EE E6820: Speech & Audio Processing & Recognition

Lecture 2: Acoustics

1. **The wave equation**
2. **Acoustic tubes: reflections & resonance**
3. **Oscillations & musical acoustics**
4. **Spherical waves & room acoustics**

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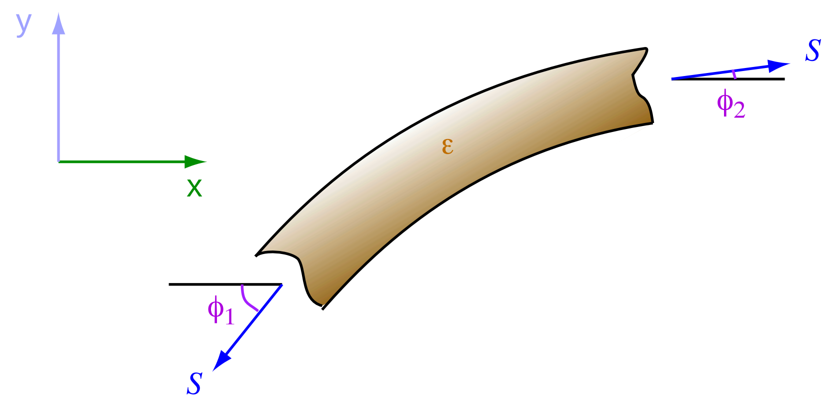
Acoustics & sound

- Acoustics is the study of physical waves
- (Acoustic) waves transmit energy without permanently displacing matter
(e.g. ocean waves)
- Same math recurs in many domains
- Intuition: pulse going down a rope

The wave equation

- **Consider a small section of the rope:**



- **displacement** $y(x)$**, tension** S**, mass** $\varepsilon \cdot dx$

→ **lateral force is** $F_y = S \cdot \sin(\phi_2) - S \cdot \sin(\phi_1)$

$$\approx S \frac{\partial^2 y}{\partial x^2} dx$$

...

Wave equation (2)

- **Newton's law:** $F = ma$

$$S \cdot \frac{\partial^2 y}{\partial x^2} \cdot dx = \varepsilon dx \cdot \frac{\partial^2 y}{\partial t^2}$$

- **Call** $c^2 = S / \varepsilon$ **(tension to mass-per-length)**

hence:

the Wave Equation: $$c^2 \cdot \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

.. partial DE relating curvature and acceleration

Solution to the wave equation

- **If** $y(x, t) = f(x - ct)$ **(any** $f(\cdot)$**)**

  then

$$\frac{\partial y}{\partial x} = f'(x - ct) \qquad \frac{\partial y}{\partial t} = -c \cdot f'(x - ct)$$

$$\frac{\partial^2 y}{\partial x^2} = f''(x - ct) \qquad \frac{\partial^2 y}{\partial t^2} = c^2 \cdot f''(x - ct)$$

  also works for $y(x, t) = f(x + ct)$

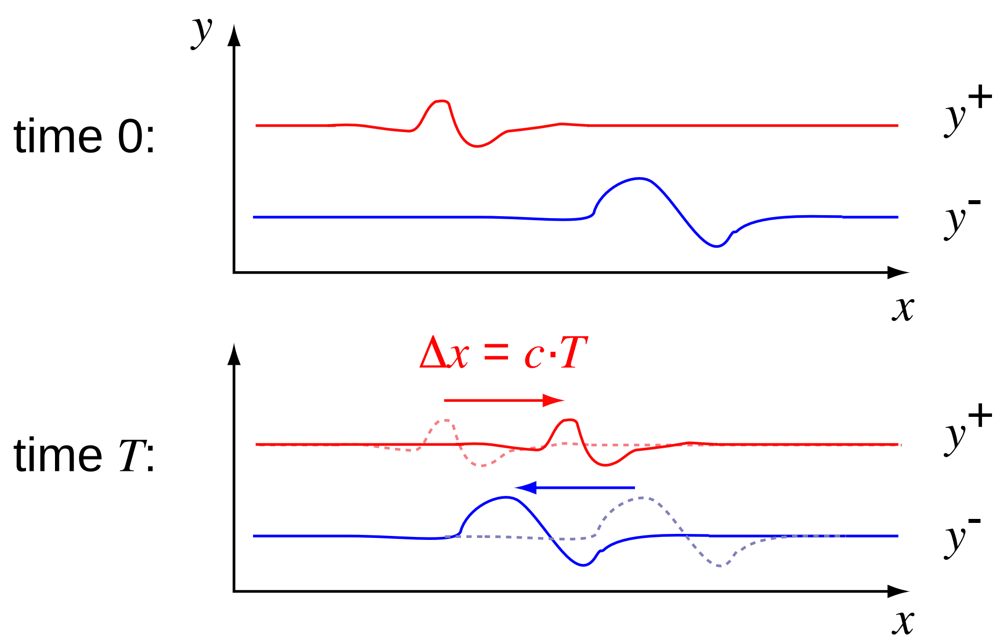
  Hence, general solution:

$$c^2 \cdot \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \quad y(x, t) = y^+(x - ct) + y^-(x + ct)$$

Solution to the wave equation (2)

