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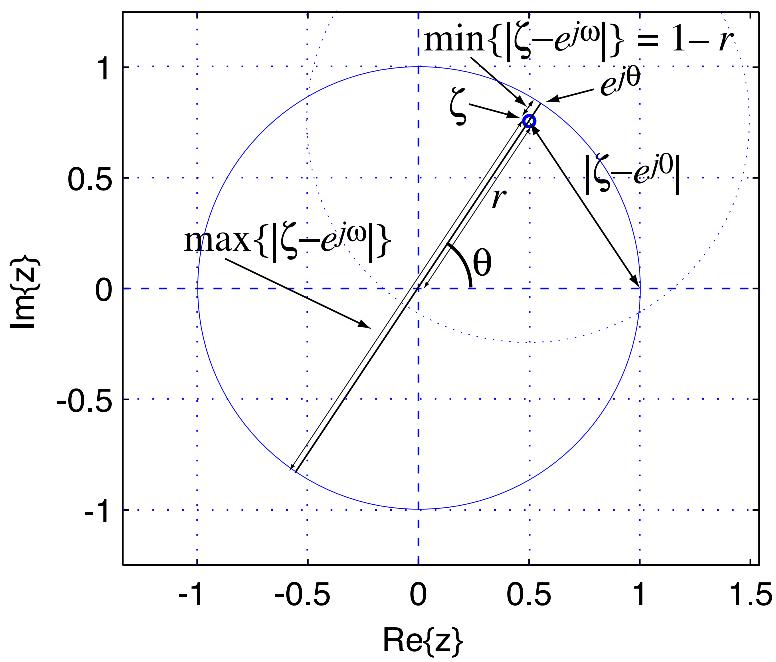
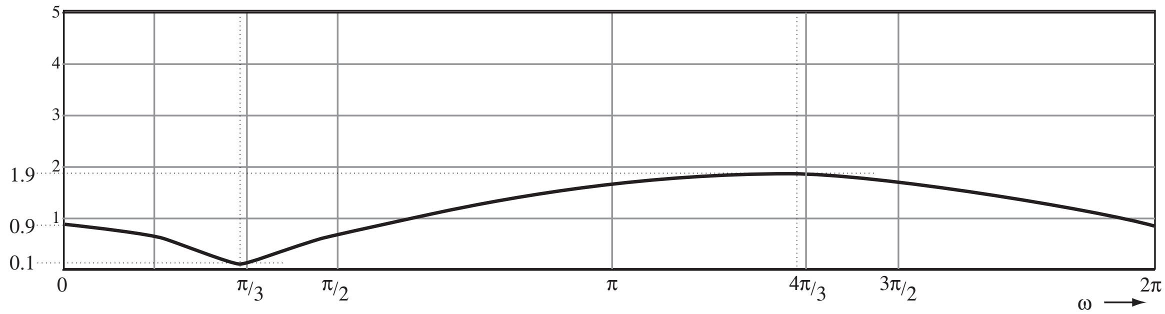
1. (a) We'll first figure out how to sketch the magnitude response of one arbitrary zero, then we'll combine pairs of zeros, and then reciprocate to get the pole responses.

A single, generic zero at $z = \zeta = re^{j\theta}$ has a magnitude response $|H(e^{j\omega})|$ that is proportional to the length of the vector from the root to the corresponding point on the unit circle: $|H(e^{j\omega})| = |\zeta - e^{j\omega}|$. By definition, this has a minimum at the point on the unit circle closest to the root, and a maximum at the point diametrically opposite (which will be a difference of π radians in ω). From the sketch below, it is geometrically obvious that the minimum occurs at $\omega = \theta$, and corresponds to distance of $1 - r$. Similarly, the maximum occurs at $\omega = \theta + \pi$ and gives a distance of $1+r$. Also marked on the sketch is $|\zeta - e^{j0}|$ (i.e., $|\zeta - 1|$), which gives the magnitude of this zero at d.c. ($\omega = 0$). We'll need to know this to achieve the overall gain normalization specified in the question, that all systems have $H(e^{j0}) = 1$.

