1}{D(e^{j\omega})}. \tag{9}$$

This approach will be called Spectral Music. $D(e^{j\omega})$, termed the null spectrum, is given by

$$D(e^{j\omega}) = V^H(\omega) \left(\sum_{l=M+1}^{L} S_l S_l^H \right) V(\omega)$$
$$= V^H(\omega) P_N V(\omega), \tag{10}$$

where $V(\omega)$ is the steering vector, i.e.,

$$V^T(\omega) = \frac{1}{\sqrt{L}} [1, e^{j\omega}, \cdots, e^{j(L-1)\omega}].$$

Noting that $P_N = I - P_s$ where $P_s = E_s E_s^H$, an alternate expression for the null spectrum is

$$D(e^{j\omega}) = V^H(\omega)(I - P_S) V(\omega)$$
$$= 1 - V^H(\omega) \left(\sum_{l=1}^{M} S_l S_l^H \right) V(\omega). \tag{11}$$

Root-Music: In Spectral Music, a primary motivation for computing the null spectra was the fact that

$$V^H(\omega_i) S_k = 0, \quad k = M+1, \cdots, L, \tag{12}$$

ω_i being a signal frequency. Therefore, if we define polynomials using the eigenvectors corresponding to the noise subspace, i.e.,

$$S_k(z) = \frac{1}{\sqrt{L}} \sum_{l=1}^{L} s_{lk} z^{-(l-1)}, \quad k = M+1, \cdots, L, \tag{13}$$

then $z_i = e^{j\omega_i}$, $i = 1, \cdots, M$, the signal zeros, are roots of each of the above polynomials. We now define

$$D(z) = \sum_{k=M+1}^{L} \left(S_k(z) S_k^*(1/z^*) \right). \tag{14}$$

Note that the null spectrum is obtained by evaluating $D(z)$ on the unit circle, i.e., $D(z)_{|z=e^{j\omega}} = D(e^{j\omega})$. Using the fact that the signal zeros are the roots of $S_k(z)$, $k = M + 1, \cdots, L$, and (14), we have

$$D(z) = c \prod_{l=1}^{L} (1 - z_l z^{-1})(1 - z_l^* z) \tag{15a}$$
$$= \prod_{l=1}^{M} (1 - z_l z^{-1})(1 - z_l^* z) c$$
$$\cdot \prod_{l=M+1}^{L} (1 - z_l z^{-1})(1 - z_l^* z)$$
$$= H_1(z) H_1^*(1/z^*) H_2(z) H_2^*(1/z^*) \tag{15b}$$

where c is a constant, $H_1(z)$ contains the signal zeros, i.e.,

$$H_1(z) = \prod_{l=1}^{M} (1 - z_l z^{-1}), \tag{16}$$

and $H_2(z)$ contains the extraneous zeros that are inside the unit circle. Let $H(z) = H_1(z)H_2(z)$, then

$$D(z) = H(z)H^*(1/z^*). \tag{17}$$

$H(z)$ can be obtained by a spectral factorization of $D(z)$ and has its roots inside or on the unit circle. The M signal zeros are also roots of the polynomial $H(z)$. However, since the signal zeros are second-order roots of the polynomial $D(z)$, it is more expedient to just root $D(z)$, and identify the signal zeros from the knowledge that they lie on the unit circle [8]. This procedure will be referred to as Root-Music.

Philosophically, Root-Music can be viewed as belonging to the same class as the Minimum-Norm method. In the Minimum-Norm method [4], the signal roots are extracted from a polynomial formed from a vector with minimum norm that lies in the noise subspace. On the other hand, Root-Music estimates the signal zeros from the roots of $H(z)$ formed from the vector h, which lies in the noise subspace, where $h = [h_0, h_1, \cdots, h_{L-1}]^T$, and

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \cdots + h_{L-1} z^{-(L-1)}. \tag{18}$$

In the presence of an estimated covariance matrix, the Root-Music procedure essentially consists of obtaining $\hat{D}(z)$ from the estimated noise subspace, and then obtaining an estimate $\hat{z}_i$ of the signal zero. An estimate of the signal frequency is obtained from the signal zero by noting that $\hat{z}_i = |\hat{z}_i| e^{j\hat{\omega}_i}$. The DOA is then estimated from the estimated signal frequency $\hat{\omega}_i$.

Root Versus Spectral Forms

In this paper, we analyze Root-Music in that we determine the effect of using an estimated covariance matrix on the signal zeros of $D(z)$ and the DOA's. Although we analyze Root-Music, it is important to realize that since the signal zeros produce the peaks in the spatial spectrum $S(e^{j\omega})$, the perturbation of the signal zeros also gives insight into the Spectral Music algorithm. In fact, it will be shown that Spectral Music has the same asymptotic mean squared error as Root-Music. In spite of that, Root-Music is preferable to Spectral Music. This can be better understood by considering the effect of an error Δz_i in the signal zero z_i on the location of the signal frequency ω_i (cf. Fig. 1). One can see that if the error Δz_i is radial, then there is no error in the estimate of the signal frequency. However, such radial errors do affect the null spectrum making the peaks in $S(e^{j\omega})$ less defined. This is particularly critical for closely spaced roots as they may result in only one peak causing an apparent loss in resolution. So Spectral methods always have less resolution compared to Root forms. This was also observed in [8]. To understand the degree to which this is true, one needs to study both $\overline{|\Delta z_i|^2}$ and $\overline{|\Delta \theta_i|^2}$. Methods with large $\overline{|\Delta z_i|^2}$ and small $\overline{|\Delta \theta_i|^2}$ suffer a larger degradation in resolution when a Spectral approach is used. Such will be found to be the case for the MUSIC algorithm.

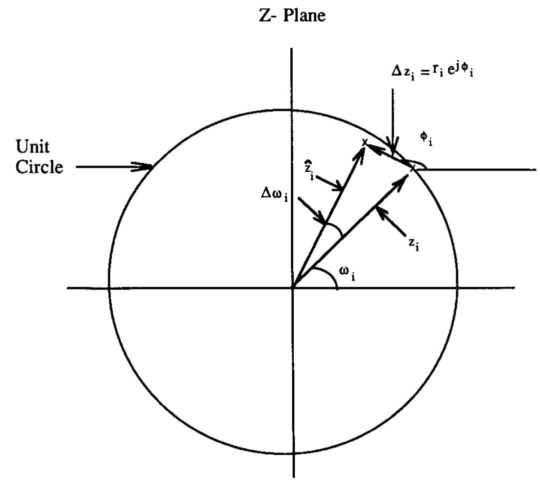

Fig. 1. Effect of error in the root location on the estimate of the angle.

