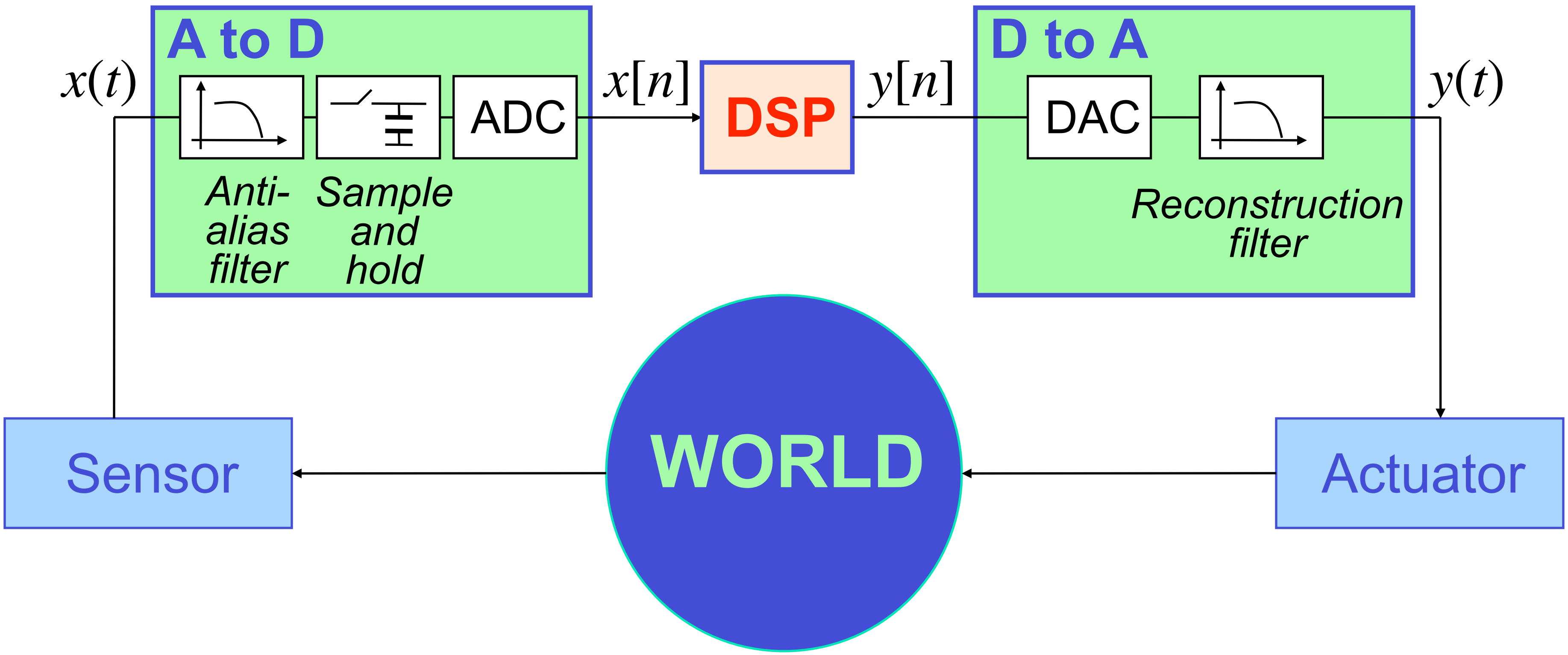
ELEN E4810: Digital Signal Processing

Topic 11: Continuous Signals

1. Sampling and Reconstruction
2. Quantization

1. Sampling & Reconstruction

- DSP must interact with an analog world:



Sampling: Frequency Domain

- Sampling: CT signal → DT signal by recording values at 'sampling instants':

Discrete $$g[n] = g_a(nT)$$ Continuous

Sampling period T

→ samp.freq. $\Omega_{samp} = 2\pi/T$ rad/sec

- What is the relationship of the spectra?

