ELEN E4810: Digital Signal Processing

Topic 9: Filter Design: FIR

1. Windowed Impulse Response
2. Window Shapes
3. Design by Iterative Optimization

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1. FIR Filter Design

- FIR filters
  - no poles (just zeros)
  - no precedent in analog filter design
- Approaches
  - windowing ideal impulse response
  - iterative (computer-aided) design

Least Integral-Squared Error

- Given desired FR $H_d(e^{j\omega})$, what is the **best** finite $h_t[n]$ to approximate it?

best in what sense?

- Can try to minimize Integral Squared Error (ISE) of frequency responses:

$$\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_d\left(e^{j\omega}\right) - H_t\left(e^{j\omega}\right)\right|^2 d\omega$$

$= \mathrm{DTFT}\{h_t[n]\}$

Least Integral-Squared Error

- Ideal IR is $h_d[n] = \mathrm{IDTFT}\{H_d(e^{j\omega})\}$, (usually infinite-extent)
- By Parseval, ISE $\phi = \sum_{n=-\infty}^{\infty}\left|h_d[n] - h_t[n]\right|^2$
- But: $h_t[n]$ only exists for $n = -M..M$,

$$\Rightarrow \phi = \sum_{n=-M}^{M}\underline{\left|h_d[n] - h_t[n]\right|^2} + \sum_{n<M\,,\,n>M}\left|h_d[n]\right|^2$$

minimized by making
$h_t[n] = h_d[n]$, $-M \le n \le M$

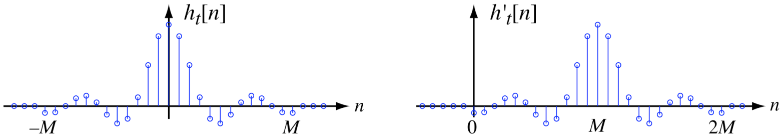
not altered by $h_t[n]$

Least Integral-Squared Error

- Thus, minimum mean-squared error approximation in $2M+1$ point FIR is truncated IDTFT:

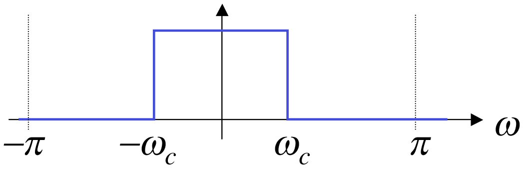
$$h_t[n] = \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right)e^{j\omega n}\,d\omega & -M \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

- Make causal by delaying by M points
$\rightarrow h'_t[n] = 0$ for $n < 0$



Approximating Ideal Filters

- From topic 6, ideal lowpass has:

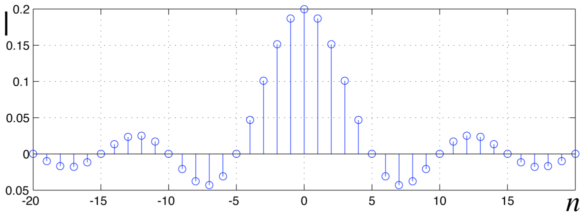


$$H_{LP}\left(e^{j\omega}\right) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

and:

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}$$

(doubly infinite)

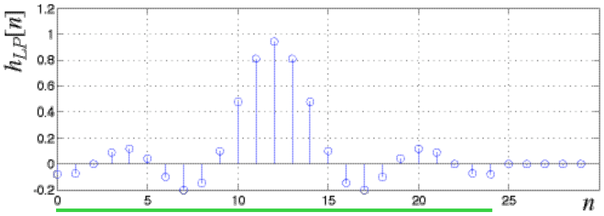


Approximating Ideal Filters

- Thus, minimum ISE causal approximation to an ideal lowpass

$$\hat{h}_{LP}[n] = \begin{cases} \dfrac{\sin\omega_c(n-M)}{\pi(n-M)} & 0 \le n \le 2M \\ 0 & \text{otherwise} \end{cases}$$

