EE E6820: Speech & Audio Processing & Recognition

Lecture 3: Pattern Classification

1. **The problem of classification**
2. **Linear and nonlinear classifiers**
3. **Probabilistic classification**
4. **Gaussians, mixtures and EM**
5. **Methodological remarks**

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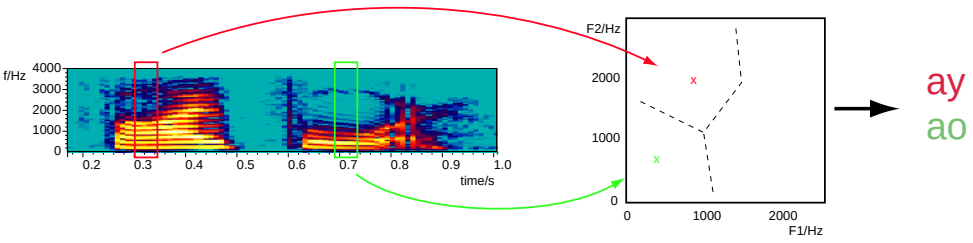
1 Pattern classification

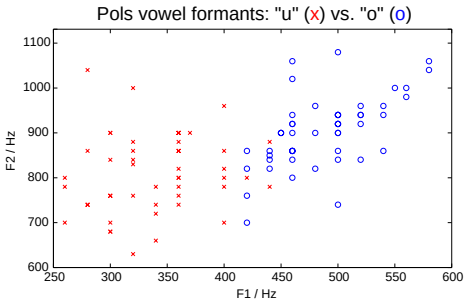
- **Classification means:**
  finding categorical (*discrete*) labels
  for real-world (*continuous*) observations



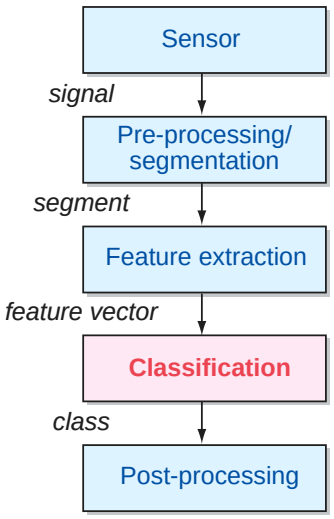
- **Why?**
  - information extraction
  - conditional processing
  - detect exceptions

Building a classifier

- **Define classes/attributes**
  - could state explicit rules
  - better to define through 'training' examples
- **Define feature space**
- **Define decision algorithm**
  - set parameters from examples
- **Measure performance**
  - calculate (weighted) error rate



Classification system parts



- STFT
- Locate vowels

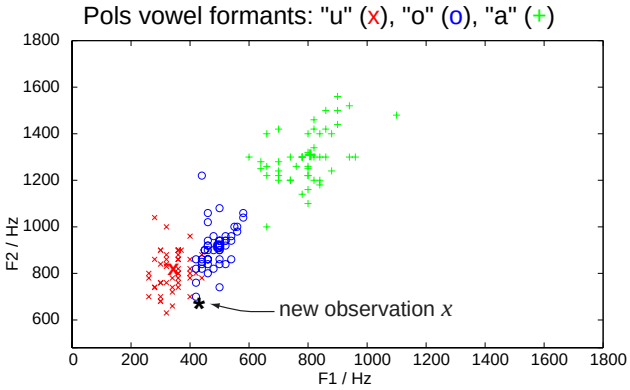
- Formant extraction

- Context constraints
- Costs/risk

Feature extraction

- **Right features are critical**
  - waveform vs. formants vs. cepstra
  - invariance under irrelevant modifications
- **Theoretically equivalent features may act very differently in practice**
  - representations make important aspects explicit
  - remove irrelevant information
- **Feature design incorporates 'domain knowledge'**
  - more data → less need for 'cleverness'?
- **Smaller 'feature space' (fewer dimensions)**
  → simpler models (fewer parameters)
  → less training data needed
  → faster training

Minimum Distance Classification



- **Find closest match (Nearest Neighbor)**
  $min[D(x, y_{\omega,i})]$
  - choice of distance metric $D(x,y)$ is important
- **Find representative match (class prototypes)**
  $min[D(x, z_{\omega})]$
  - class data $\{y_{\omega,i}\}$ → class *model* z_{ω}