Ω in rad/**second**

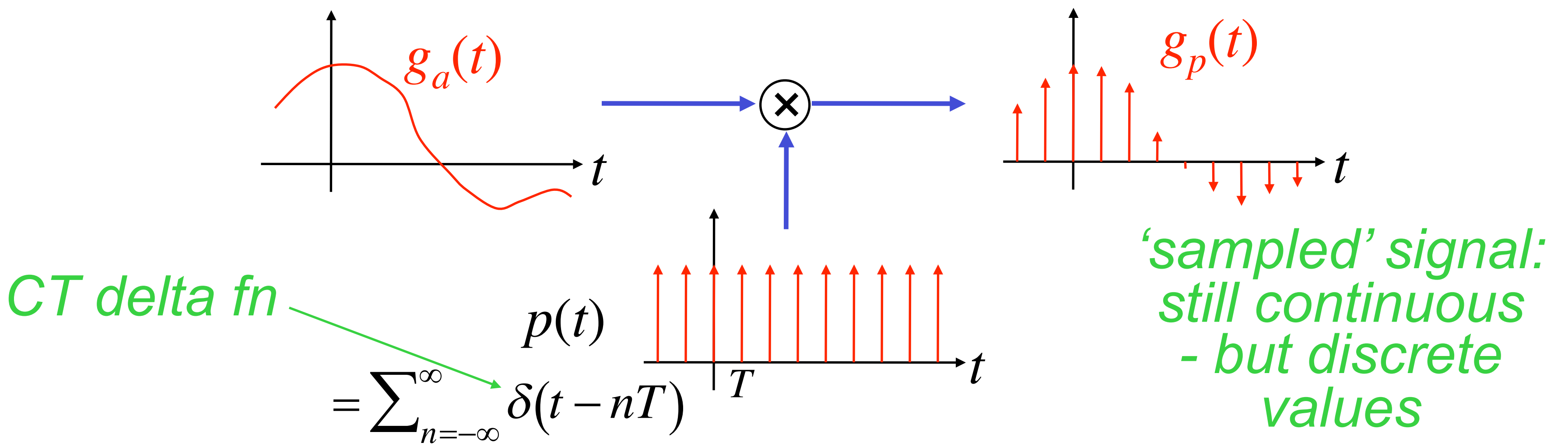
- i.e. relate $$G_a(j\Omega) = \int_{-\infty}^{\infty} g_a(t) e^{-j\Omega t} dt$$ CTFT

and $$G(e^{j\omega}) = \sum_{-\infty}^{\infty} g[n] e^{-j\omega n}$$ DTFT

ω in rad/**sample**

Sampling

- DT signals have same 'content' as CT signals gated by an impulse train:



- $g_p(t) = g_a(t) \cdot p(t)$ is a CT signal with the same information as DT sequence $g[n]$

Spectra of sampled signals

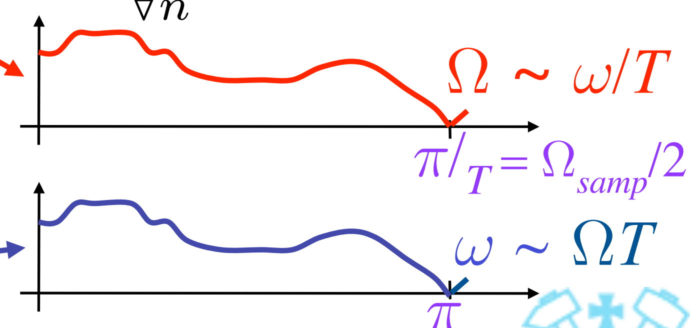
- Given CT $\quad g_p(t) = \sum_{n=-\infty}^{\infty} g_a(nT) \cdot \delta(t - nT)$
- CTFT Spectrum

by linearity

$$G_p(j\Omega) = \mathcal{F}\{g_p(t)\} = \sum_{\forall n} g_a(nT)\mathcal{F}\{\delta(t - nT)\}$$

$$\Rightarrow G_p(j\Omega) = \sum_{\forall n} g_a(nT)e^{-j\Omega nT}$$

- Compare to DTFT $\quad G(e^{j\omega}) = \sum_{\forall n} g[n]e^{-j\omega n}$
- i.e. $G(e^{j\omega}) = G_p(j\Omega)|_{\Omega T = \omega}$