Causal shift



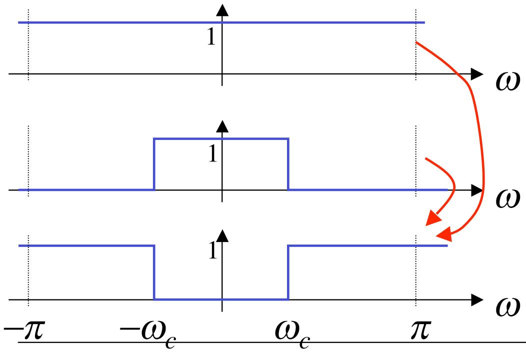
2M+1 points

Freq. Resp. (FR) Arithmetic

- Ideal LPF has pure-real FR i.e.
  $\theta(\omega) = 0,\ H(e^{j\omega}) = |H(e^{j\omega})|$

→ Can build piecewise-constant FRs by combining ideal responses, e.g. HPF:

wouldn't work if phases were nonzero!



$\delta[n]$ i.e. $H(e^{j\omega}) = 1$

$-$

$h_{LP}[n]$ $H_{LP}(e^{j\omega}) = 1$ for $|\omega| < \omega_c$

$=$

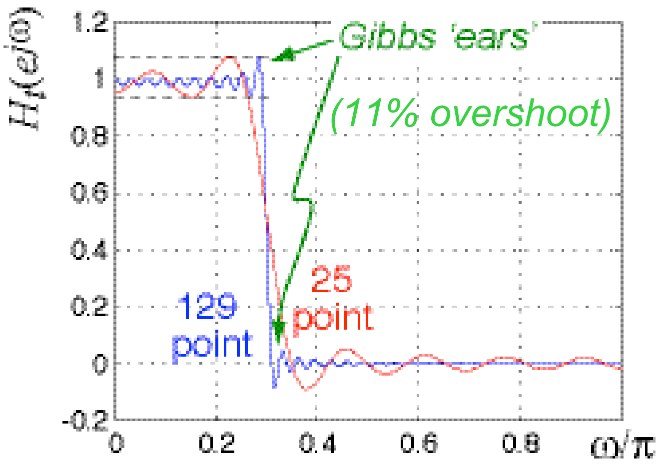
$h_{HP}[n] = \delta[n] - (\sin\omega_c n)/\pi n$

Gibbs Phenomenon

- Truncated ideal filters have *Gibbs' Ears*:

Increasing filter length
→ narrower ears
(reduces ISE)
***but** height the same*

*→ **not** optimal by*
minimax criterion

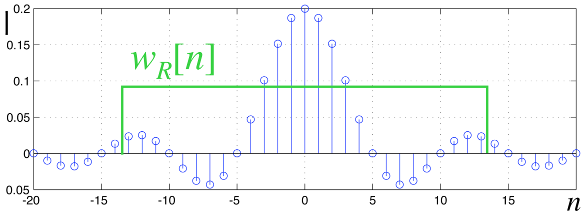


Where Gibbs comes from

- Truncation of $h_d[n]$ to $2M+1$ points is multiplication by a rectangular window:

$$h_t[n] = h_d[n]\cdot w_R[n]$$

$$w_R[n] = \begin{cases}1 & -M \le n \le M\\ 0 & \text{otherwise}\end{cases}$$

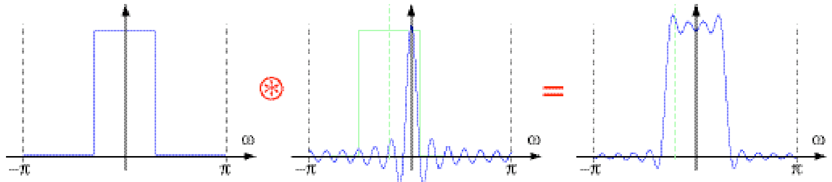


- Multiplication in time domain is convolution in frequency domain:

$$g[n]\cdot h[n] \quad \leftrightarrow \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$$

Where Gibbs comes from

- Thus, FR of truncated response is convolution of ideal FR and FR of rectangual window (pdc.sinc):



$H_d(e^{j\omega})$

$\mathrm{DTFT}\{w_R[n]\}$

periodic sinc...