2 Linear and nonlinear classifiers

- **Minimum Euclidean distance is equivalent to a linear discriminant function:**
  - examples $\{y_{\omega,i}\}$ → template $z_\omega = \text{mean}\{y_{\omega,i}\}$
  - observed feature vector $x = [x_1\ x_2\ ...]^T$
  - Euclidean distance metric

    $$D[x, y] = \sqrt{(x-y)^T(x-y)}$$
  - minimum distance rule:

    $$\text{class } i = \underset{i}{\text{argmin}}\ D[x, z_i]$$

    $$= \underset{i}{\text{argmin}}\ D^2[x, z_i]$$
  - i.e. choose class O over U if

    $$D^2[x, z_O] < D^2[x, z_U] \quad ...$$

Linear classifier (cont'd)

- **Decision rule: Choose class O over U if:**

$$D^2[x, z_O] < D^2[x, z_U]$$

$$(x-z_O)^T(x-z_O) < (x-z_U)^T(x-z_U)$$

$$x^Tx + z_O^Tz_O - 2x^Tz_O < x^Tx + z_U^Tz_U - 2x^Tz_U$$

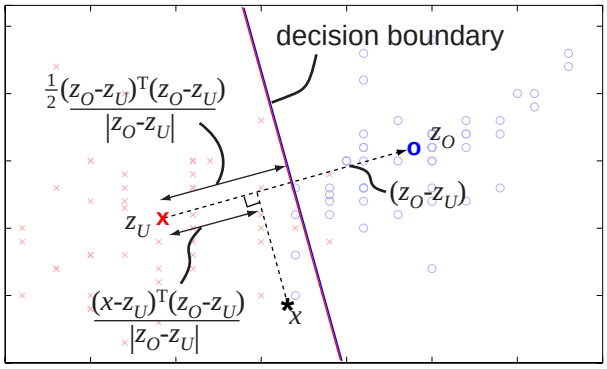
Projection → $2x^T(z_O - z_U) > z_O^Tz_O - z_U^Tz_U$ ← *Threshold*

$$(x-z_U)^T(z_O-z_U) \underset{z_U}{\overset{z_O}{\gtrless}} \frac{1}{2}(z_O-z_U)^T(z_O-z_U)$$

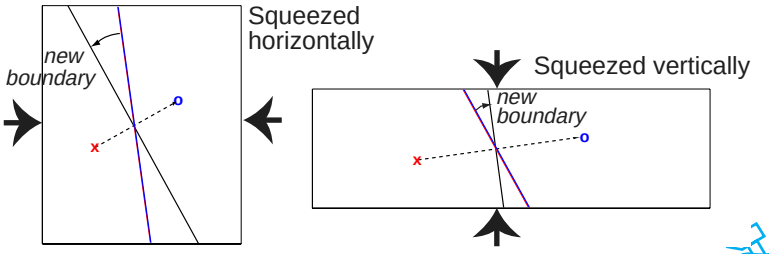
- **i.e. distance from z_U to x is less than half the distance to z_O ...**

Linear classification boundaries

- **Min-distance divides normal to class centers:**



- **Scaling axes changes boundary:**



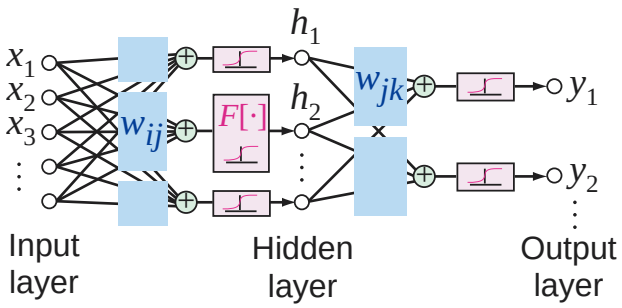
Decision boundaries

- Linear functions **give** linear boundaries
  - planes and hyperplanes as dimensions increase
- **Linear boundaries can become curves under feature transformations**
  - e.g. a linear discriminant in $\boldsymbol{x}' = [x_1^2\ x_2^2\ ...]^T$
  - support vector machines...
- **What about more complex boundaries?**
  - i.e. if a linear discriminant is $\sum w_i x_i$ how about a nonlinear function?
  - $F[\sum w_i x_i]$ changes only threshold space, but $\sum_j F[\sum_i w_{ij} x_i]$ offers more flexibility, and $F[\sum_j w_{jk} \cdot F[\sum_i w_{ij} x_i]]$ has even more ...