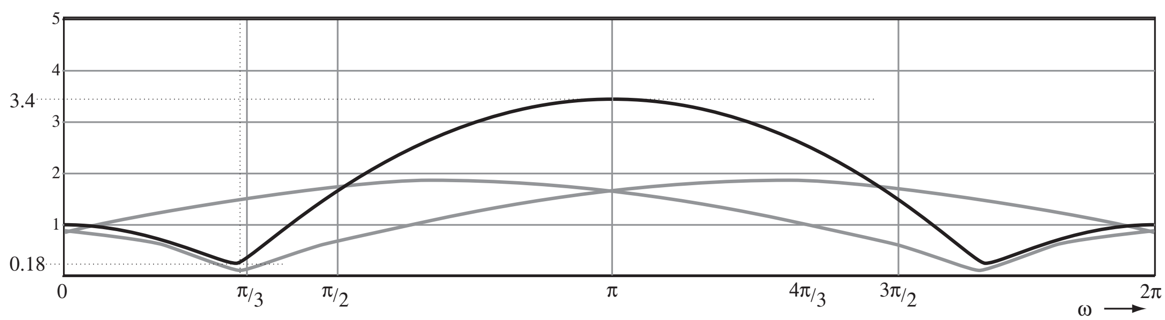


To get a few more points for our sketch, we can note that the (unnormalized) zero will have a gain of 1 where the $e^{j\omega}$ unit circle intersects a unit-radius circle centered on ζ (shown thin and dotted in the sketch). Finally, we can tell that the minimum is going to be fairly narrow: again, by geometry, the minimum distance has grown by a factor of $\sqrt{2}$ approximately $1 - r$ radians either side of the minimum (i.e., making a right-angle triangle bisected by the closest approach). Now we can sketch the overall unnormalized magnitude response as a function of ω. We should do this for a full revolution around the unit circle, to make sure we include both minimum and maximum. For the sketch, $\zeta \approx 0.5 + 0.75j$, so $r = \sqrt{(1/2)^2 + (3/4)^2} \approx 0.9$, and θ is a little less than $\pi/3$, so the minimum is $1 - 0.9 = 0.1$, the maximum is 1.9, and the d.c. value is equal to r (by symmetry about $\Re\{z\} = 0.5$), i.e. also 0.9. The gain passes through 1 at $\omega \approx 2\pi/3$ and

$-\pi/16$. Note that because $|H(e^{j\omega})|$ is periodic in ω, the values (and slopes) at $\omega = 0$ and 2π must match.



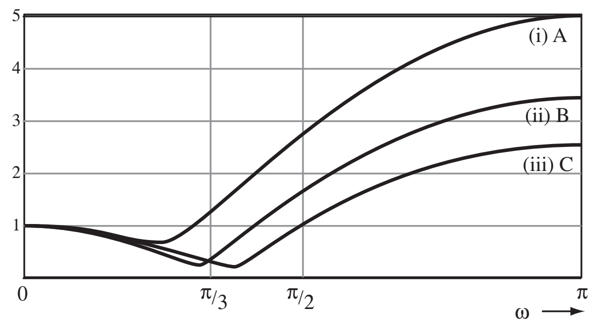
The conjugate zero will have the same effect, only reflected around $\omega = 0$ (or equivalently $\omega = 2\pi$). The net result of the conjugate pair will then be the product of these two; its minimum will occur at a frequency just slightly lower than for the single zero (due to the effect of the slope of the conjugate zero at that point), and will be around 0.1×1.5 (estimating the value of the sketch above around $\omega = 2\pi - \pi/3 = 5\pi/3$), i.e., 0.15 (before gain normalization). By symmetry, there will be a maximum at $\omega = \pi$, that will, by considering the right-angle triangle, be $(\sqrt{(3/4)^2 + (3/2)^2})^2 = 45/16 \approx 2.8$. The gain at $\omega = 0$ will be $r^2 = 13/16 \approx 0.8$. To achieve the specified d.c. gain of 1, the entire curve must be divided by this value (i.e., scaled up by $16/13 \approx 1.23$), so the complete sketch of the answer to part (ii) will have minima of around $0.15 \times 1.23 \approx 0.18$ just below $\omega = \pi/3$ (and another just above $\omega = 5\pi/3$, but that's outside the range we're asked to plot), a maximum at $\omega = \pi$ of around $2.8 \times 1.23 \approx 3.4$, and a local maximum of 1 at $\omega = 0$. The sketch below shows the two unnormalized single-zero responses, along with their product normalized to be 1 at $\omega = 0$. I plotted the entire range $\omega = 0 \ldots 2\pi$ to emphasize the symmetry of the second zero's response.