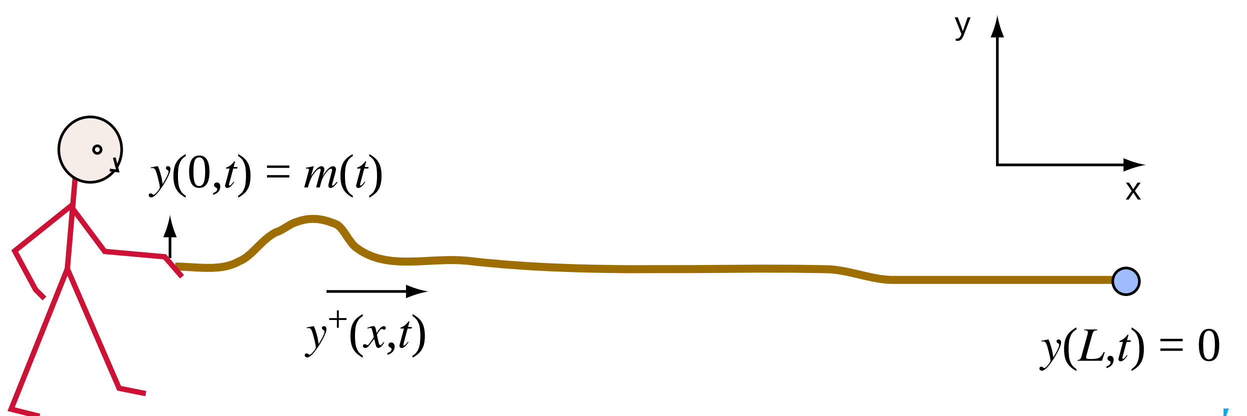
- $y^{+}(x - ct)$ **and** $y^{-}(x + ct)$ **are travelling waves**
  - **shape stays constant but changes position:**



- c **is travelling wave velocity (** $\Delta x / \Delta t$ **)**
- y^{+} **moves right,** y^{-} **moves left**
- **resultant** $y(x)$ **is sum of the two waves**

Wave equation solutions (3)

- **What is the form of y^+, y^- ?**
  - any doubly-differentiable function will satisfy wave equation
- **Actual waveshapes dictated by boundary conditions**
  - e.g. $y(x)$ at $t = 0$
  - plus constraints on y at particular x's
    e.g. input motion $y(0, t) = m(t)$
    rigid termination $y(L, t) = 0$



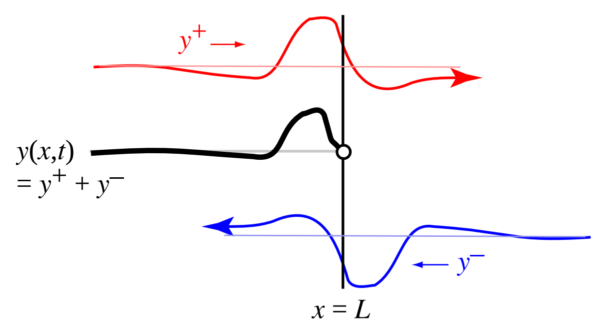
Terminations and reflections

- **System constraints:**
  - initial $y(x, 0) = 0$ (flat rope)
  - input $y(0, t) = m(t)$ (at agent's hand) ($\rightarrow y^+$)
  - termination $y(L, t) = 0$ (fixed end)
  - wave equation $y(x,t) = y^+(x - ct) + y^-(x + ct)$
- **At termination:**

  $y(L, t) = 0 \;\rightarrow\; y^+(L - ct) = -y^-(L + ct)$

  i.e. y^+ and y^- are mirrored in time and amplitude around $x = L$

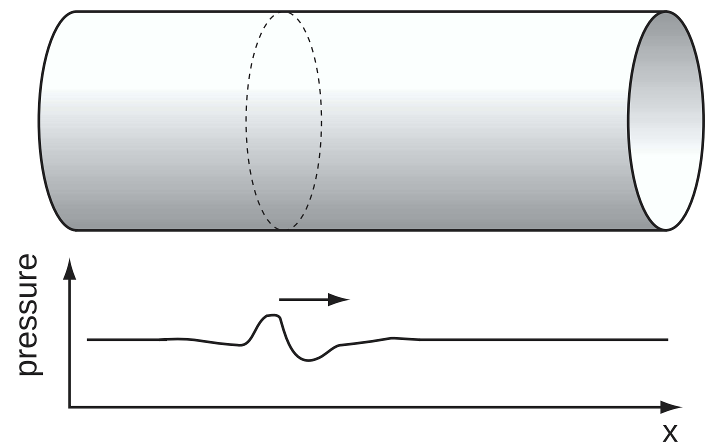
$\rightarrow$ inverted reflection at termination



simulation [travel1.m]