Neural networks

- **Sums over nonlinear functions of sums → large range of decision surfaces**
- **e.g. Multi-layer perceptron (MLP) with 1 hidden layer:**

$$y_k = F[\sum_j w_{jk} \cdot F[\sum_j w_{ij} x_i]]$$



- **Problem is finding the weights w_{ij} ... (*training*)**

Back-propagation training

- **Find w_{ij} to minimize e.g. $E = \sum_n |y(x_n) - t_n|^2$ for training set of patterns $\{x_n\}$ and desired (target) outputs $\{t_n\}$**
- **Differentiate error with respect to weights**
  - contribution from each pattern x_n, t_n:

$$y_k = F[\sum_j w_{jk} \cdot h_j]$$

$$\frac{\partial E}{\partial w_{jk}} = \frac{\partial E}{\partial y} \cdot \frac{\partial y}{\partial \Sigma} \cdot \frac{\partial}{\partial w_{jk}} (\sum_j w_{jk} h_j)$$

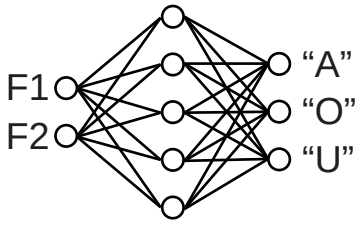
$$= 2(y - t) \cdot F'[\Sigma] \cdot h_j$$

$$w_{jk} = w_{jk} - \alpha \cdot \frac{\partial E}{\partial w_{jk}}$$

- **i.e. gradient descent with learning rate α**

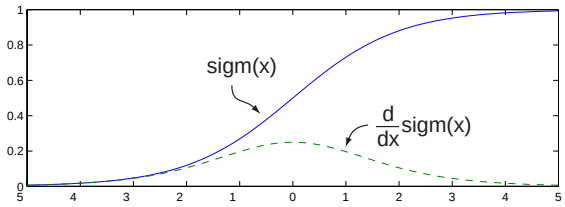
Neural net example

- **2 input units (normalized F1, F2)**
- **5 hidden units, 3 output units ("U", "O", "A")**