IV. Mean Squared Error in the Signal Zeros

We first analyze the mean squared error $\overline{|\Delta z_i|^2}$ in the estimates of the signal zeros. The analysis is approximate in that higher order terms are neglected, and only terms that give rise to terms of $o(N^{-1})$ in the mean squared error are retained. The errors in the eigenvectors result in errors in $D(z)$, which in turn give rise to errors in the signal zeros. From (15a), we have the following relationship between the errors in the signal zeros and the estimated $D(z)$:

$$\hat{D}(z) = \hat{c} \sum_{l=1}^{L-1} (1 - (z_l + \Delta z_l) z^{-1})(1 - (z_l + \Delta z_l)^* z).$$

Substituting $z = z_i$, z_i being a signal zero, and noting that $|z_i| = 1$, we have

$$\begin{aligned} \hat{D}(e^{j\omega_i}) &= \hat{c} \sum_{l=1}^{L-1} (1 - (z_l + \Delta z_l) z_i^{-1}) \\ &\quad \cdot (1 - (z_l + \Delta z_l)^* z_i) \\ &= \hat{c} |\Delta z_i|^2 \prod_{\substack{l=1 \\ l \neq i}}^{L-1} \left|(1 - (z_l + \Delta z_l) z_i^{-1})\right|^2 \\ &\approx c |\Delta z_i|^2 \prod_{\substack{l=1 \\ l \neq i}}^{L-1} \left|(1 - z_l z_i^{-1})\right|^2, \end{aligned}$$

where higher terms have been neglected. Taking expectation of both sides, we have

$$\overline{|\Delta z_i|^2} = \frac{\overline{\hat{D}}(e^{j\omega_i})}{c \prod_{\substack{l=1 \\ l \neq i}}^{L-1} \left|(1 - z_l z_i^{-1})\right|^2} = S_{MU} \frac{\overline{\hat{D}}(e^{j\omega_i})}{L} \tag{19}$$

where

$$S_{MU} = \frac{L}{c \prod_{\substack{l=1 \\ l \neq i}}^{L-1} \left| (1 - z_l z_i^{-1}) \right|^2} = \sum_{k=0}^{L-1} \left| \frac{\partial z_i}{\partial h_k} \right|^2. \quad (20)$$

h_k's are the coefficients of the polynomial $H(z)$ [cf. (18)]. The term S_{MU} can be interpreted as the parameter sensitivity of the Root-Music method, i.e., the sensitivity of the zeros of $H(z)$ to perturbations in its coefficients [11], [12]. The similarity with the expression obtained in the Minimum-Norm method is worth noting [7]. An alternate expression for the parameter sensitivity, useful from a computational standpoint, is

$$S_{MU} = \frac{L}{c \prod_{\substack{l=1 \\ l \neq i}}^{L-1} \left| (1 - z_l z_i^{-1}) \right|^2} = L \lim_{\omega \to \omega_i} \frac{\left| 1 - e^{j\omega_i} e^{-j\omega} \right|^2}{D(e^{j\omega})}$$

$$= \frac{L}{V_1^H(\omega_i) P_N V_1(\omega_i)} \quad (21)$$

where $V_1(\omega)$ is the derivative of $V(\omega)$, i.e.,

$$V_1^T(\omega) = \frac{1}{\sqrt{L}} \left(0, je^{j\omega}, 2je^{j2\omega}, \cdots, j(L-1)e^{j(L-1)\omega}\right).$$

Equation (21) can be obtained using (10) for $D(e^{j\omega})$, and applying L'Hospital's rule.

We now need an expression for $\overline{\hat{D}(e^{j\omega_i})}$. In the presence of noise, from (11) and (5), we have

$$\hat{D}(e^{j\omega}) = 1 - V^H(\omega) \left(\sum_{l=1}^{M} (S_l + \eta_l)(S_l + \eta_l)^H \right) V(\omega). \quad (22)$$

Noting that $D(e^{j\omega_i}) = 0$ and using the statistics, it can be shown that [10], [11]

$$\overline{\hat{D}(e^{j\omega_i})} = \frac{(L-M)\sigma_n^2}{N} \left(\sum_{k=1}^{M} \frac{\lambda_k}{(\lambda_k - \sigma_n^2)^2} \left| V^H(\omega_i) S_k \right|^2 \right). \quad (23)$$

Substituting for $\overline{\hat{D}(e^{j\omega_i})}$ in (19), we have

$$\overline{|\Delta z_i|^2} = \frac{S_{MU}}{L} \frac{(L-M)\sigma_n^2}{N} \left(\sum_{k=1}^{M} \frac{\lambda_k}{(\lambda_k^s)^2} \left| V^H(\omega_i) S_k \right|^2 \right). \quad (24)$$

Note that $\overline{\hat{D}(e^{j\omega_i})}$ was used in [10] and [11] to compare MUSIC and the Minimum-Norm method.

V. Mean Squared Error in the DOA Estimate

Now we characterize the error in the DOA's as a result of using an estimated covariance matrix. Let $\Delta z_i = r_i e^{j\phi_i}$, then $\hat{z}_i = z_i + \Delta z_i = e^{j\omega_i} + r_i e^{j\phi_i} = |\hat{z}_i| e^{j\hat{\omega}_i}$. Then it can be shown that [7]

$$\Delta\omega_i = \hat{\omega}_i - \omega_i = r_i \sin(\phi_i - \omega_i),$$

$$\Delta\theta_i = \frac{\lambda}{2\pi d \cos\theta_i} r_i \sin(\phi_i - \omega_i).$$

Hence, the mean squared deviation in the estimate of the DOA is given by

$$\overline{\Delta\theta_i^2} = \left(\frac{\lambda}{2\pi d \cos\theta_i} \right)^2 \overline{r_i^2 \sin^2(\phi_i - \omega_i)}. \quad (25)$$

To obtain an expression for (25), we need an expression for $r_i \sin(\phi_i - \omega_i)$. This can be obtained in the following manner. Differentiating the null spectrum $D(e^{j\omega})$ [(15a)] with respect to ω, we have

$$D_1(e^{j\omega}) = \frac{dD(e^{j\omega})}{d\omega}$$

$$= jc \sum_{l=1}^{L-1} \Bigg[\left(z_l e^{-j\omega}(1 - z_l^* e^{j\omega}) - z_l^* e^{j\omega}(1 - z_l e^{-j\omega}) \right) \prod_{\substack{k=1 \\ k \neq l}}^{L-1} (1 - z_k e^{-j\omega}) \cdot (1 - z_k^* e^{j\omega}) \Bigg].$$