Spectra of sampled signals

- Also, note that $p(t) = \sum_{\forall n} \delta(t - nT)$ is periodic, thus has Fourier Series:

$$p(t) = \boxed{\frac{1}{T}} \sum_{k=-\infty}^{\infty} e^{j\left(\frac{2\pi}{T}\right)kt}$$

$$\because c_k = \frac{1}{T}\int_{-T/2}^{T/2} p(t) e^{-j2\pi kt/T}\,dt = \frac{1}{T}$$

- But $F\left\{e^{j\Omega_0 t}x(t)\right\} = X\left(j(\Omega - \Omega_0)\right)$ *shift in frequency*

so $$G_p(j\Omega) = \frac{1}{T}\sum_{\forall k} \underline{G_a\left(j\left(\Omega - k\Omega_{samp}\right)\right)}$$

- scaled sum of replicas of $G_a(j\Omega)$ shifted by multiples of sampling frequency Ω_{samp}

CT and DT Spectra

- So: $G(e^{j\omega}) = G_p(j\Omega)\Big|_{\Omega T=\omega} = \frac{1}{T}\sum_{\forall k} G_a\left(j\left(\frac{\omega}{T} - k\frac{2\pi}{T}\right)\right)$

DTFT $\quad G(e^{jT\Omega}) = \frac{1}{T}\sum_{\forall k} G_a\left(j\left(\Omega - k\Omega_{samp}\right)\right) \quad$ CTFT