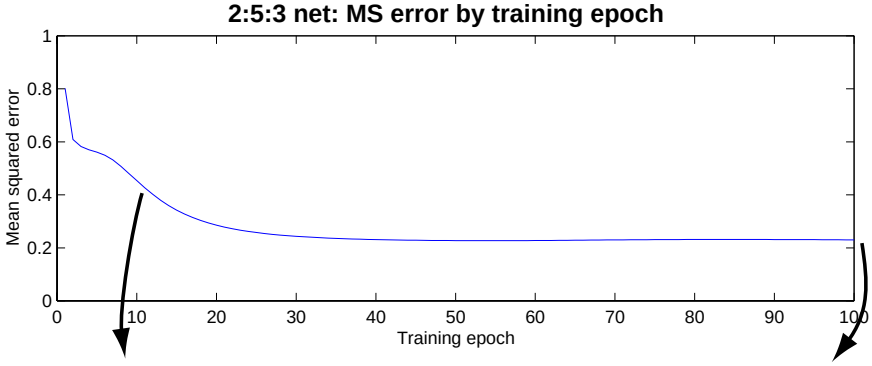
- **Sigmoid nonlinearity:**

$$F[x] = \frac{1}{1+e^{-x}} \quad \Rightarrow \quad \frac{dF}{dx} = F(1-F)$$

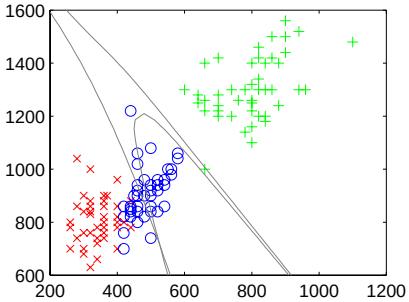


Neural net training

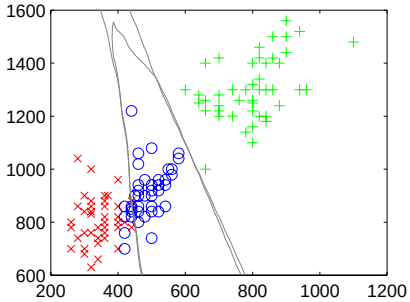
2:5:3 net: MS error by training epoch



Contours @ 10 iterations



Contours @ 100 iterations



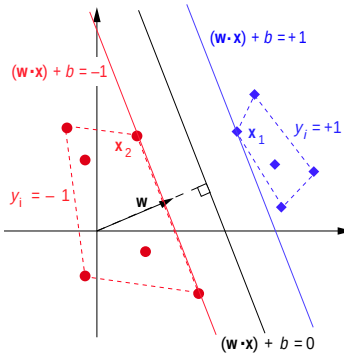
example...

Support Vector Machines (SVMs)

- **Principled nonlinear decision boundaries...**

- **Key idea: Boundary can be linear in a higher-dimensional space**

e.g. $$\Phi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} x_1^2 \\ \sqrt{2}x_1x_2 \\ x_2^2 \end{bmatrix}$$

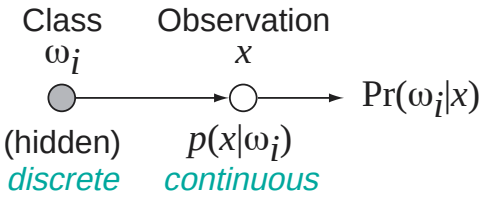


  - depends only on training points at edges (**support vectors**)
  - unique optimal solution for a given projection Φ
  - solution involves only inner products in projected space

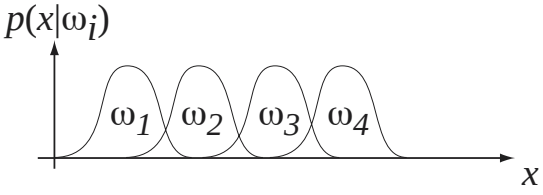
    e.g. via kernel $k(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x}) \cdot \Phi(\mathbf{y}) = (\mathbf{x} \cdot \mathbf{y})^2$

3 Statistical Interpretation

- **Observations are random variables whose distribution depends on the class:**

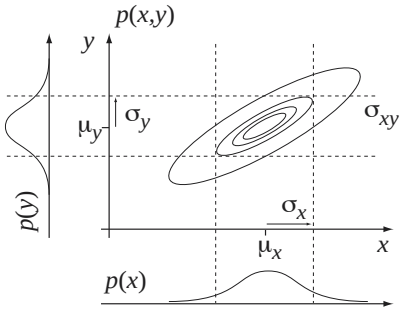


- **Source distributions** $p(x|\omega_i)$
  - reflect variability in feature
  - reflect noise in observation
  - generally have to be estimated from data (rather than known in advance)



Random Variables review

- **Random vars have joint distributions (pdf's):**

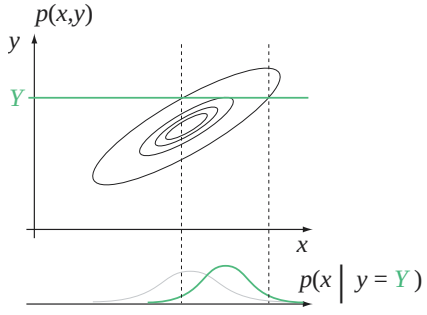


- marginal $p(x) = \int p(x, y)dy$

- covariance $\Sigma = E[(\mathbf{o} - \mu)(\mathbf{o} - \mu)^T] = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix}$

Conditional probability

- **Knowing one value in a joint distribution constrains the remainder:**



- **Bayes' rule:** $p(x, y) = p(x|y) \cdot p(y)$

$$\Rightarrow p(y|x) = \frac{p(x|y) \cdot p(y)}{p(x)}$$

→ **can reverse conditioning given priors/marginal**

- either term can be discrete or continuous

Priors and posteriors

- **Bayesian inference can be interpreted as updating prior beliefs with new information, x:**