Due to estimation errors, we have

$$\hat{D}_1(e^{j\omega}) = j\hat{c} \sum_{l=1}^{L-1} \Bigg[\left((z_l + \Delta z_l) e^{-j\omega}(1 - (z_l + \Delta z_l)^* e^{j\omega}) - (z_l + \Delta z_l)^* e^{j\omega}(1 - (z_l + \Delta z_l) e^{-j\omega}) \right) \cdot \prod_{\substack{k=1 \\ k \neq l}}^{L-1} (1 - (z_k + \Delta z_k) e^{-j\omega}) \cdot (1 - (z_k + \Delta z_k)^* e^{j\omega}) \Bigg].$$

Substituting $\omega = \omega_i$, the signal frequency, and neglecting higher order terms, we have

$$\hat{D}_1(e^{j\omega_i}) \approx jc \Bigg[\left((z_i + \Delta z_i) e^{-j\omega_i}(1 - (z_i + \Delta z_i)^* e^{j\omega_i}) - (z_i + \Delta z_i)^* e^{j\omega_i}(1 - (z_i + \Delta z_i) e^{-j\omega_i}) \right) \cdot \prod_{\substack{k=1 \\ k \neq i}}^{L-1} (1 - z_k e^{-j\omega_i})(1 - z_k^* e^{j\omega_i}) \Bigg]$$

$$\approx jc \Bigg[\left((1 + \Delta z_i e^{-j\omega_i})(-\Delta z_i^* e^{j\omega_i}) - (1 + \Delta z_i^* e^{j\omega_i})(-\Delta z_i e^{-j\omega_i}) \right) \cdot \prod_{\substack{k=1 \\ k \neq i}}^{L-1} \left| 1 - z_k e^{-j\omega_i} \right|^2 \Bigg]$$

$$= j(-\Delta z_i^* e^{j\omega_i} + \Delta z_i e^{-j\omega_i}) V_1^H(\omega_i) P_N V_1(\omega_i)$$

$$= j(-r_i e^{-j\phi_i} + r_i e^{j\phi_i} e^{-j\omega_i}) V_1^H(\omega_i) P_N V_1(\omega_i)$$

$$= 2r_i \sin(\omega_i - \phi_i) V_1^H(\omega_i) P_N V_1(\omega_i).$$

Hence,

$$r_i \sin(\omega_i - \phi_i) = \frac{\hat{D}_1(e^{j\omega_i})}{2(V_1^H(\omega_i)P_N V_1(\omega_i))}.$$

Substituting for $r_i \sin(\phi_i - \omega_i)$ from the above equation in (25), we have

$$\overline{|\Delta\theta_i|^2} = \left(\frac{\lambda}{2\pi d \cos\theta_i}\right)^2 \frac{\overline{\hat{D}_1^2(e^{j\omega_i})}}{4(V_1^H(\omega_i)P_N V_1(\omega_i))^2}. \tag{26}$$

We now need an expression for $\hat{D}_1^2(e^{j\omega_i})$. Starting from (11),

$$D_1(e^{j\omega}) = -V_1^H(\omega)\left(\sum_{k=1}^{M} S_k S_k^H\right)V(\omega) - V^H(\omega)\left(\sum_{k=1}^{M} S_k S_k^H\right)V_1(\omega).$$

Due to the error in the eigenvectors [cf. (5)], we have

$$\begin{aligned}\hat{D}_1(e^{j\omega_i}) = &- V_1^H(\omega_i)\left(\sum_{k=1}^{M}(S_k\eta_k^H + \eta_k S_k^H + \eta_k\eta_k^H)\right) \cdot V(\omega_i)\\ &- V^H(\omega_i)\left(\sum_{k=1}^{M}(S_k\eta_k^H + \eta_k S_k^H + \eta_k\eta_k^H)\right) \cdot V_1(\omega_i)\end{aligned}$$

$$\begin{aligned}\hat{D}_1^2(e^{j\omega_i}) &= \Bigg(V_1^H(\omega_i)\left(\sum_{k=1}^{M}(S_k\eta_k^H + \eta_k S_k^H + \eta_k\eta_k^H)\right) \cdot V(\omega_i)\\ &\quad + V^H(\omega_i)\left(\sum_{k=1}^{M}(S_k\eta_k^H + \eta_k S_k^H + \eta_k\eta_k^H)\right) \cdot V_1(\omega_i)\Bigg)^2\\ &= \Bigg(V_1^H(\omega_i)\left(\sum_{k=1}^{M}(S_k\eta_k^H + \eta_k S_k^H)\right)V(\omega_i)\\ &\quad + V^H(\omega_i)\left(\sum_{k=1}^{M}(S_k\eta_k^H + \eta_k S_k^H)\right)V_1(\omega_i)\Bigg)^2\\ &= 2\,\mathrm{Re}\left(V_1^H(\omega_i)\left(\sum_{k=1}^{M}(S_k\eta_k^H + \eta_k S_k^H)\right)V(\omega_i)\right)^2\\ &\quad + 2\,\mathrm{Re}\Bigg(V_1^H(\omega_i)\left(\sum_{k=1}^{M}(S_k\eta_k^H + \eta_k S_k^H)\right)\\ &\quad \cdot V(\omega_i)V^H(\omega_i)\left(\sum_{l=1}^{M}(S_l\eta_l^H + \eta_l S_l^H)\right)V_1(\omega_i)\Bigg).\end{aligned} \tag{27}$$

Taking expectation and after some manipulations, it can be shown that (Appendix A)

$$\overline{\hat{D}_1^2(e^{j\omega_i})} = V_1^H(\omega_i)P_N V_1(\omega_i)\frac{2\sigma_n^2}{N} \cdot \left(\sum_{k=1}^{M}\frac{\lambda_k}{(\lambda_k^s)^2}\left|S_k^H V(\omega_i)\right|^2\right). \tag{28}$$

Substituting this in (26), and using (21), we have

$$\overline{|\Delta\theta_i|^2} = \left(\frac{\lambda}{2\pi d\cos\theta_i}\right)^2 S_{MU}\frac{\sigma_n^2}{2LN} \cdot \left(\sum_{k=1}^{M}\frac{\lambda_k}{(\lambda_k^s)^2}\left|S_k^H V(\omega_i)\right|^2\right). \tag{29a}$$

$$= \left(\frac{\lambda}{2\pi d\cos\theta_i}\right)^2 \frac{\overline{|\Delta z_i|^2}}{2L}. \tag{29b}$$