2 Acoustic tubes

- **Sound waves travel down acoustic tubes:**

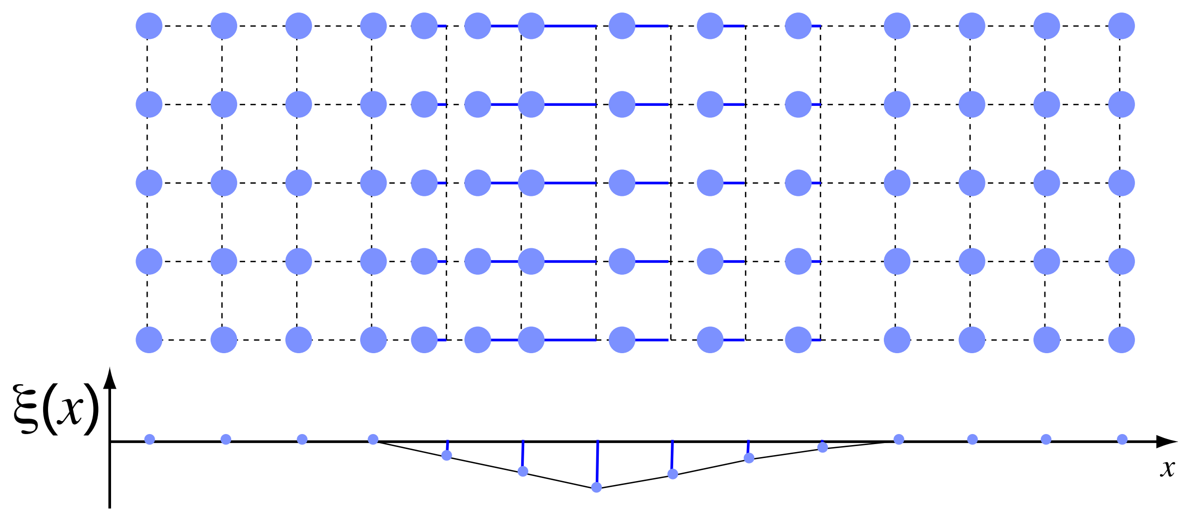


  - 1-dimensional; very similar to strings

- **Common situation:**
  - wind instrument bores
  - ear canal
  - vocal tract

Pressure and velocity

- **Consider air particle displacement $\xi(x,t)$:**



- **Particle velocity** $v(x,t) = \dfrac{\partial \xi}{\partial t}$

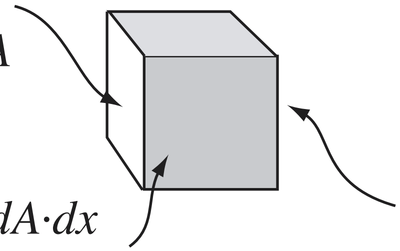
  hence volume velocity $u(x,t) = A \cdot v(x,t)$

- **(Relative) air pressure** $p(x,t) = -\dfrac{1}{\kappa} \cdot \dfrac{\partial \xi}{\partial x}$

Wave equation for a tube

- **Consider elemental volume:**

Area dA
Force $p \cdot dA$



Force $(p + {}^{\partial p}/_{\partial x} \cdot dx) \cdot dA$

Volume $dA \cdot dx$
Mass $\rho \cdot dA \cdot dx$

- **Newton's law:** $F = ma$

$$-\frac{\partial p}{\partial x} \cdot dx \cdot dA = \rho dA dx \cdot \frac{\partial v}{\partial t}$$

$$\Rightarrow \frac{\partial p}{\partial x} = -\rho \frac{\partial v}{\partial t}$$

- **Hence** $c^2 \cdot \dfrac{\partial^2 \xi}{\partial x^2} = \dfrac{\partial^2 \xi}{\partial t^2}$ $\qquad c = \dfrac{1}{\sqrt{\rho\kappa}}$

Acoustic tube traveling waves

- **Traveling waves in particle displacement:**

$$\xi(x,t) = \xi^{+}(x-ct) + \xi^{-}(x+ct)$$

- **Call** $u^{+}(\alpha) = -cA\frac{\partial}{\partial\alpha}\xi^{+}(\alpha)$

$$Z_0 = \frac{\rho c}{A}$$

- **Then volume velocity:**

$$u(x,t) = A\cdot\frac{\partial\xi}{\partial t} = u^{+}(x-ct) - u^{-}(x+ct)$$

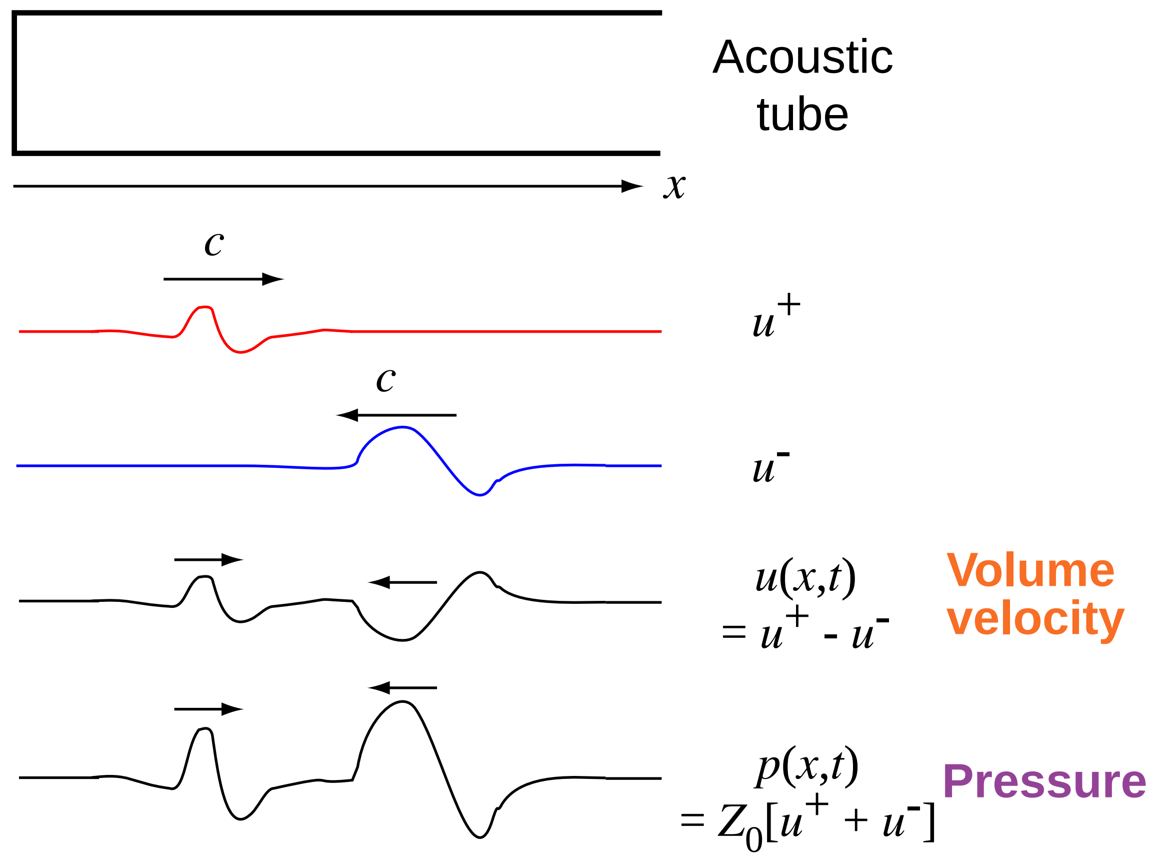
- **And pressure:**

$$p(x,t) = -\frac{1}{\kappa}\cdot\frac{\partial\xi}{\partial x} = Z_0\cdot[u^{+}(x-ct) + u^{-}(x+ct)]$$