Bayes' Rule:

$$Pr(\omega_i) \cdot \frac{p(x|\omega_i)}{\sum_j p(x|\omega_j) \cdot Pr(\omega_j)} = Pr(\omega_i|x)$$

Likelihood

Prior probability

'Evidence' = p(x)

Posterior probability

- **Posterior is prior scaled by likelihood & normalized by evidence (so Σ(posteriors) = 1)**
- **Objection: priors are often unknown**
  - ...but omitting them amounts to assuming they are all equal

Bayesian (MAP) classifier

- **Minimize the probability of error by choosing maximum a posteriori (MAP) class:**

$$\hat{\omega} = \underset{\omega_i}{\text{argmax}}\, Pr(\omega_i|x)$$

- **Intuitively right - choose most probable class in light of the observation**
- **Formally, expected probability of error,**

$$E[Pr(err)] = \int p(x) \cdot Pr(err|x) dx$$

$$= \sum_i \int_{X_{\hat{\omega}_i}} p(x)(1 - Pr(\omega_i|x)) dx$$

where $X_{\hat{\omega}_i}$ is the region of X where ω_i is chosen

.. but $Pr(\omega_i|x)$ is largest in that region, so $Pr(err)$ is minimized.

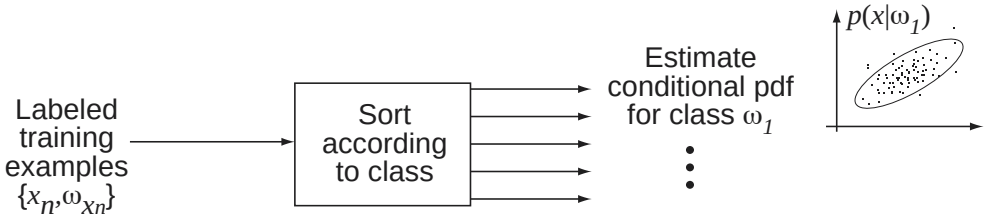
Practical implementation

- **Optimal classifier is** $\hat{\omega} = \underset{\omega_i}{\operatorname{argmax}} Pr(\omega_i|x)$

  but we don't know $Pr(\omega_i|x)$

- **Can model directly e.g. train a neural net / SVM to map from inputs** x **to a set of outputs** $Pr(\omega_i)$ **- a discriminative model**

- **Often easier to model conditional distributions** $p(x|\omega_i)$ **then use Bayes' rule to find MAP class**



Likelihood models

- **Given models for each distribution** $p(x|\omega_i)$**,**

  the search for $\hat{\omega} = \underset{\omega_i}{\operatorname{argmax}} Pr(\omega_i|x)$

  becomes $\underset{\omega_i}{\operatorname{argmax}} \dfrac{p(x|\omega_i) \cdot Pr(\omega_i)}{\sum_j p(x|\omega_j) \cdot Pr(\omega_j)}$

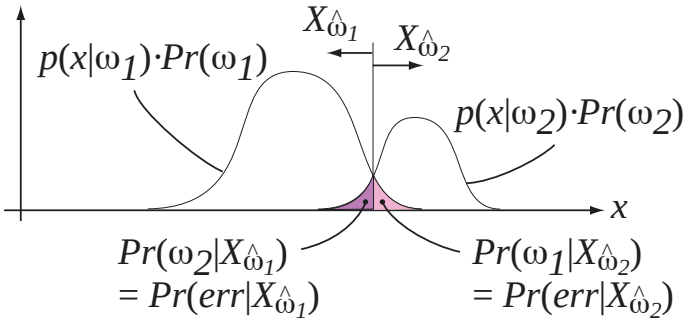
  but denominator = $p(x)$ **is the same over all** ω_i

  hence $\hat{\omega} = \underset{\omega_i}{\operatorname{argmax}}\ p(x|\omega_i) \cdot Pr(\omega_i)$

  or even $\underset{\omega_i}{\operatorname{argmax}} [\log p(x|\omega_i) + \log Pr(\omega_i)]$