Relationship to Spectral Music: Starting with the null spectrum and using a first-order Taylor series expansion of its derivative as in [12] and [7], an expression for the mean squared error in the DOA estimate obtained using Spectral Music can be obtained. It can be shown to be exactly equal to (29). Hence, both Spectral and Root Music have the same asymptotic mean squared error. However, implicit in the derivation of the mean squared error for Spectral Music is the assumption that corresponding to each source (signal zero) there is a peak in the spatial spectrum [12], [7]. This, as explained in Section III, is a stronger assumption than distinct z-plane roots, implying that (29) is valid in the case of Spectral Music for smaller errors. However, an idea regarding the effectiveness of the spectral approach can be obtained by comparing (29) and (24) [7]. Other than the factor $(\lambda/2\pi d \cos\theta_i)^2$, the remaining term in (29) is smaller than that in (24) by a factor of $2L$. For the other methods, it is usually a factor of 2 [6], [7]. This implies that although, in general, Spectral forms are less effective than Root forms, this is more so for MUSIC. Further evidence of this is provided in the next section.

VI. One and Two Source Case

Here we specialize the general expressions for the one and two source case. They are compared to those obtained for Least Squares ESPRIT. Least Squares ESPRIT is very similar to the method in [3] except that instead of using a Total Least Squares approach, a Least Squares approach is used to estimate the state transition matrix [6]. For the rest of the discussion, ESPRIT refers to Least Squares ESPRIT. Also, the comparison to ESPRIT helps contrast the two methods and provides additional support to the superiority of Root-Music over the popular Spectral Music algorithm in the case of a linear array.

One Source Case

For the case of a single source with power level P_1, $\lambda_1^s = LP_1$, $\lambda_1 = \lambda_1^s + \sigma_n^2$, and $S_1 = V(\omega_1)$. In order to evaluate (24), an expression for the sensitivity of MUSIC is needed. To obtain this we note that

$$\begin{aligned} V_1^H(\omega_1)P_N V_1(\omega_1) &= V_1^H(\omega_1)(I - P_S)V_1(\omega_1) \\ &= V_1^H(\omega_1)(I - S_1 S_1^H)V_1(\omega_1) \\ &= \frac{(L-1)(L+1)}{12}. \end{aligned}$$

Hence, the sensitivity of MUSIC is given by

$$S_{MU} = \frac{L}{V_1^H(\omega_1)P_N V_1(\omega_1)} = \frac{12L}{(L-1)(L+1)}.$$

As in [10], substituting for the terms in (23) for $\overline{\hat{D}(e^{j\omega_i})}$, and using (24), we have

$$\overline{|\Delta z_1|^2_{MU}} = \frac{12L}{(L-1)(L+1)} \frac{\lambda_1 \sigma_n^2 (L-1)}{LN(LP_1)^2} \approx \frac{12\sigma_n^2}{L^2 P_1 N}. \tag{30a}$$

Similarly from (29),

$$\overline{|\Delta\theta_1|^2_{MU}} = \left(\frac{\lambda}{2\pi d \cos\theta_1}\right)^2 \frac{6}{L^2}\left(\frac{\sigma_n^2}{LP_1 N}\right). \tag{30b}$$

These expressions can be compared to those obtained for ESPRIT in [6]

$$\overline{|\Delta z_1|^2_{ES}} \approx \frac{2\sigma_n^2}{L^2 P_1 N}, \tag{31a}$$

and

$$\overline{|\Delta\theta_1|^2_{ES}} = \left(\frac{\lambda}{2\pi d \cos\theta_1}\right)^2 \frac{1}{L}\left(\frac{\sigma_n^2}{LP_1 N}\right). \tag{31b}$$

Note that the mean squared error in the signal zeros for ESPRIT [(31a)] is smaller than Root-Music [(30a)] by a factor of 6. On the other hand, the mean squared error in the DOA estimates of Root-Music is smaller [(30b)] than that of ESPRIT [(31b)] for values of L greater than 6. Noteworthy is the additional $2L$, compared to 2 for ESPRIT, in the denominator of $\overline{|\Delta\theta_1|^2_{MU}}$ compared to $\overline{|\Delta z_1|^2_{MU}}$.

Two Source Case

General conclusions are hard to make for the two source case. To get further insights, we consider the situation of two uncorrelated sources with equal power level P. For the analysis, we will consider $L >> 1$, $ASNR = PL/\sigma_n^2$ to be large, and $\Delta = L\omega_d/\sqrt{3} << 1$, where $\omega_d = (\omega_1 - \omega_2)/2$, and $ASNR\ \Delta << 1$. $|\epsilon|$, where $\epsilon = V^H(\omega_1)V(\omega_2)$, is also chosen close to one. The above assumptions result in a scenario of high resolution in a high-to-moderate signal-to-noise ratio environment. Substituting for the eigenvalues and eigenvectors, $\overline{\hat{D}(e^{j\omega_i})}$ can be simplified [10]. A brief summary is provided in Appendix B. This results in

$$\overline{|\Delta z_i|^2_{MU}} \approx S_{MU} \frac{1}{N}\left[\frac{1}{ASNR} + \frac{1}{(ASNR)^2\Delta^2}\right]. \tag{32a}$$

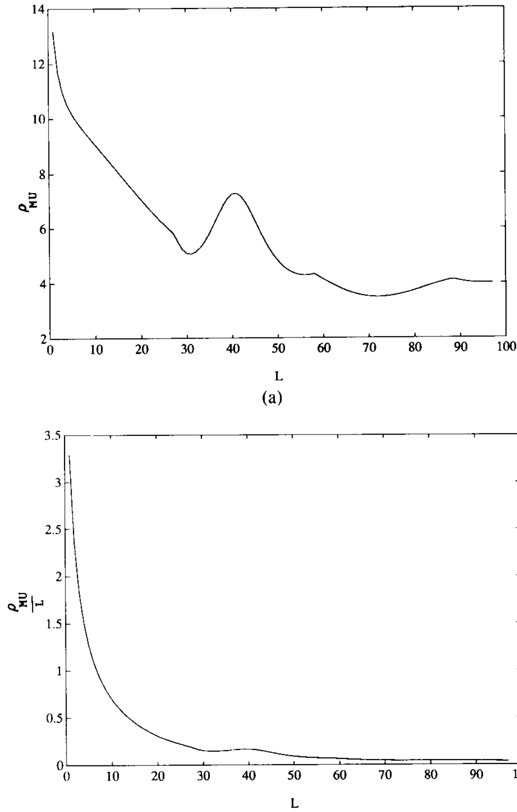

Fig. 2. (a) ρ_{MU} as a function of array length L. (b) ρ_{MU}/L as a function of array length L.