- **(Scaled) sum & diff. of traveling waves**

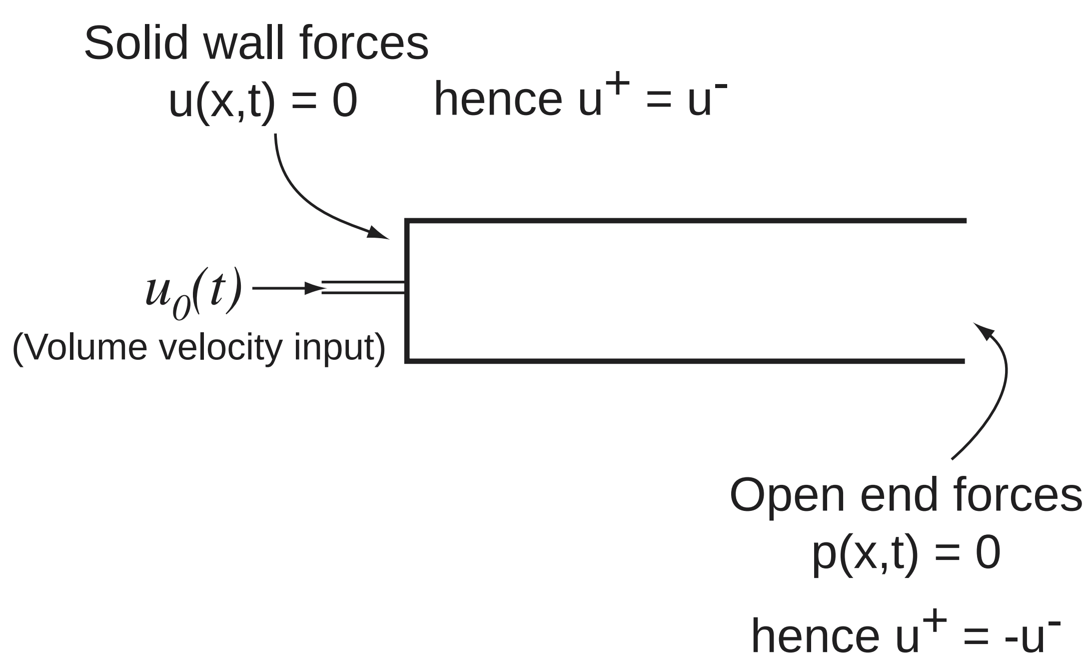
Acoustic tube traveling waves (2)

- **Different resultants for pressure and volume velocity:**



Terminations in tubes

- **Equivalent of fixed point for tubes?**



- **Open end is like fixed point for rope: reflects wave back inverted**
- **Unlike fixed point, solid wall reflects traveling wave without inversion**

Standing waves

- **Consider (complex) sinusoidal input:**

$$u_0(t) = U_0 \cdot e^{j\omega t}$$

- **Pressure/volume must have form** $Ke^{j(\omega t + \phi)}$
- **Hence traveling waves:**

$$u^+(x - ct) = |A|e^{j(-kx + \omega t + \phi_A)}$$

$$u^-(x + ct) = |B|e^{j(kx + \omega t + \phi_B)}$$

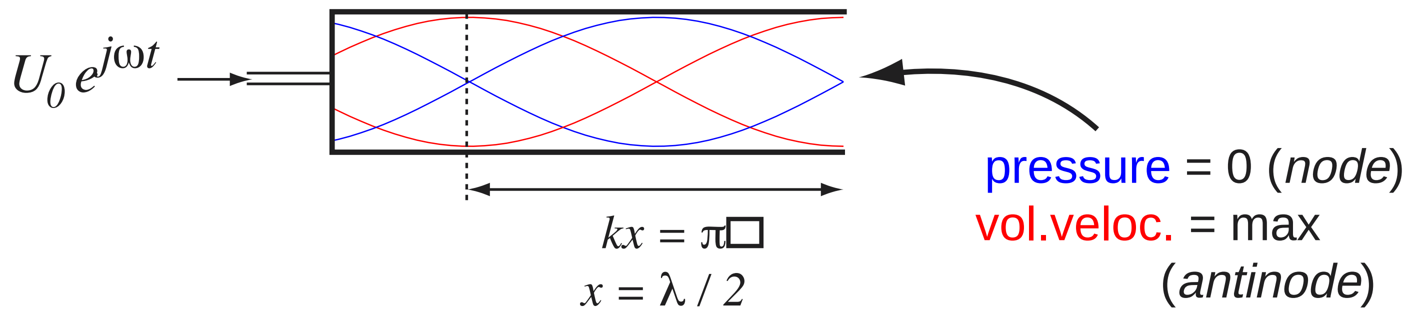
where $k = \omega / c$ **(spatial frequency, rad/m)**

(wavelength $\lambda = c/f = 2\pi c/\omega$ **)**

- **Pressure, vol. veloc. resultants show stationary pattern: standing waves**
  - even when $|A| \neq |B|$

→simulation [sintwavemov.m]

Standing waves (2)



- **For lossless termination ($|u^+| = |u^-|$), have true nodes & antinodes**
- **Pressure and vol. veloc. are phase shifted**
  - in space and in time

*

Transfer function

- **Consider tube excited by** $u_0(t) = U_0 \cdot e^{j\omega t}$**:**
  - sinusoidal traveling waves must satisfy termination 'boundary conditions'
  - satisfied by complex constants A and B in

$$
\begin{aligned}
u(x, t) &= u^{+}(x - ct) + u^{-}(x + ct) \\
&= Ae^{j(-kx + \omega t)} + Be^{j(kx + \omega t)} \\
&= e^{j\omega t} \cdot (Ae^{-jkx} + Be^{jkx})
\end{aligned}
$$

  - standing wave pattern will scale with input magnitude
  - point of excitation makes a big difference...