- **Choose parameters to maximize data likelihood** $p(x_{train}|\omega_i)$

Sources of error



- **Suboptimal threshold / regions (bias error)**
  - use a Bayes classifier!
- **Incorrect distributions (model error)**
  - better distribution models/more training data
- **Misleading features ('Bayes error')**
  - *irreducible* for a given feature set regardless of classification scheme

4

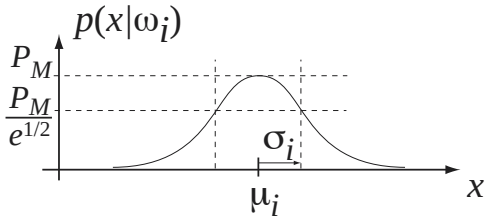
Gaussian models

- **Easiest way to model distributions is via parametric model**
  - assume known form, estimate a few parameters
- **Gaussian model is simple & useful:**

in 1-D: $$p(x|\omega_i) = \frac{1}{\sqrt{2\pi}\sigma_i} \cdot \exp\left[-\frac{1}{2}\left(\frac{x-\mu_i}{\sigma_i}\right)^2\right]$$

normalization to make it sum to 1

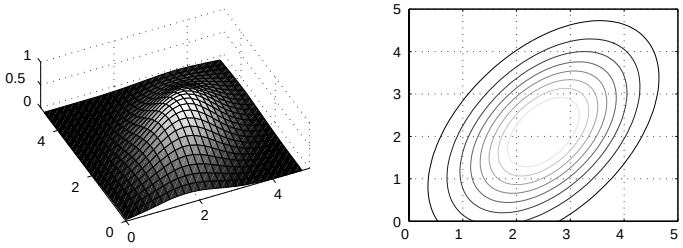
- **Parameters mean μ_i and variance σ_i^2 → fit**



Gaussians in *d* dimensions:

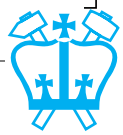
$$p(\mathbf{x}|\omega_i) = \frac{1}{(\sqrt{2\pi})^d |\Sigma_i|^{1/2}} \cdot \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^T \Sigma_i^{-1}(\mathbf{x}-\mu_i)\right]$$

- **Described by d dimensional mean vector μ_i and d x d covariance matrix Σ_i**



- **Classify by maximizing log likelihood i.e.**

$$\underset{\omega_i}{\text{argmax}} \left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^T \Sigma_i^{-1}(\mathbf{x}-\mu_i) - \frac{1}{2}\log|\Sigma_i| + \log Pr(\omega_i)\right]$$



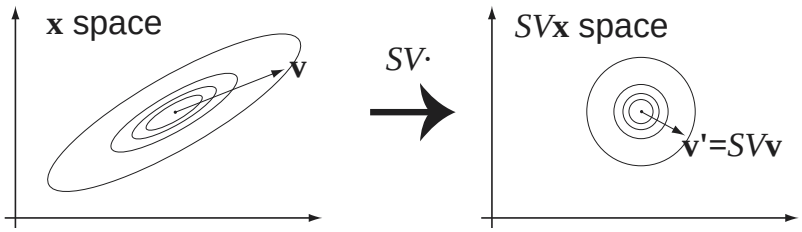
Multivariate Gaussian covariance

- **Eigen analysis of inverse of covariance gives**

$$\Sigma_i^{-1} = (SV)^T(SV) \quad \text{with diagonal } S = \Lambda^{1/2}$$

→ **Gaus'n ~** $\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^T(SV)^T(SV)(\mathbf{x}-\mu_i)\right)$

i.e. (SV) transforms x into uncorrelated, normalized-variance space



- $(\mathbf{x}-\boldsymbol{\mu}_i)^T \Sigma_i^{-1}(\mathbf{x}-\mu_i)$ **= Mahalanobis distance** ^{2}
  - Euclidean distance in normalized space

example...