Similarly, for the mean squared error in the DOA estimate, we have (Appendix B)

$$\overline{|\Delta\theta_i|^2_{MU}} = \left(\frac{\lambda}{2\pi d \cos\theta_i}\right)^2 \frac{S_{MU}}{2L}\frac{1}{N}\left(\frac{1}{ASNR} + \frac{1}{(ASNR)^2\Delta^2}\right). \tag{32b}$$

For the ESPRIT case, after some approximations, it was shown that [6]

$$\begin{aligned} \overline{|\Delta z_i|^2_{ES}} &\approx \frac{3}{L}\frac{1}{1-|\epsilon|}\frac{1}{N}\frac{1}{ASNR}\left(1 + \frac{1}{4ASNR} + \frac{1}{ASNR\,\Delta^2}\right) \\ &\approx \frac{3}{L}\frac{1}{1-|\epsilon|}\frac{1}{N}\frac{1}{ASNR^2\Delta^2}, \end{aligned} \tag{33a}$$

and

$$\overline{|\Delta\theta_i|^2_{ES}} = \left(\frac{\lambda}{2\pi d \cos\theta_i}\right)^2 \frac{3}{2L}\frac{1}{1-|\epsilon|}\frac{1}{N}\frac{1}{ASNR^2\Delta^2}. \tag{33b}$$

A comparison of (32a) and (33a) suggests that we need S_{MU} to be of the order $(3/L)(1/1 - |\epsilon|)$ for the mean squared error in the signal zeros obtained by Root-Music to be comparable to ESPRIT. A comparison of (32b) and

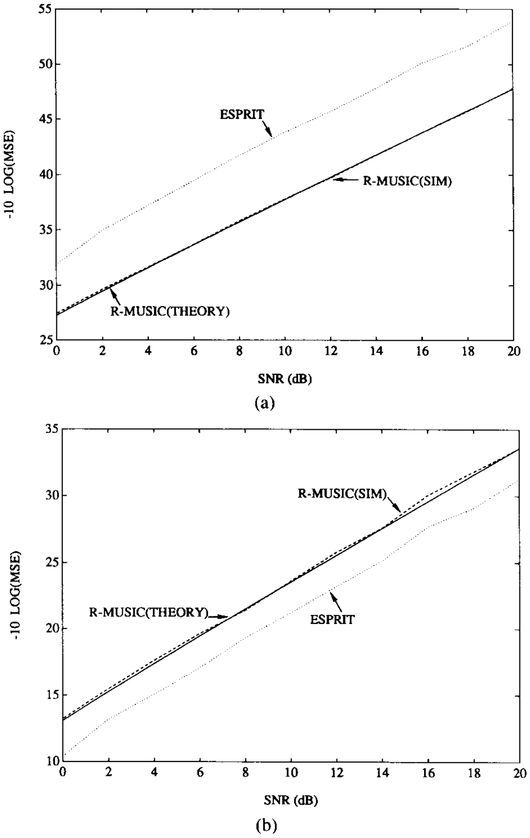

Fig. 3. (a) Mean squared error of signal zero as a function of signal-to-noise ratio (in dB): Root-Music versus ESPRIT (L = 8, N = 100, DOA = 18°). (b) Mean squared error of DOA as a function of signal-to-noise ratio (in dB): Root-Music versus ESPRIT (L = 8, N = 100, DOA = 18°).

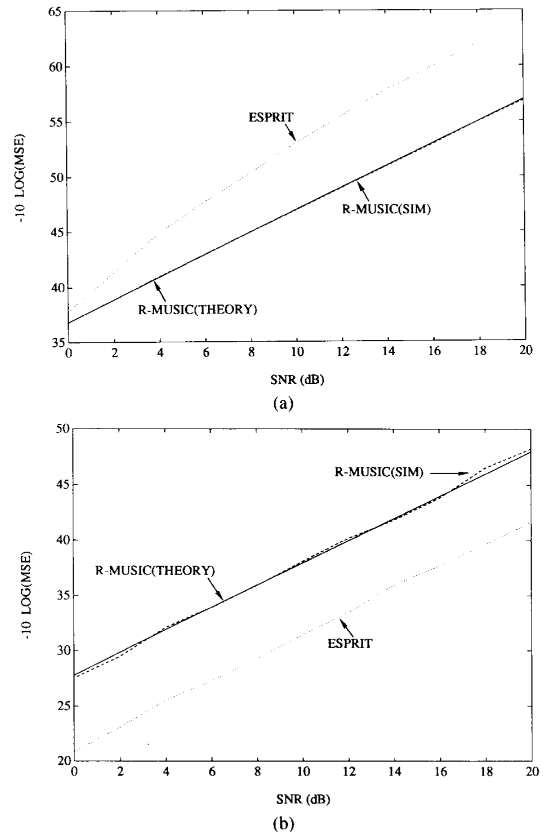

Fig. 4. (a) Mean squared error of signal zero as a function of signal-to-noise ratio (in dB): Root-Music versus ESPRIT (L = 24, N = 100, DOA = 18°). (b) Mean squared error of DOA as a function of signal-to-noise ratio (in dB): Root-Music versus ESPRIT (L = 24, N = 100, DOA = 18°).

(33b) suggests that we need S_{MU}/L to be of the order $(3/L)(1/1 - |\epsilon|)$ in order for the mean squared error in the DOA's obtained by Root-Music to be comparable to ESPRIT. Again note the extra L in the denominator of the expression for the mean squared error in the DOA of Root-Music. Regarding the sensitivity of the MUSIC method S_{MU}, the following general observations can be made. From (18), we have $\|h\|^2 = \|H(z)\|_2 = 1 - (M/L)$, and tends to one as L increases. This observation is similar to that observed in the Minimum-Norm method [15]. As in the Minimum-Norm case, this fact is very useful in the reduction of sensitivity [15]. More quantitatively, the factor ρ_{MU}, where $\rho_{MU} = \frac{1}{3} S_{MU} L(1 - |\epsilon|)$, determines the relative performance of the methods. If $\rho_{MU} \geq 1$, then the signal zeros of ESPRIT have a lower mean squared error compared to Root-Music and vice versa. On the other hand, $\rho_{MU}/L \geq 1$, then the DOA estimates of ESPRIT have a lower mean squared error compared to Root-Music and vice versa. For the high resolution case, i.e., $|\epsilon| \approx 1$, ρ_{MU} through computer simulations [Fig. 2(a)] has been found to be independent of DOA and only dependent on the array length L and the angular separation. For most values of L, ρ_{MU} was found to be greater than 1, implying that the signal zeros of ESPRIT have a lower mean squared error. On the other hand, ρ_{MU}/L was found to be less than 1 [Fig. 2(b)]. This implies that the DOA estimates of Root-Music have a lower mean squared error.