$g_a(t) \leftrightarrow G_a(j\Omega)$

$\times$

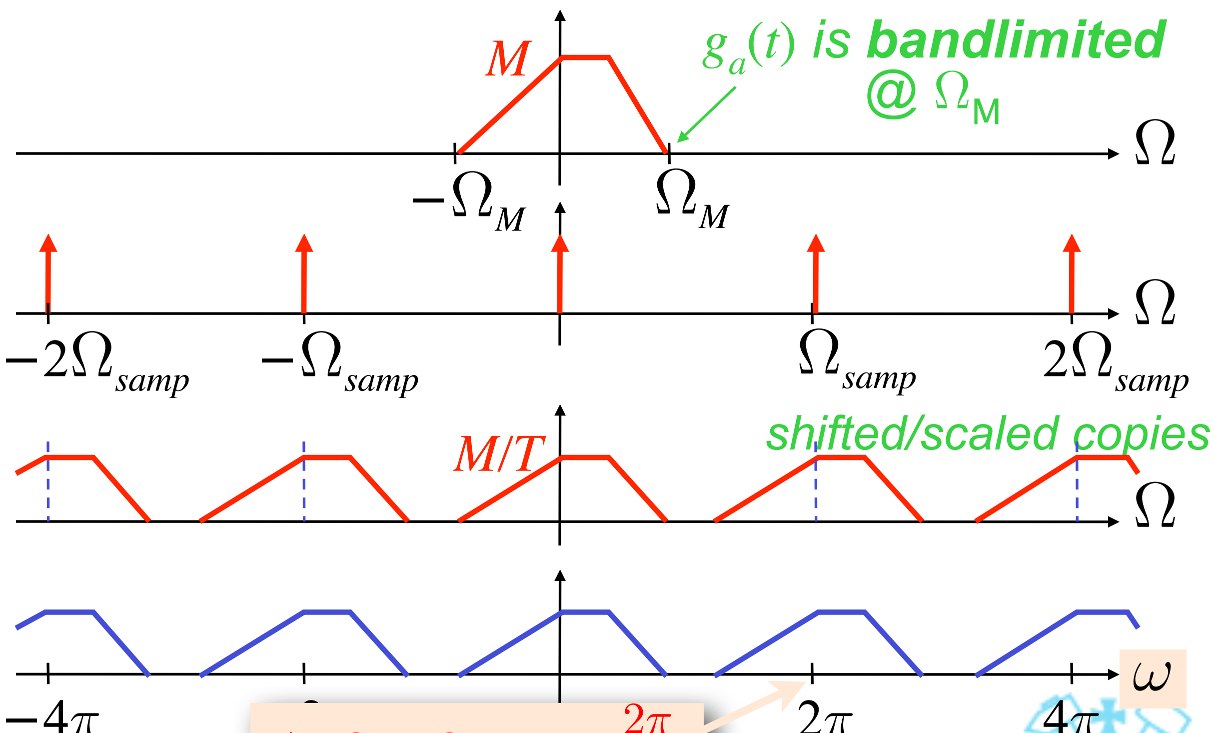
$p(t) \leftrightarrow P(j\Omega)$

$=$

$g_p(t) \leftrightarrow G_p(j\Omega)$

$\downarrow$

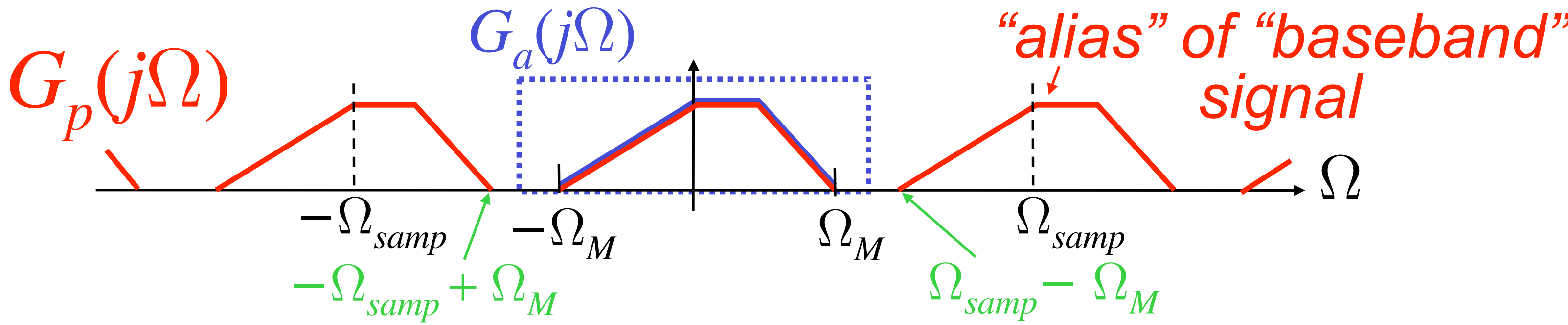
$g[n] \leftrightarrow G(e^{j\omega})$



At $\Omega = \Omega_{samp} = \frac{2\pi}{T}$ have $\omega = \Omega T = 2\pi$

Avoiding aliasing

- Sampled analog signal has spectrum:



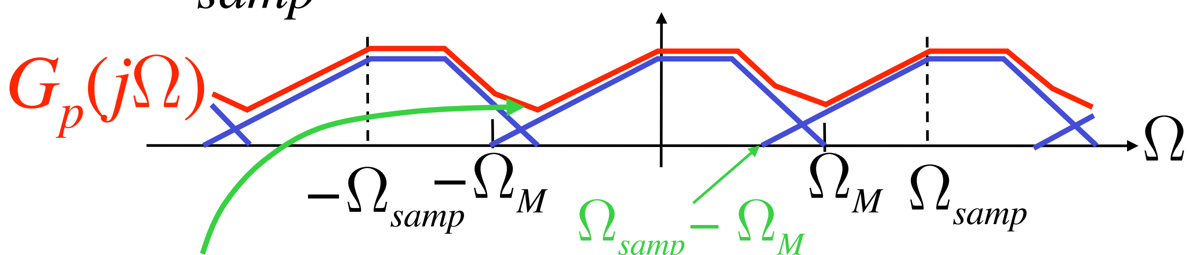
- $g_a(t)$ is bandlimited to ± Ω_M rad/sec
- When sampling frequency Ω_{samp} is large...