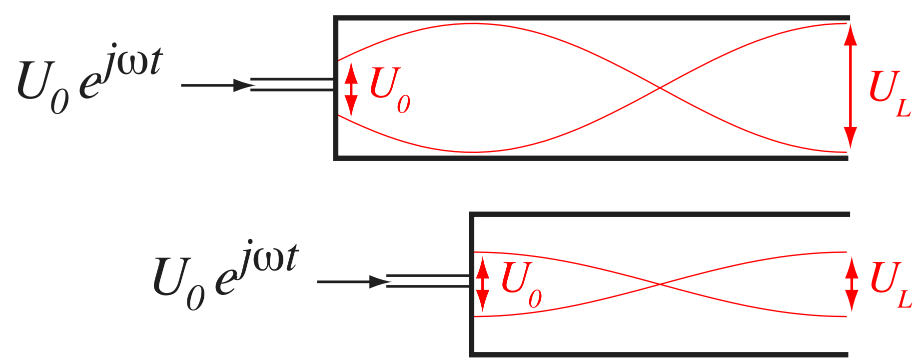
Transfer function (2)

- **For open-ended tube of length L excited at $x = 0$ by $U_0 e^{j\omega t}$:**

$$u(x, t) = U_0 e^{j\omega t} \cdot \frac{\cos k(L - x)}{\cos kL} \qquad \left(k = \frac{\omega}{c}\right)$$

(matches at $x = 0$, maximum at $x = L$)

- **i.e. standing wave pattern**
e.g. varying L for a given ω (and hence k):

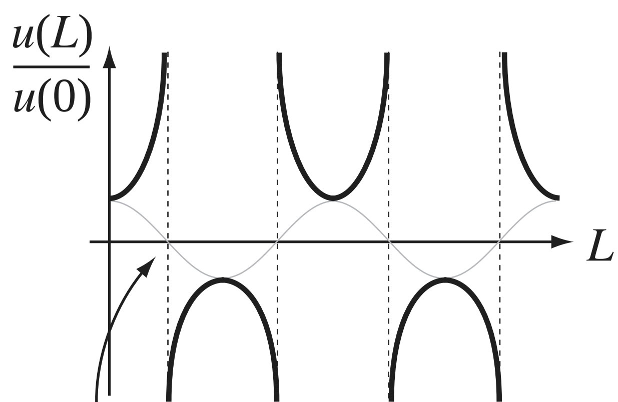


magnitude of U_L depends on L (and ω)...

Transfer function (3)

- **Varying ω for a given L, i.e. at $x = L$:**

$$\frac{U_L}{U_0} = \frac{u(L,t)}{u(0,t)} = \frac{1}{\cos kL} = \frac{1}{\cos(\omega L/c)}$$



$\left|\frac{u(L)}{u(0)}\right| \longrightarrow \infty$ at $\omega L/c = (2r+1)\pi/2$, $r = 0,1,2...$

- **Output vol. veloc. always larger than input**

- **Unbounded for $L = (2r+1)\frac{\pi c}{2\omega} = (2r+1)\frac{\lambda}{4}$**

  i.e. resonance (amplitude grows w/o bound)

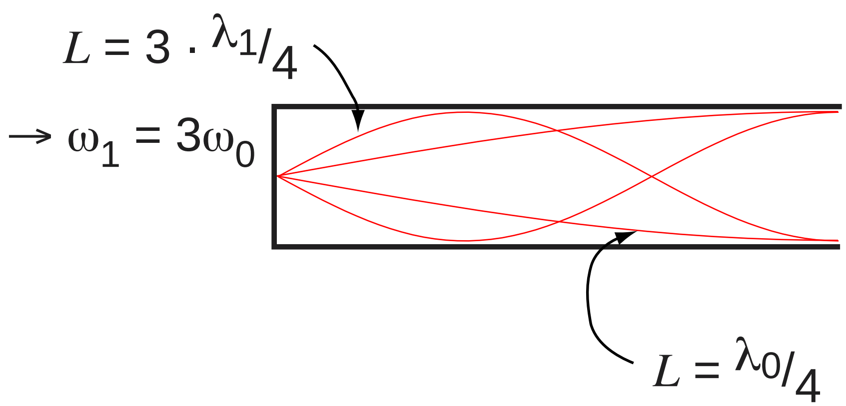
Resonant modes

- **For lossless tube**

  with $L = m \cdot \frac{\lambda}{4}$, m **odd,** λ **wavelength,**

  $\left|\frac{u(L)}{u(0)}\right|$ **is unbounded, meaning:**

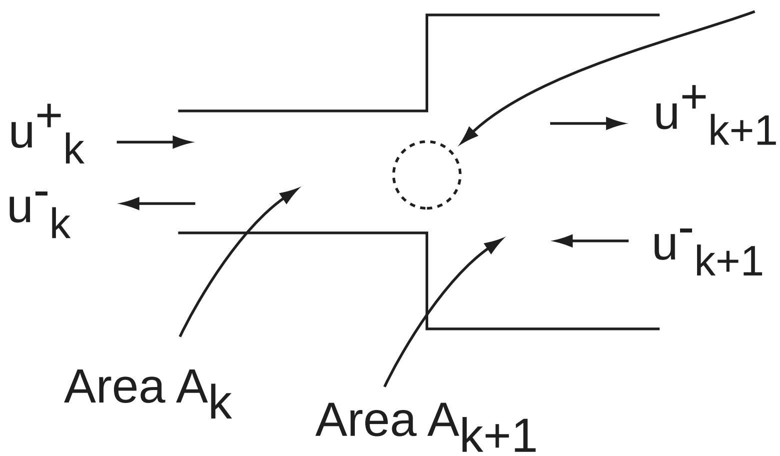
  - transfer function has pole on frequency axis
  - energy at that frequency sustains indefinitely



  - compare to time domain...