As in the one source case, although the estimates of the signal zeros of Root-Music have a larger mean squared error, the DOA estimates have a smaller mean squared error. This is a result of the extra factor L in the denominator of the expression for the mean squared error in the DOA estimate. This is also true for the more general case as evident by comparing (24) and (29). A large $\overline{|\Delta z_i|^2}$ and small $\overline{|\Delta\theta_i|^2}$ implies that the errors in the signal zeros have a largely radial component. The largely radial nature of the errors makes the peaks in the spatial spectrum less distinct rendering Spectral Music less attractive. Locating the roots, and using its angular location to obtain the DOA as is done in Root-Music, is preferable.

VII. Simulations

In this section, some computer simulations are presented that were done to test the validity of the theory. In all cases, there was very close agreement between theory

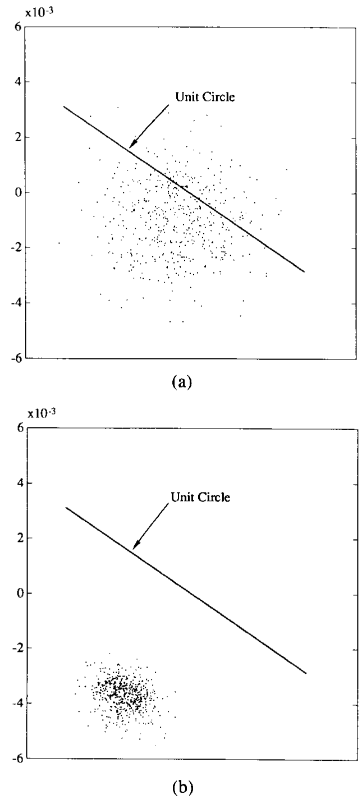

Fig. 5. (a) Estimates of signal zero obtained from 200 trials for ESPRIT ($L = 24$, $N = 100$, $SNR = 10$ dB $DOA = 18°$). (b) Estimates of signal zero obtained from 200 trials for Root-Music ($L = 24$, $N = 100$, $SNR = 10$ dB, $DOA = 18°$).

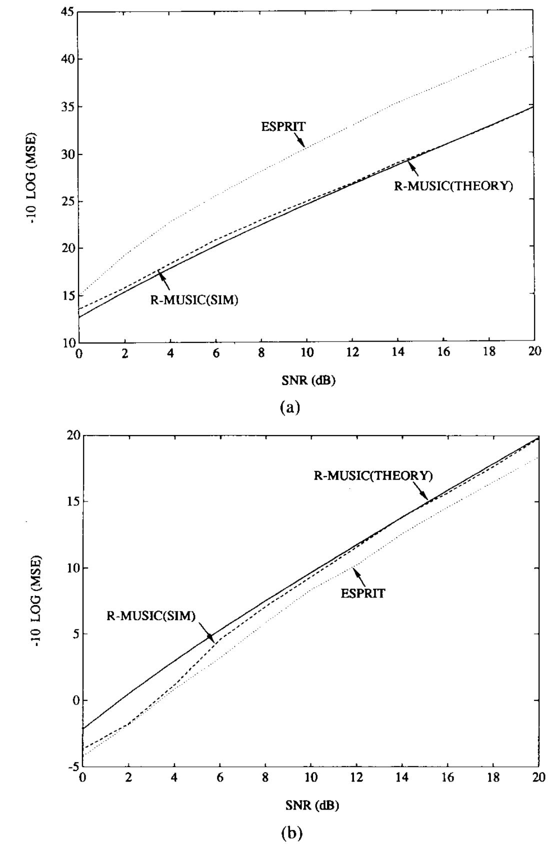

Fig. 6. (a) Mean squared error of signal zero corresponding to source at 18° as a function of signal-to-noise ratio (in dB) for an 8 element array: Root-Music versus ESPRIT ($N = 100$, $L = 8$, $DOA = 18°, 22°$). (b) Mean squared error of DOA corresponding to source at 18° as a function of signal-to-noise ratio (in dB) for an 8 element array: Root-Music versus ESPRIT ($N = 100$, $L = 8$, $DOA = 18°, 22°$).

and the computer simulations. Theory refers to the values computed using (24) and (29).

Example 1: Here we examine Root-Music for the single source case and compare it to ESPRIT. The parameters for this example are $d/\lambda = 0.5$, array size $L = 8$, $\theta_1 = 18°$, and the number of snapshots $N = 100$. The results for various signal-to-noise ratios are summarized in Fig. 3(a) and (b). In all the figures, the mean squared error (MSE) in decibels (10 Log (1/MSE)) as a function of SNR is shown. In Fig. 3(a), the MSE in the estimates of the signal zeros obtained using ESPRIT and Root-Music are shown along with the theoretical predictions for Root-Music. As predicted by theory, ESPRIT is better. Fig. 3(b) compares the MSE in the DOA estimates. Again, as expected, Root-Music has a smaller mean squared error. The theoretical predictions are also close to those obtained by computer simulations.

Example 2: The parameters are the same as in Example 1, except that the array length is increased to 24, i.e., $L = 24$. The results are summarized in Fig. 4(a) and (b). The conclusions are the same as in Example 1. The superiority of using Root-Music for estimating the DOA's is further borne out by the results in Fig. 5(a) and (b). These figures show a close-up view of the location of the estimated signal zeros $\hat{z}_i$ (200 trials) at 10 dB relative to the desired signal zero indicated by a cross. Root-Music results in a tighter cluster, and the error also has a largely radial component.