→ no overlap between aliases

→ can recover $g_a(t)$ from $g_p(t)$ by low-pass filtering

Aliasing & The Nyquist Limit

- If bandlimit Ω_M is too large, or sampling rate Ω_{samp} is too small, aliases will overlap:



- Spectral effect cannot be filtered out
  → cannot recover $g_a(t)$

Sampling theorum

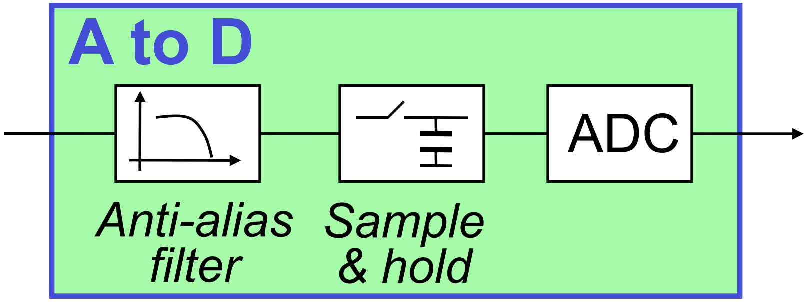
- Avoid by: $\Omega_{samp} - \Omega_M \geq \Omega_M \Rightarrow \boxed{\Omega_{samp} \geq 2\Omega_M}$
- i.e. bandlimit $g_a(t)$ at $\leq \dfrac{\Omega_{samp}}{2}$ Nyquist frequency

Anti-Alias Filter

- To understand speech, need ~ 3.4 kHz
  → 8 kHz sampling rate (i.e. up to 4 kHz)
- Limit of hearing ~20 kHz
  → 44.1 kHz sampling rate for CDs

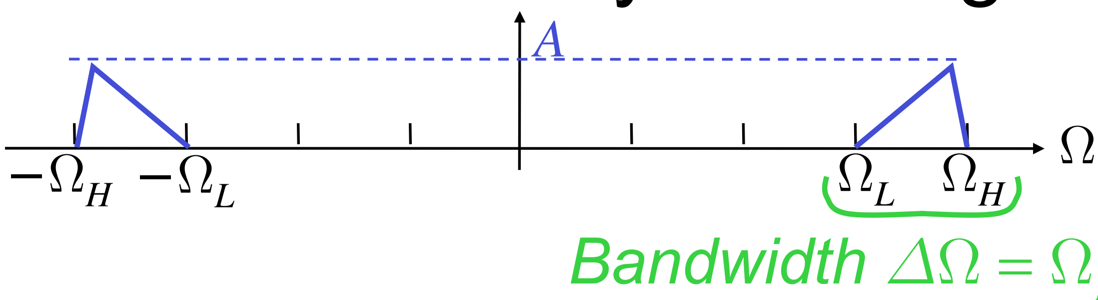
'space' for filter rolloff

- Must remove energy above Nyquist with LPF before sampling: "Anti-alias" filter

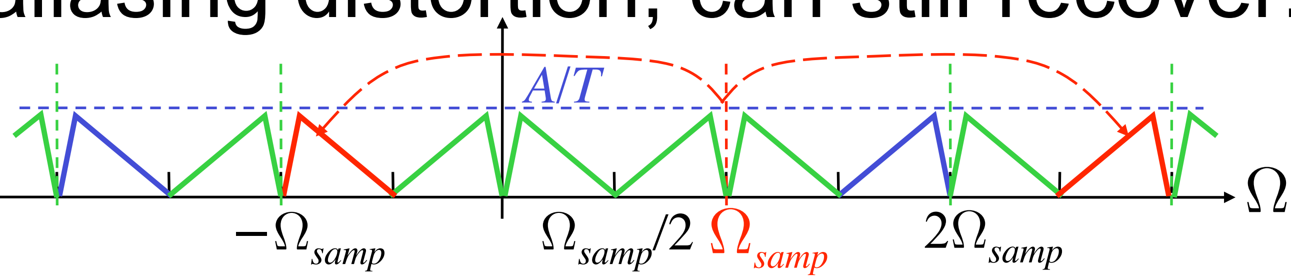


Sampling Bandpass Signals

- Signal is not always in 'baseband' around $\Omega = 0$ ... may be at higher Ω:



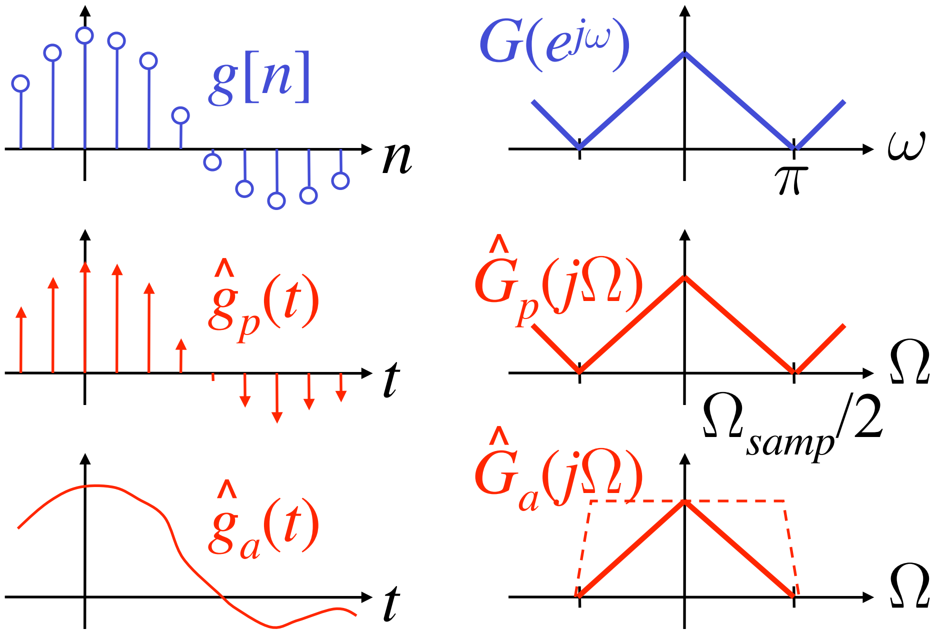
- If aliases from sampling don't overlap, no aliasing distortion; can still recover:



- Basic limit: $\Omega_{samp}/2 \geq$ bandwidth $\Delta\Omega$

Reconstruction

- To turn $g[n]$ back to $\hat{g}_a(t)$:
  - make a continuous impulse train $\hat{g}_p(t)$
  - lowpass filter to extract baseband $\rightarrow \hat{g}_a(t)$



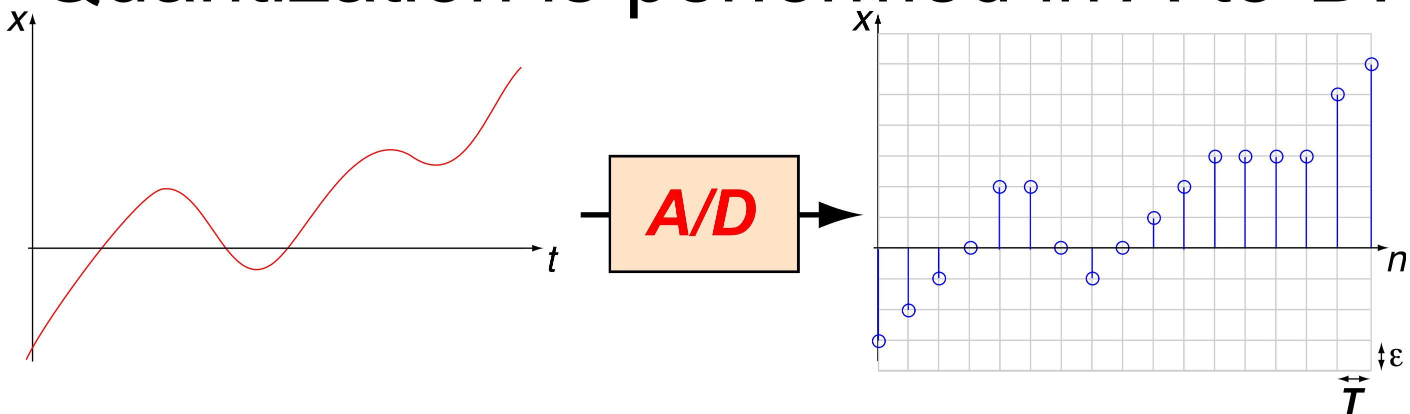
- Ideal reconstruction filter is *brickwall*
  - i.e. sinc - not realizable (especially analog!)
  - use something with finite transition band...