Gaussian Mixture models (GMMs)

- **Single Gaussians cannot model**
  - distributions with multiple modes
  - distributions with nonlinear correlation
- **What about a weighted sum?**

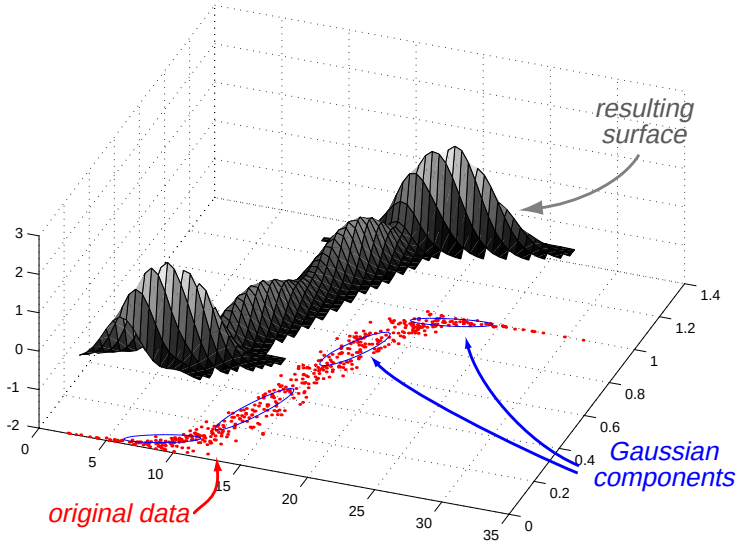
  i.e. $p(x) \approx \sum_k c_k p(x|m_k)$

  where $\{c_k\}$ is a set of weights and

  $\{p(x|m_k)\}$ is a set of Gaussian components
  - can fit anything given enough components
- **Interpretation:**
  Each observation is generated by one of the Gaussians, chosen at random, with priors:
  $c_k = Pr(m_k)$

Gaussian mixtures (2)

- **e.g. nonlinear correlation:**



- **Problem: finding c_k and m_k parameters**
  - easy if we knew which m_k generated each x

Expectation-maximization (EM)

- **General procedure for estimating a model when some parameters are unknown**
  - e.g. which component generated a value in a Gaussian mixture model
- **Procedure:**
  Iteratively update fixed model parameters Θ to maximize Q=expected value of log-probability of known training data x_{trn}
  and unknown parameters u:

$$Q(\Theta, \Theta_{old}) = \sum_u Pr(u|x_{trn}, \Theta_{old}) \log p(u, x_{trn}|\Theta)$$

  - iterative because $Pr(u|x_{trn}, \Theta_{old})$ changes each time we change Θ
  - can prove this increases data likelihood, hence maximum-likelihood (ML) model
  - local optimum - depends on initialization

Fitting GMMs with EM

- **Want to find**
  component Gaussians $m_k = \{\mu_k, \Sigma_k\}$
  plus weights $c_k = Pr(m_k)$
  to maximize likelihood of training data x_{trn}
- **If we could assign each x to a particular m_k, estimation would be direct**
- **Hence, treat ks (mixture indices) as unknown, form $Q = E[\log p(x, k|\Theta)]$,**
  differentiate with respect to model parameters
  → equations for μ_k, Σ_k and c_k to maximize Q...

GMM EM update equations:

- **Solve for maximizing Q:**

$$\mu_k = \frac{\sum_n p(k|x_n, \Theta) \cdot x_n}{\sum_n p(k|x_n, \Theta)}$$

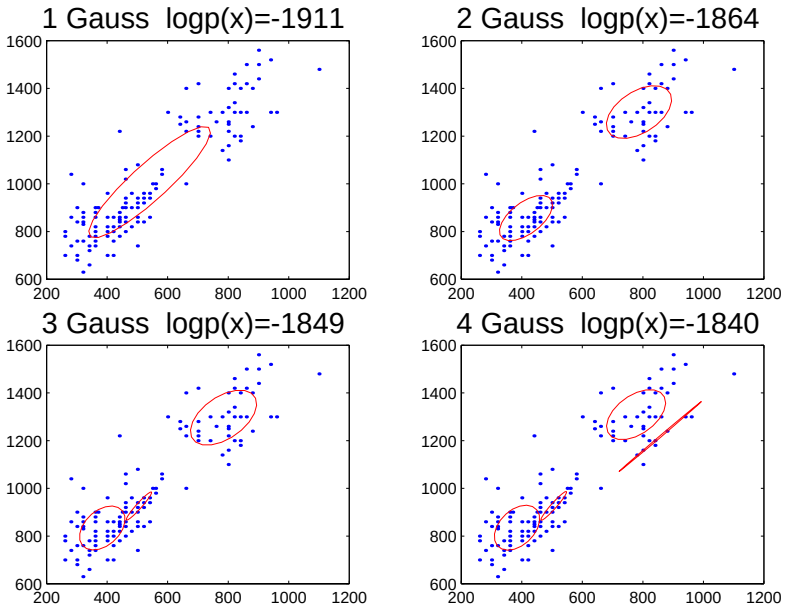
$$\Sigma_k = \frac{\sum_n p(k|x_n, \Theta)(x_n - \mu_k)(x_n - \mu_k)^T}{\sum_n p(k|x_n, \Theta)}$$