Example 3: Here two uncorrelated sources with equal power are considered. The parameters for this example are $L = 8$, $N = 100$, and the DOA's are 18° and 22°. Fig. 6(a) and (b) summarizes the results of this experiment. The theoretical results for Root-Music are also shown in Fig. 6(a) and (b), and in general they are in good agreement with the computer simulations. Fig. 6(a) also compares the MSE in the estimate of the signal zero, corresponding to the DOA 18°, obtained using ESPRIT and Root-Music. For $L = 8$, from Fig. 2(a), $\rho_{MU} = 9.86$ implying that ESPRIT will be better. This is supported by the computer simulations. Fig. 6(b) compares the MSE in the estimate of the DOA obtained using ESPRIT and Root-Music. For $L = 8$, from Fig. 2(b), $\rho_{MU}/L = 1.22$ implying that ESPRIT will be slightly better then Root-Music. However, from the computer simulations, Root-Music is slightly better. This discrepancy can be attributed to the optimistic assumptions, particularly regarding L being

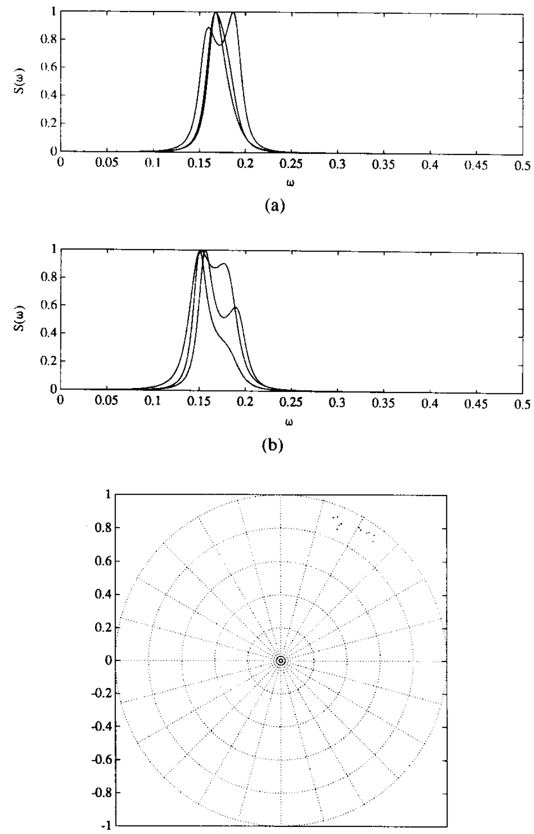

Fig. 7. (a) and (b) $S(\omega)$ for MUSIC for six trials ($N = 100$, $L = 8$, $SNR = 6$ dB, $DOA = 18°, 22°$). (c) Estimates of the signal zeros obtained from the same six trials for Root-Music ($N = 100$, $L = 8$, $SNR = 6$ dB, $DOA = 18°, 22°$).

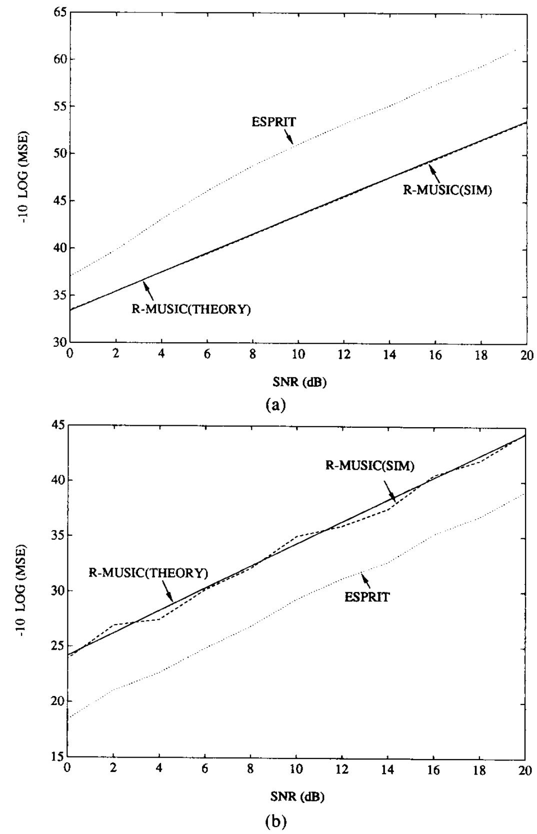

Fig. 8. (a) Mean squared error of signal zero corresponding to source at 18° as a function of signal-to-noise ratio (in dB) for a 24 element array: Root-Music versus ESPRIT ($N = 100$, $L = 24$, $DOA = 18°, 22°$). (b) Mean squared error of DOA corresponding to source at 18° as a function of signal-to-noise ratio (in dB) for a 24 element array: Root-Music versus ESPRIT ($N = 100$, $L = 24$, $DOA = 18°, 22°$).

large, that were made in arriving at the approximate expressions (33a) and (33b). Fig. 7(a)–(c) illustrates the usefulness of a rooting procedure as opposed to plotting the spatial spectrum. Fig. 7(a) and (b) shows the plot of the spatial spectrum for six trials, and the corresponding zero locations are indicated in Fig. 7(c). It can be seen that although the roots have been clearly resolved, the plots of the spatial spectrum do not necessarily indicate two distinct peaks.

Example 4: The same experiment as in Example 3 is conducted except that the array size is increased to 24. The results are summarized in Fig. 8(a) and (b). The theoretical and computer simulations are in close agreement, and the conclusions are similar to those reached in Example 3.

VIII. Summary

In this paper, we have analyzed the asymptotic properties of the estimates obtained by using Root-Music. In particular, closed-form expressions for the mean squared error in the estimates of the signal zeros, and DOA's, have been derived. Simplified expressions are also presented for the one and two source case, and compared to those obtained for Least Squares ESPRIT. Computer simulations were also presented, and they are in close agreement with the theory. An important outcome of this analysis is the fact that the error in the signal zeros had a largely radial component. This provided an explanation as to why Root-Music is superior to the (Spectral) MUSIC algorithm.