The responses for zeros A and C are similar, but with different values of r and θ, and consequently different maxima and minima. For A, $\zeta = 0.5 + 0.5j$, $r \approx 0.71$, $\theta = \pi/4$, so the single-zero's minimum is around 0.29, its maximum is around 1.71, and its conjugate's value at the minimum is around 1.2 (so the unnormalized value at the minimum will be around $1.2 \times 0.29 \approx 0.35$). The single zero gives a d.c. magnitude of 0.71 and a magnitude at $\omega = \pi/2$ of $\sqrt{(3/2)^2 + (1/2)^2} \approx 1.6$, so the unnormalized maxima of the product of the two responses is 0.5 at $\omega = 0$ and 2.5 at $\omega = \pi$. Thus, the normalized minima are around $0.35/0.5 = 0.7$ and the maximum at $2.5/0.5 = 5$.

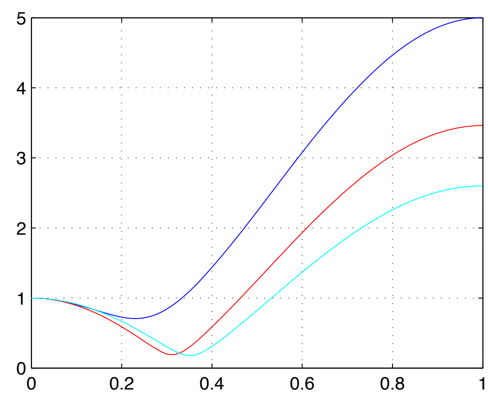
For C, $\zeta = 0.5 + 1j$, $r \approx 1.12$, θ is a little larger than $\pi/3$, so the unnormalized combined response has minimum around $0.12 \times 1.9 \approx 2.1$ and is 1.25 at $\omega = 0$ and $13/4 = 3.25$

at $\omega = \pi$. Thus, the gain-normalized response has minima of about $2.1/1.25 \approx 1.7$ and a maximum around $3.25/1.25 = 2.6$. Thus, the sketches for all three of the zero pairs (over the specified interval $\omega = 0 \ldots \pi$) are shown below:

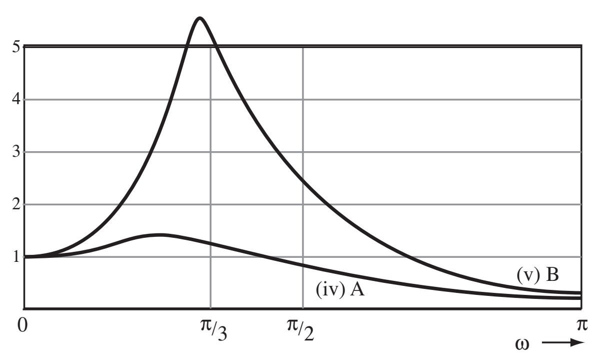


For confirmation, here are the actual responses plotted by Matlab:

```
>> zA = 0.5 + 0.5j;
>> zB = 0.5 + 0.75j;
>> zC = 0.5 + 1j;
>> GzA = 1/abs((zA-1).^2);
>> GzB = 1/abs((zB-1).^2);
>> GzC = 1/abs((zC-1).^2);
>> w = 0:0.01:pi;
>> ejw = exp(w*1j);
>> plot(w/pi, GzA*abs((ejw-zA).*(ejw-zA')))
>> grid
>> hold on
>> plot(w/pi, GzB*abs((ejw-zB).*(ejw-zB')),'-r')
>> plot(w/pi, GzC*abs((ejw-zC).*(ejw-zC')),'-c')
>> hold off
```

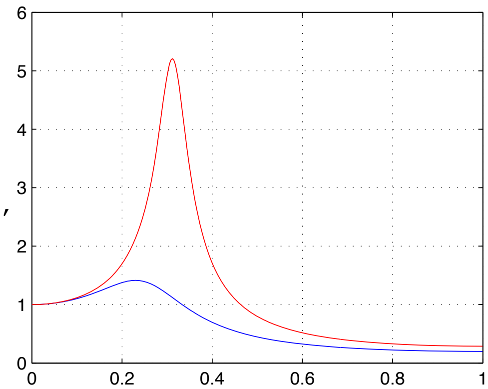


For the poles, the responses will be simply the reciprocals of the zeros, except of course for C, which is a system with poles outside the unit circle, meaning that the unit circle is not in the region of convergence, and the Fourier transform is not finite (an unstable system). But for A and B, the maxima in the sketch above become minima at $1/5 = 0.2$ and $1/3.6 \approx 0.28$ respectively, and (relatively sharp) maxima at $1/0.7 \approx 1.4$ and $1/0.18 \approx 5.5$ respectively:



And in Matlab (following on from the code above):

```
>> w = omega;
>> plot(w/pi, 1./(GzA*abs((ejw-zA).*(ejw-zA'))))
>> hold on
>> plot(w/pi, 1./(GzB*abs((ejw-zB).*(ejw-zB'))),'-r'
>> grid
```



Grading: To get full marks, you had to show some actual calculation for the values of the frequencies and magnitudes at the maxima and minima. Getting the precise "shape" of the curves was not critical.