$$c_k = \frac{1}{N}\sum_n p(k|x_n, \Theta)$$

- **Each involves $p(k|x_n, \Theta)$,**
  'fuzzy membership' of x_n to Gaussian k
- **Parameter is just sample average, weighted by fuzzy membership**

GMM examples

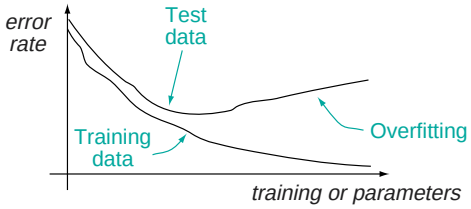
- **Vowel data fit with different mixture counts:**



example...

5 Methodological Remarks: 1: Training and test data

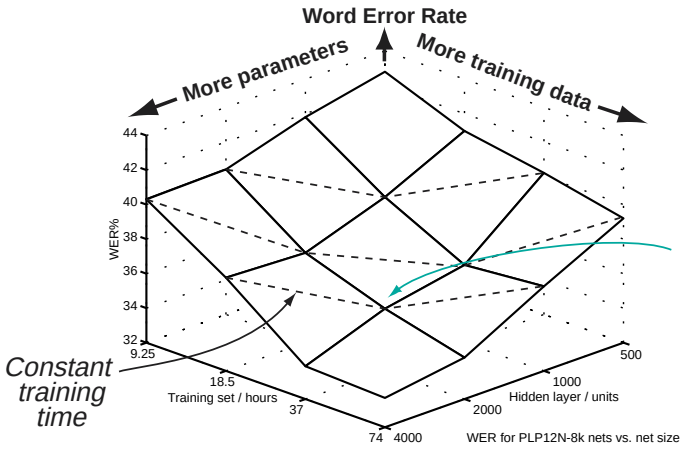
- **A rich model can learn every training example (overtraining)**



- **But, goal is to classify new, unseen data i.e. generalization**
  - sometimes use 'cross validation' set to decide when to stop training
- **For evaluation results to be meaningful:**
  - don't test with training data!
  - don't train on test data (even indirectly...)

Model complexity

- **More model parameters → better fit**
  - more Gaussian mixture components
  - more MLP hidden units
- **More training data → can use larger models**
- **For fixed training set size, there will be some optimal model size (to avoid overfitting):**



Model combination

- **No single model is always 'best'**
  But different models have different strengths

→ **Benefits from combining several models e.g.**

$$Pr(\omega) = \sum_k w_k \cdot Pr(\omega|m_k)$$

where each $Pr(\omega|m_k)$ comes from a different model / feature set

and $w_k = Pr(m_k)$ estimates probability that model k is correct

- **Should be able to incorporate into one model, but may be easier to combine in practice**
- **Trick is to have good estimates for w_k:**
  - static, from training data
  - dynamic, from test data...

Summary

- **Classification is making discrete (hard) decisions**
- **Basis is comparison with known examples**
  - explicitly, or via a model
- **Probabilistic interpretation for principled decisions**
- **Ways to build models:**
  - neural nets can learn posteriors $Pr(\omega|x)$
  - Gaussians can model pdfs $p(x|\omega)$
  - other approaches...
- **EM allows estimation of complex models**

Parting thought:

- **How do you know what's going on?**