- **e.g 17.5 cm vocal tract, c = 350 m/s**
  $\rightarrow \omega_0 = 2\pi \cdot 500$ Hz (then 1500, 2500 ...)

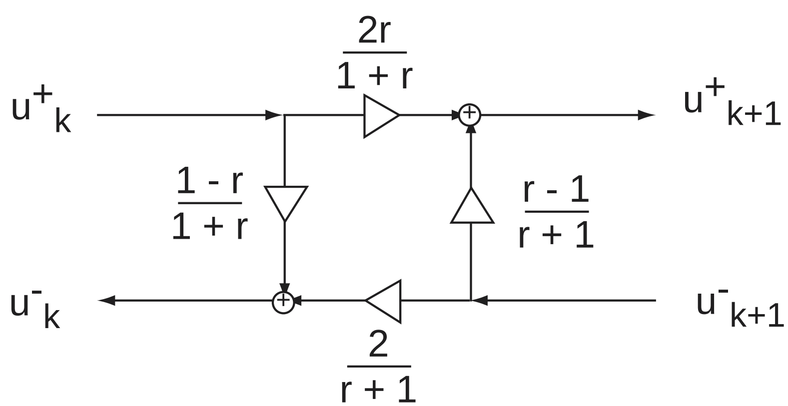
Scattering junctions



At abrupt change in area:

- pressure must be continuous
  $p_k(x, t) = p_{k+1}(x, t)$
- vol. veloc. must be continuous
  $u_k(x, t) = u_{k+1}(x, t)$
- traveling waves
  $u^+_k, u^-_k, u^+_{k+1}, u^-_{k+1}$
  will be different

- **Solve e.g. for u^-_k and u^+_{k+1}: (generalized term.)**

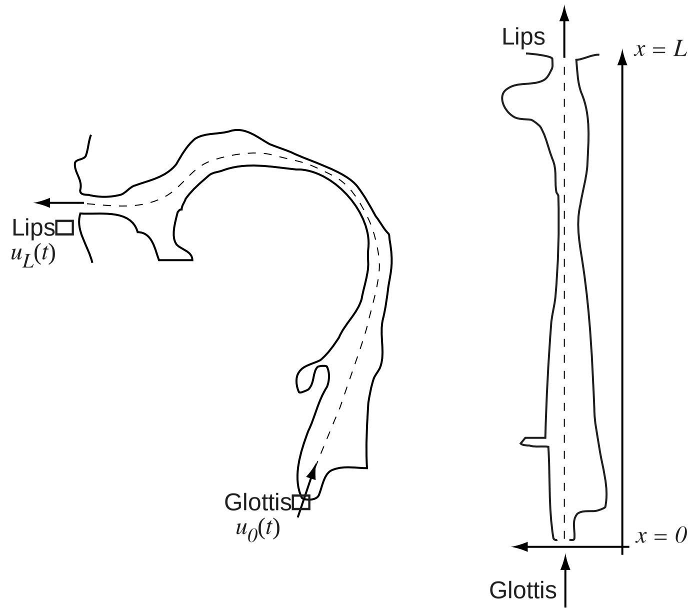


$$r = \frac{A_{k+1}}{A_k}$$

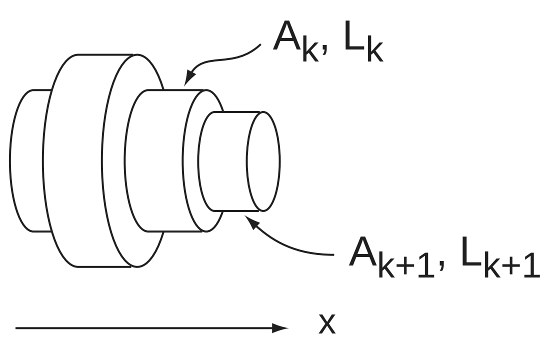
"Area ratio"

Concatenated tube model

- **Vocal tract acts as a waveguide**

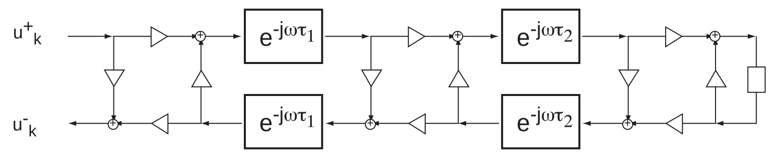


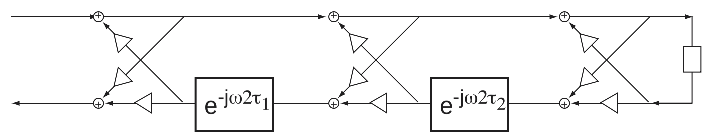
- **Discrete approx. as varying-diameter tube:**