(b) The energy of the impulse response $h[n]$ is $\sum_n |h[n]|^2$. The two-zero systems have simple three-point impulse responses, but the two-pole systems have infinite impulse responses (decaying sinusoids). However, by Parseval's relation, the energy also equals $\frac{1}{2\pi}\int_{\pi}^{\pi} |H(e^{j\omega})|^2 d\omega$. We can "eyeball" our sketches of the magnitude responses to see that (iv) (poles at A) clearly has the smallest squared integral (i.e., the average value of the curve after we square it) – it never even exceeds 1.5. Although the magnitudes of two of the systems with zeros do have minima below the minimum of this curve, their broad and large maxima give them a much larger integrated energy. Matlab confirms (doing a numerical approximation to the integral, which is just the average level of the squared curves):

```
>> mean((GzA*abs((ejw-zA).*(ejw-zA'))).^2)
ans =     9.0046
>> mean((GzB*abs((ejw-zB).*(ejw-zB'))).^2)
ans =     4.0334
>> mean((GzC*abs((ejw-zC).*(ejw-zC'))).^2)
ans =     2.2828
>> mean((1./(GzA*abs((ejw-zA).*(ejw-zA')))).^2)
ans =     0.6000
>> mean((1./(GzB*abs((ejw-zB).*(ejw-zB')))).^2)
ans =     2.7868
```

Thus, we need to find the impulse response of the system with the conjugate poles at A. However, we haven't been asked for the closed-form response, but only the first five values; it's probably easier simply to evaluate these by "running the machine" for a few steps. The z-transform of this system is $H(z) = \frac{G}{(1-\zeta z^{-1})(1-\zeta^* z^{-1})}$ where $\zeta = 0.5 + 0.5j = re^{j\theta}$ with $r = \sqrt{2}/2$ and $\theta = \pi/4$ (so $\cos\theta = \sqrt{2}/2$ too). Expanding the denominator to be $1 - 2r\cos(\theta)z^{-1} + r^2 z^{-2}$ gives $H(z) = \frac{G}{1-z^{-1}+0.5z^{-2}}$; now it is clear that $G = 0.5$ to ensure $H(1) = 1$. Thus, our difference equation is $y[n] = 0.5x[n] + y[n-1] - 0.5y[n-2]$; working this forward from zero initial state and with $x[n] = \delta[n]$ gives $y[n] = \{0.5, 0.5, 0.25, 0, -0.125\}$ for $n = 0 \ldots 4$. Note that this sequence is decaying exponentially, and the total energy of these first four points is about 0.58, so it's plausible that it's going to asymptote at 0.6, Matlab's numerical estimate of the total energy from above.

Grading: You got roughly half the points for figuring out the system with the lowest energy IR, and the rest for finding the IR. You got some points for a correct approach to finding the impulse response, even if it didn't give you the right answer, or if you applied it to the wrong system. Of course, you were welcome to find the IR by solving the LCCDE, but that would probably have taken longer than the approach above.

2. As it happens, this is just a type-IV even-length, anti-symmetric FIR filter as we covered in class on Tuesday, but the question was posed without assuming you'd be familiar with this.

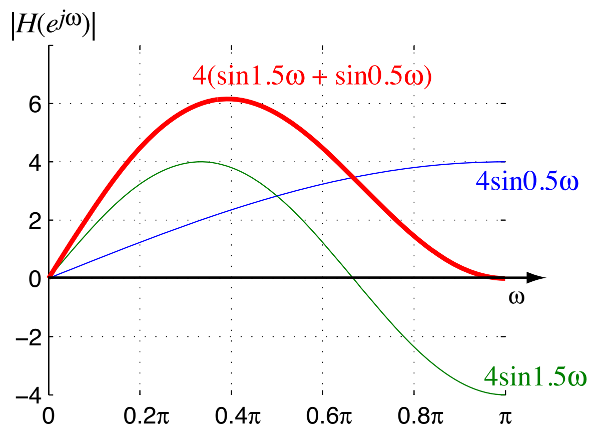
   (a) We have $h[n] = \{\underline{2}, 2, -2, -2\}$, so $H(z) = 2(1+z^{-1}-z^{-2}-z^{-3})$ and $H(e^{j\omega})$ can be found in the usual way by dividing out $e^{j(-1.5)\omega}$ and combining the resulting conjugate pairs of complex exponentials into sins:

$$\begin{aligned}H(e^{j\omega}) =& 2\left(1+e^{-j\omega}-e^{-2j\omega}-e^{-3j\omega}\right)\\ =& 2e^{-1.5j\omega}\left(e^{1.5j\omega}-e^{-1.5j\omega}+e^{0.5j\omega}-e^{-0.5j\omega}\right)\\ =& 2e^{-1.5j\omega}\left(2j\sin 1.5\omega+2j\sin 0.5\omega\right)\\ =& 4je^{-1.5j\omega}\left(\sin 1.5\omega+\sin 0.5\omega\right)\end{aligned}$$

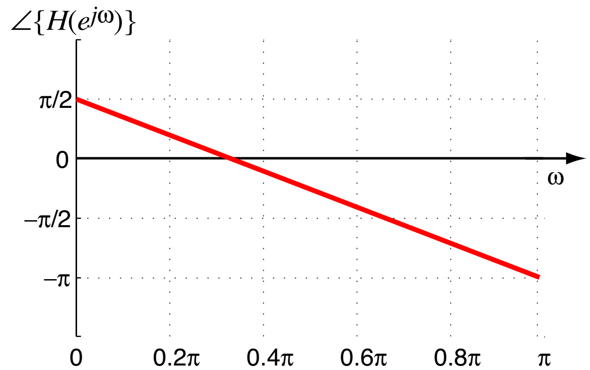
   The 4-point DFT is simply $H(e^{j\omega})$ evaluated at $\omega = 2\pi k/4$ for $k = 0, 1, 2, 3$ which gives $H[k] = \{0, x, 0, x^*\}$, where

$$\begin{aligned}x &= 4je^{-(3/2)\pi/2}(\sin 3\pi/4+\sin\pi/4)\\ &= 4e^{j\pi/2}e^{-j3\pi/4}(\sqrt{2}/2+\sqrt{2}/2)\\ &= 4e^{-j\pi/4}\sqrt{2}\\ &= 4-4j\end{aligned}$$

   (b) Since $H(e^{j\omega}) = 4je^{-1.5j\omega}\left(\sin 1.5\omega+\sin 0.5\omega\right)$, the magnitude response is given by $4(\sin 1.5\omega+\sin 0.5\omega)$, which never quite goes negative in the interval $\omega = 0\ldots\pi$:



The phase response is then simply the phase of $je^{-1.5j\omega}$, which is just $\pi/2 - 1.5\omega$:



(c) We notice that $x[n]$ is a delayed version of $h[n]$ with a pure sinusoid added. We're given no information about gating or sidedness, so we can assume the sinusoid is of infinite extent. At $\omega = \pi$, $H(e^{j\omega}) = 0$, so this frequency is entirely removed from the output. All we're left with is the convolution of h with a delayed version of itself, $y[n] = h[n] \circledast h[n - n_0]$, or equivalently $h[n] \circledast h[n]$ delayed by n_0 samples. For a finite-length sequence this short, the easiest thing is probably to figure out all 7 nonzero points of the convolution one by one, as $\sum_m h[m]h[n-m]$, giving $h[n] \circledast h[n] = \{4, (4+4) = 8, (-4+4-4) = -4, (-4-4-4-4) = -16, -4, 8, 4\}$ (exploiting the observation that the result must be symmetric around its midpoint since it is a negated autocorrelation, $h[n] = -h[4-n]$). Thus, starting from $n = 0$ (and assuming $n_0 > 0$), $y[n] = \{[n_0 \text{ zeros}], 4, 8, -4, -16, -4, 8, 4, 0, 0, 0, \ldots\}$.

(d) Because $y[n]$ is simply $h[n]$ convolved with itself and delayed by n_0 samples, $Y(e^{j\omega})$ is simply $e^{-jn_0}H(e^{j\omega})^2$, and thus $|Y[k]|$ is just $|H[k]|^2$, even though $y[n]$ itself is too long to fit into a 4 point sequence, and would have to be time aliased to directly calculate its DFT. Thus, $|Y[k]| = \{0, 32, 0, 32\}$.

Grading: All subsections received equal proportions of the points; the magnitude sketch just had to be based on the correct sinusoids, it didn't matter if you got the shape or location of the bump quite right. The key parts for (c) were to eliminate the sinusoid and try to describe a delayed self-convolution; mistakes in actually calculating the convolution lost minimal points.