2. Quantization

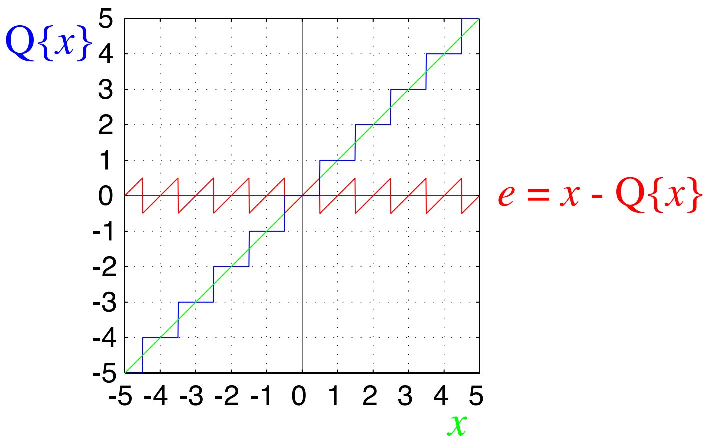
- Course so far has been about discrete-time i.e. quantization of time
- Computer representation of signals also quantizes level (e.g. 16 bit integer word)
- Level quantization introduces an error between ideal & actual signal → noise
- Resolution (# bits) affects data size → quantization critical for compression
  - smallest data ↔ coarsest quantization

Quantization

- Quantization is performed in A-to-D:



- Quantization has simple transfer curve:



Quantized signal

$$\hat{x} = \mathrm{Q}\{x\}$$

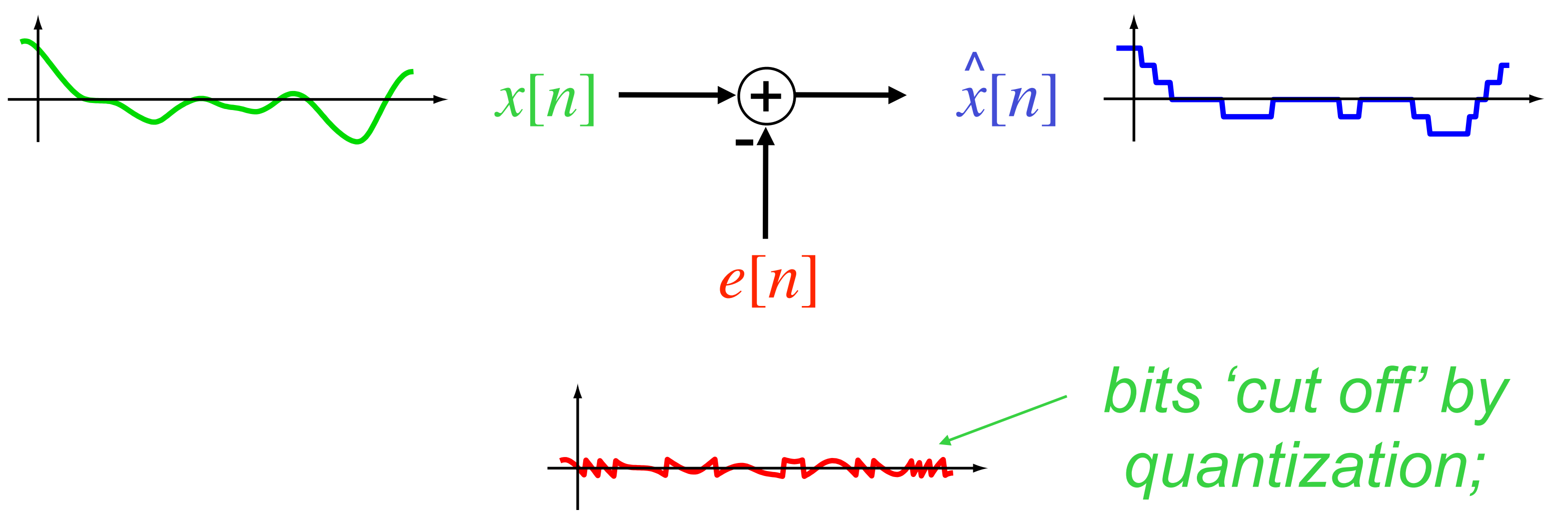
Quantization error

$$e = x - \hat{x}$$

Quantization noise

- Can usually model quantization as additive white noise:

i.e. uncorrelated with self or signal x



bits 'cut off' by quantization; hard amplitude limit

Quantization SNR

- Common measure of noise is Signal-to-Noise ratio (SNR) in dB:

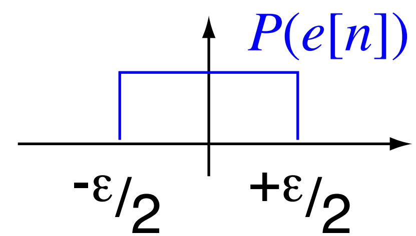
$$SNR = 10 \cdot \log_{10} \frac{\sigma_x^2}{\sigma_e^2} \text{ dB}$$

signal power

noise power

- When $|x|$ >> 1 LSB, quantization noise has ~ uniform distribution:

$$\Rightarrow \sigma_e^2 = \frac{\varepsilon^2}{12}$$



(quantizer step = ε)

Quantization SNR

- Now, $\sigma_x{}^2$ is limited by dynamic range of converter (to avoid clipping)
- e.g. b+1 bit resolution (including sign) output levels vary $-2^b\cdot\varepsilon$ .. $(2^b-1)\varepsilon$

$$= \frac{-R_{FS}}{2} \ldots \frac{R_{FS}}{2} - \varepsilon$$ where full-scale range $R_{FS} = 2^{b+1}\cdot\varepsilon$

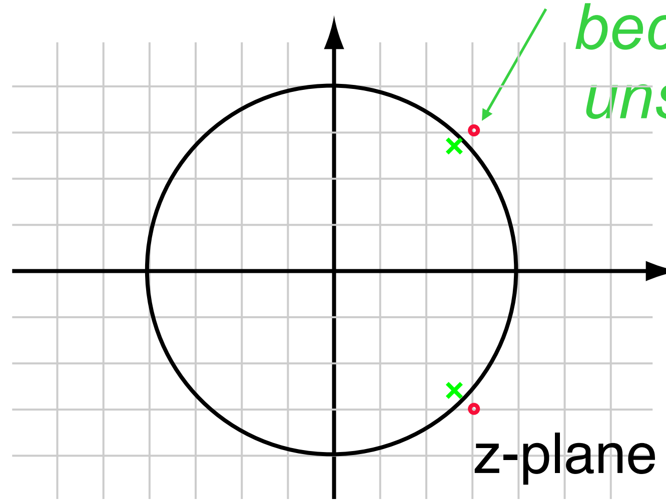
$$\Rightarrow SNR = 10\log_{10}\frac{\sigma_x^2}{\left[\frac{R_{FS}^2}{2^{2(b+1)}\cdot 12}\right]} \approx 6b + 16.8 - 20\log_{10}\frac{R_{FS}}{\sigma_x}$$

i.e. ~ 6 dB SNR per bit

depends on signal

Coefficient Quantization

- Quantization affects not just signal but filter constants too
  - .. depending on implementation
  - .. may have different resolution
- Some coefficients are very sensitive to small changes
  - e.g. poles near unit circle



12/9 Project Presentations

10:15 Arthur Argall: **Genomic Signal Processing**

10:25 Yue Hou, Dongxue Liu:
Blind Signal Separation

10:35 Nathan Lin:
Spectral Domain Phase Microscopy

10:45 Minda Yang, Yitong Li:
Reconstruction from Neural Measurements

10:55 Yinan Wu: **Mixed Channel Music**

11:05 Maja Rudolph, Preston Conley:
Walking Pace Extraction from Video

11:15 Alan Zambeli-Ljepović, Tanya Shah, Sophie Wang:
Lung Sounds Analysis