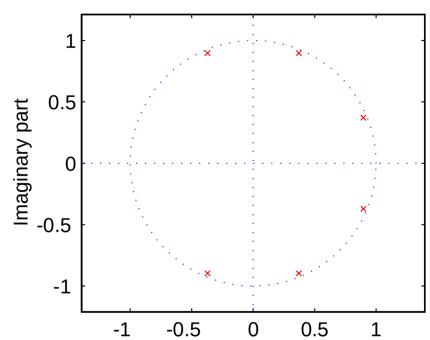
Concatenated tube resonances

- **Concatenated tubes → scattering junctions → lattice filter**





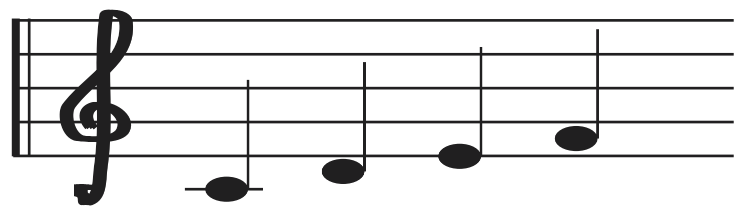
- **Can solve for transfer function - all-pole**



- **Approximate vowel synthesis from resonances**
  sound example: ah ee oo

3 Oscillations & musical acoustics

- **Pitch (periodicity) is essence of music:**



  - why? why music?

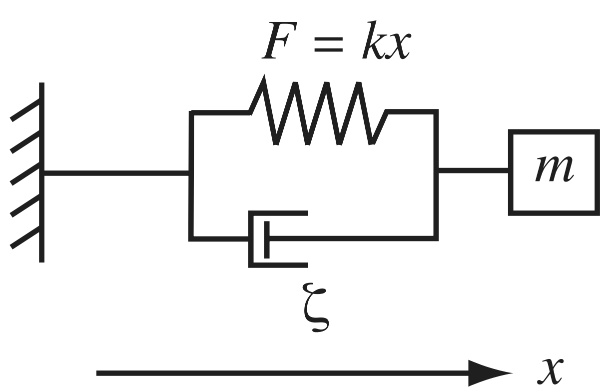
- **Different kinds of oscillators:**
  - simple harmonic motion (tuning fork)
  - relaxation oscillator (voice)
  - string traveling wave (plucked/struck/bowed)
  - air column (nonlinear energy element)

Simple harmonic motion

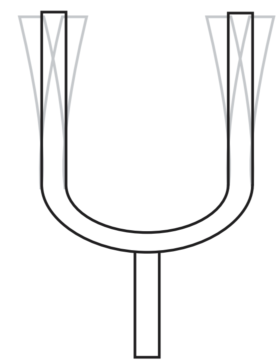
- **Basic mechanical oscillation:**

$$\ddot{x} = -\omega^2 x \qquad x = A\cos(\omega t + \varphi)$$

- **Spring + mass (+ damper)**



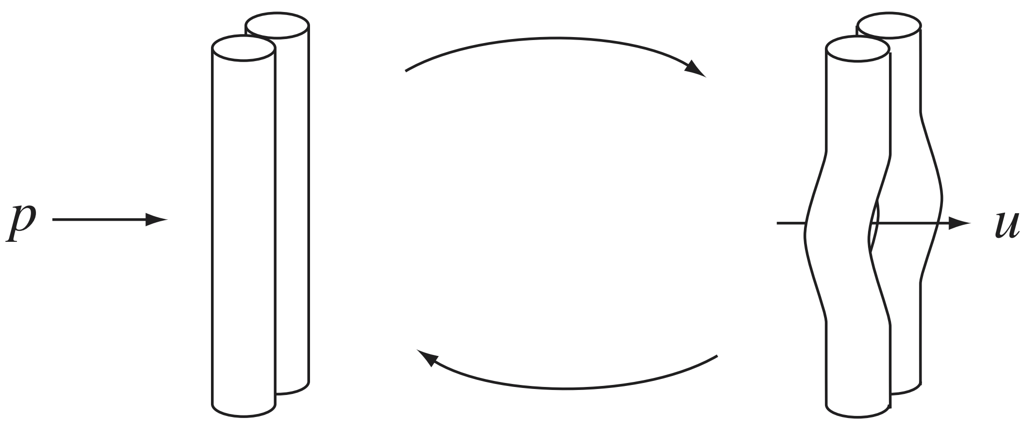
$$\omega^2 = \frac{k}{m}$$



- **e.g. tuning fork**
- **Not great for music:**
  - fundamental (cosωt) only
  - relatively low energy

Relaxation oscillator

- **Multi-state process:**
  - one state builds up potential (e.g. pressure)
  - switch to second (release) state
  - revert to first state etc.
- **e.g. vocal folds:**



(http://www.medicine.uiowa.edu/otolaryngology/cases/normal/normal2.htm)

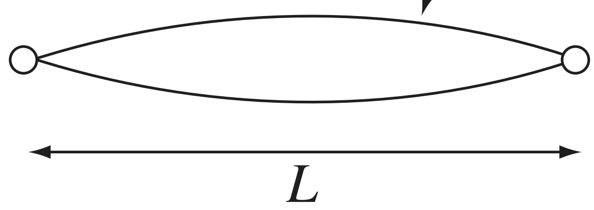
- **Oscillation period depends on force (tension)**
  - easy to change
  - hard to keep stable

  →less used in music

Ringing string

- **e.g. our original 'rope' example**

tension S□
mass/length ε



$$\omega^2 = \frac{\pi^2 S□}{L^2 \varepsilon}$$

- **Many musical instruments**
  - guitar (plucked)
  - piano (struck)
  - violin (bowed)
- **Control period (pitch):**
  - change length (fretting)
  - change tension (tuning piano)
  - change mass (piano strings)
- **Influence of excitation ... [pluck1a.m]**

Wind tube

- **Resonant tube + energy input**

$$\omega = \frac{\pi c\Box}{2L}$$ (quarter wavelength)

- **e.g. clarinet**
  - lip pressure keeps reed closed
  - reflected pressure wave opens reed
  - reinforced pressure wave passes through
- **Finger holes determine first reflection**
  → effective waveguide length