Appendix A

Here we simplify (27) and derive an expression for the mean squared error in the DOA estimate. Starting from (27),

$$\begin{aligned}\hat{D}_1^2(e^{j\omega_i}) &= 2 \text{ Re } \left(V_1^H(\omega_i) \sum_{k=1}^{M} (S_k \eta_k^H + \eta_k S_k^H) V(\omega_i) \right)^2 \\ &\quad + 2 \text{ Re } \left(V_1^H(\omega_i) \sum_{k=1}^{M} (S_k \eta_k^H + \eta_k S_k^H) V(\omega_i) \right. \\ &\quad \left. \cdot V^H(\omega_i) \sum_{l=1}^{M} (S_l \eta_l^H + \eta_l S_l^H) V_1(\omega_i) \right) \\ &= 2 \text{ Re } T_1 + 2 \text{ Re } T_2.\end{aligned}$$

Taking expectations of both sides,

$$\overline{\hat{D}_1^2(e^{j\omega_i})} = 2 \text{ Re } \overline{T}_1 + 2 \text{ Re } \overline{T}_2. \qquad (A.1)$$

It can be shown that $\overline{T}_1 = 0$ and that 2 Re $\overline{T}_2$ reduces to (28). Here we will only show the manipulations as related to $\overline{T}_2$. The proof of $\overline{T}_1 = 0$ is similar. To simplify the notation, we will use V and V_1 to denote $V(\omega_i)$ and $V_1(\omega_i)$, respectively.

$$T_2 = \sum_{k=1}^{M}\sum_{l=1}^{M} ((V_1^H S_k \eta_k^H V + V_1^H \eta_k S_k^H V) \cdot (V^H \eta_l S_l^H V_1 + V^H S_l \eta_l^H V_1))$$
$$= \sum_{k=1}^{M}\sum_{l=1}^{M} (V_1^H S_k \eta_k^H V V^H \eta_l S_l^H V_1 + V_1^H S_k \eta_k^H V V^H S_l \eta_l^H V_1 + V_1^H \eta_k S_k^H V V^H \eta_l S_l^H V_1 + V_1^H \eta_k S_k^H V V^H S_l \eta_l^H V_1).$$

Taking expectations of both sides, and using (6) and (7), we have

$$\overline{T}_2 = \sum_{k=1}^{M}\sum_{l=1}^{M} (V_1^H S_k S_l^H V_1 V^H \overline{\eta_l \eta_k^H} V + V_1^H S_k V^H S_l V^T \overline{\eta_k^* \eta_l^H} V_1 + S_k^H V S_l^H V_1 V_1^H \overline{\eta_k \eta_l^T} V^* + S_k^H V V^H S_l V_1 \overline{\eta_k \eta_l^H} V_1)$$
$$= \sum_{k=1}^{M} \left(V_1^H S_k S_k^H V_1 V^H \frac{\lambda_k}{N} \sum_{\substack{l=1\\ l\neq k}}^{L} \frac{\lambda_l}{(\lambda_k - \lambda_l)^2} S_l S_l^H V \right) + \sum_{k=1}^{M}\sum_{\substack{l=1\\ l\neq k}}^{M} \left(V_1^H S_k V^H S_l V^T \cdot \left(-\frac{\lambda_k \lambda_l}{N(\lambda_k - \lambda_l)^2} S_l^* S_k^H \right) V_1 \right)$$
$$+ \sum_{k=1}^{M}\sum_{\substack{l=1\\ l\neq k}}^{M} \left(S_k^H V S_l^H V_1 V_1^H \cdot \left(-\frac{\lambda_k \lambda_l}{N(\lambda_k - \lambda_l)^2} S_l S_k^T \right) V^* \right) + \sum_{k=1}^{M} \left(S_k^H V V^H S_k V_1^H \frac{\lambda_k}{N} \sum_{\substack{l=1\\ l\neq k}}^{L} \frac{\lambda_l}{(\lambda_k - \lambda_l)^2} S_l S_l^H V_1 \right).$$

Cancelling out the common terms results in

$$\overline{T}_2 = \sum_{k=1}^{M} \left(V_1^H S_k S_k^H V_1 V^H \frac{\lambda_k}{N} \sum_{l=M+1}^{L} \frac{\lambda_l}{(\lambda_k - \lambda_l)^2} S_l S_l^H V \right) + \sum_{k=1}^{M} \left(S_k^H V V^H S_k V_1^H \frac{\lambda_k}{N} \sum_{l=M+1}^{L} \frac{\lambda_l}{(\lambda_k - \lambda_l)^2} S_l S_l^H V_1 \right)$$
$$= \sum_{k=1}^{M} \left(S_k^H V V^H S_k V_1^H \frac{\lambda_k}{N} \sum_{l=M+1}^{L} \frac{\lambda_l}{(\lambda_k - \lambda_l)^2} S_l S_l^H V_1 \right).$$

The last equality is a result of the orthogonality of V and the noise subspace, i.e., (12). The above expression then can be further simplified as shown below.

$$\overline{T}_2 = \sum_{k=1}^{M} \left(|S_k^H V|^2 V_1^H \frac{\lambda_k}{N} \frac{\sigma_n^2}{(\lambda_k - \sigma_n^2)^2} P_N V_1 \right) = (V_1^H P_N V_1) \frac{\sigma_n^2}{N} \left(\sum_{k=1}^{M} \frac{\lambda_k}{(\lambda_k^s)^2} |S_k^H V|^2 \right). \qquad (A.2)$$

Substituting (A.2) in (A.1) results in (28).

Appendix B

The expressions necessary to specialize (24) and (29) for the case of two uncorrelated sources with equal power P are considered here. It can be shown that the eigenvalues and eigenvectors are [10]

$$\lambda_{1,(2)}^s = PL(1 \underset{(-)}{+} |\epsilon|) \qquad \text{and } S_{1,(2)} = \frac{U_1 \underset{(-)}{+} U_2}{\sqrt{2(1 \underset{(-)}{+} |\epsilon|)}}$$

where

$$\epsilon = V^H(\omega_1)V(\omega_2) = \frac{1}{L} e^{-j(L-1)\omega_d} \frac{\sin(L\omega_d)}{\sin(\omega_d)},$$

with $\omega_d = (\omega_1 - \omega_2)/2$, and $U_i = e^{-j[(L-1)/2]\omega_i} V(\omega_i)$, $i = 1, 2$. If we assume that $(L\omega_d)^2 << 1$, then

$$\epsilon = e^{-j(L-1)\omega_d} \left[1 - \frac{1}{6} L^2 \omega_d^2 + \frac{1}{120} L^4 \omega_d^4 + \cdots \right].$$

For the high resolution case $|\epsilon| \approx 1$, and it was shown in [10] that $1 - |\epsilon| \approx (\Delta^2/2)$,

$$|S_1^H V(\omega_i)|^2 \approx 1, \quad \text{and } |S_2^H V(\omega_i)|^2 \approx \frac{\Delta^2}{4}.$$

Substituting these relationships in (24) and (29) results in (32a) and (32b).

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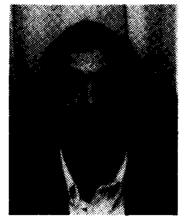

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