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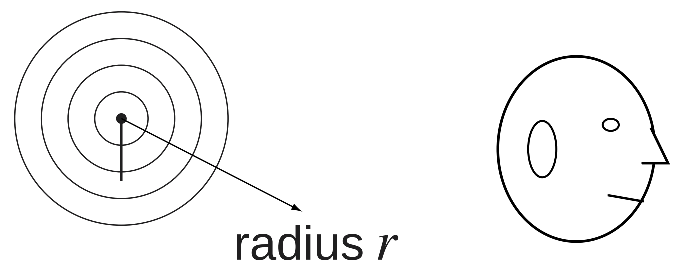
Room acoustics

- **Sound in free air expands spherically:**



- **Spherical wave equation:**

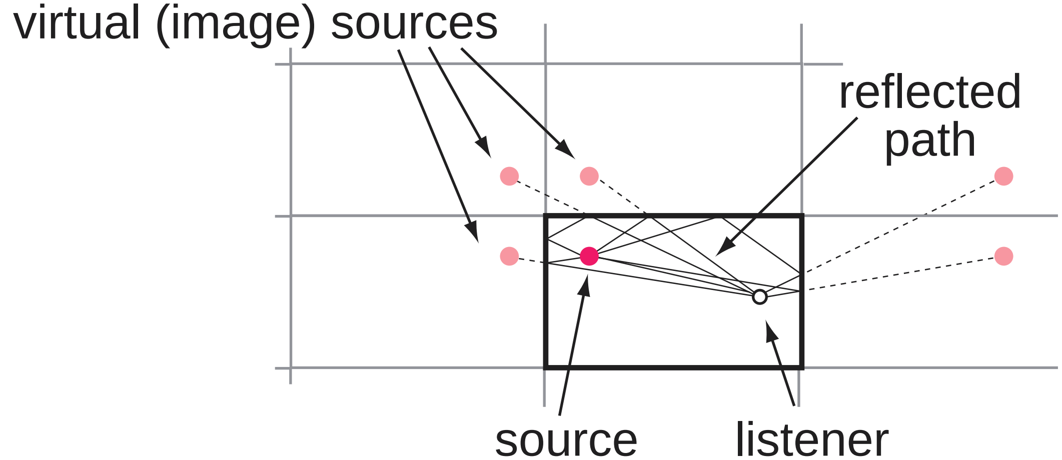
$$\frac{\partial^2 p}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial p}{\partial r} = \frac{1}{c^2} \cdot \frac{\partial^2 p}{\partial t^2}$$

solved by $p(r, t) = \frac{P_0}{r} \cdot e^{j(\omega t - kr)}$

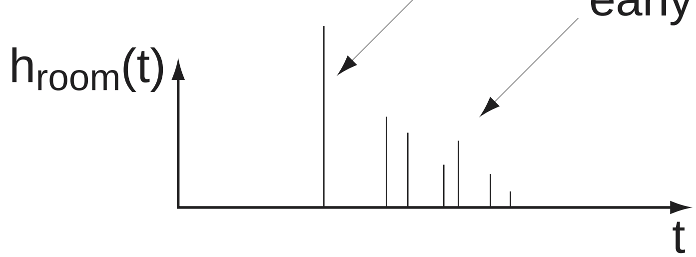
- **Intensity** $\propto p^2$ **falls as** $\frac{1}{r^2}$

Effect of rooms (1): Images

- **Ideal reflections are like multiple sources:**



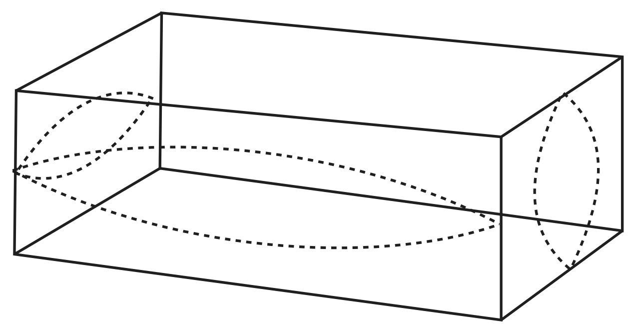
- **'Early echoes' in room impulse response:**



  - actual reflections may be $h_r(t)$, not $\delta(t)$

Effect of rooms (2): modes

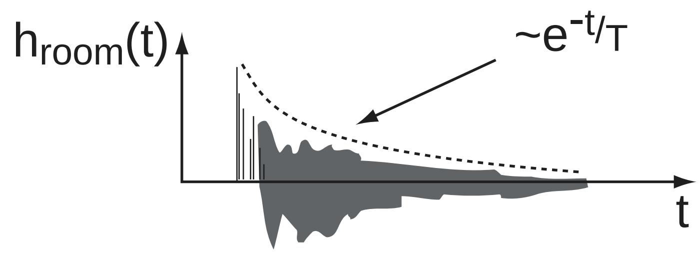
- **Regularly-spaced echoes behave like acoustic tubes:**



- **Real rooms have lots of modes!**
  - dense, sustained echoes in impulse response
  - complex pattern of peaks in frequency response

Reverberation

- **Exponential decay of reflections:**



- **Frequency-dependent**
  - greater absorption at high frequencies $\rightarrow$ faster decay
- **Size-dependent**
  - larger rooms $\rightarrow$ longer delays $\rightarrow$ slower decay
- **Sabine's equation:**

$$RT_{60} = \frac{0.049V}{S\bar{\alpha}}$$

- **Time constant varies with size, absorption**

Summary

- **Travelling waves**
- **Acoustic tubes & resonance**
- **Musical acoustics & periodicity**
- **Room acoustics & reverberation**

Parting Thought:

- **Musical bottles**