$$\frac{\sin Nx}{\sin x}$$

$H_t(e^{j\omega})$

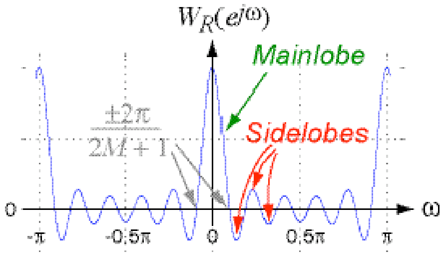
Where Gibbs comes from

- Rectangular window:

$$w_R[n] = \begin{cases} 1 & -M \le n \le M \\ 0 & \text{otherwise} \end{cases} \Rightarrow \quad W_R\left(e^{j\omega}\right) = \sum_{n=-M}^{M} e^{-j\omega n} = \frac{\sin\left(\left[2M+1\right]\frac{\omega}{2}\right)}{\sin\frac{\omega}{2}}$$

- Mainlobe width ($\propto 1/L$) determines transition band
- Sidelobe height determines ripples

doesn't vary with length



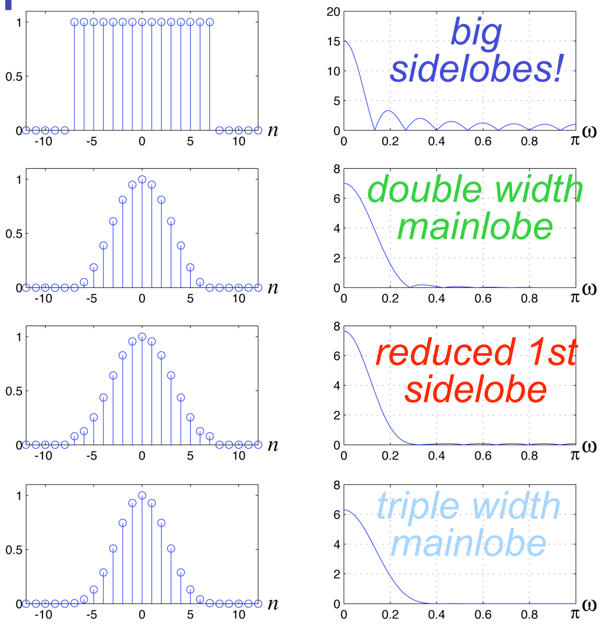
"periodic sinc"

2. Window Shapes for Filters

- Windowing (infinite) ideal response
  → FIR filter: $h_t[n] = h_d[n] \cdot w[n]$
- Rectangular window has best ISE error
- Other “tapered windows” vary in:
  - mainlobe → transition band width
  - sidelobes → size of ripples near transition
- Variety of ‘classic’ windows...

Window Shapes for FIR Filters

- Rectangular:
  $w[n] = 1 \quad -M \le n \le M$
- Hann:
  $0.5 + 0.5\cos(2\pi \frac{n}{2M+1})$
- Hamming:
  $0.54 + 0.46\cos(2\pi \frac{n}{2M+1})$
- Blackman:
  $0.42 + 0.46\cos(2\pi \frac{n}{2M+1})$
  $+0.08\cos(2\pi \frac{2n}{2M+1})$



big sidelobes!

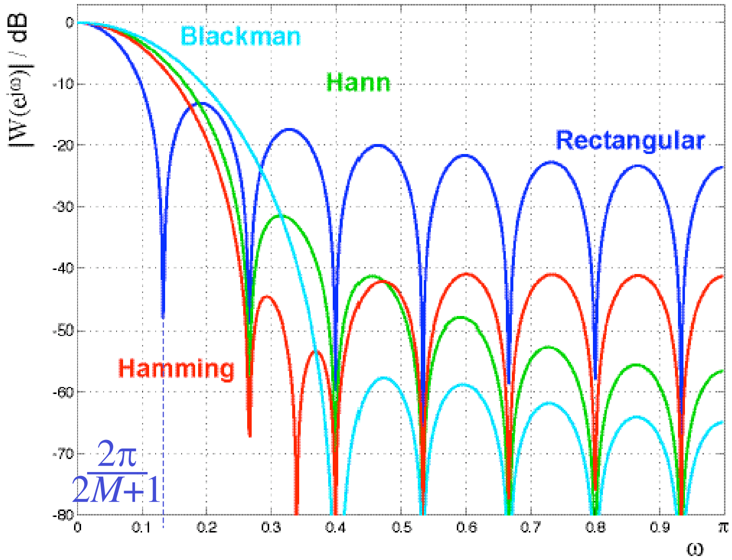
double width mainlobe

reduced 1st sidelobe

triple width mainlobe

Window Shapes for FIR Filters

- Comparison on dB scale:



Adjustable Windows

- Have discrete main-sidelobe tradeoffs...
- Kaiser window = parametric, continuous tradeoff:

modified zero-order Bessel function

$$w[n] = \frac{I_0\left(\beta\sqrt{1-(\tfrac{n}{M})^2}\right)}{I_0(\beta)} \qquad -M \le n \le M$$

- Empirically, for min. SB atten. of α dB:

$$\beta = \begin{cases} 0.11(\alpha - 8.7) & 50 < \alpha \\ 0.58(\alpha - 21)^{0.4} + 0.08(\alpha - 21) & 21 \le \alpha \le 50 \\ 0 & \alpha < 21 \end{cases}$$

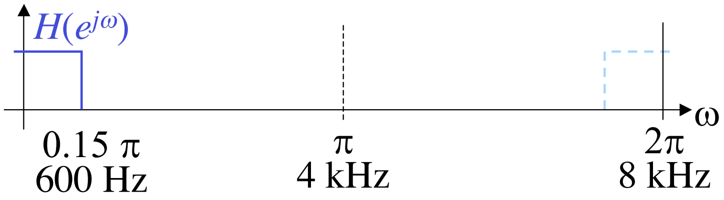
required order

$$N = \frac{\alpha - 8}{2.3\Delta\omega}$$

transition width

Windowed Filter Example

- Design a 25 point FIR low-pass filter with a cutoff of 600 Hz (SR = 8 kHz)
- No specific transition/ripple req's
  → compromise: use Hamming window
- Convert the frequency to radians/sample: $\omega_c = \frac{600}{8000} \times 2\pi = 0.15\pi$



Windowed Filter Example

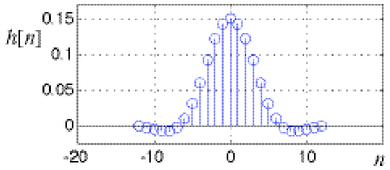
1. Get ideal filter impulse response:

$$\omega_c = 0.15\pi \quad \Rightarrow h_d[n] = \frac{\sin 0.15\pi n}{\pi n}$$

2. Get window:
   Hamming @ $N = 25 \rightarrow M = 12$ $(N = 2M+1)$

$$\Rightarrow w[n] = 0.54 + 0.46\cos\left(2\pi \tfrac{n}{25}\right) \quad -12 \le n \le 12$$

3. Apply window:

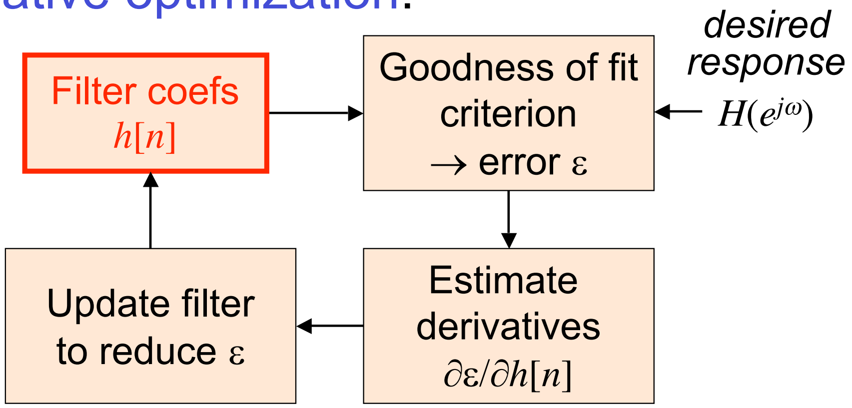


$$h[n] = h_d[n] \cdot w[n]$$

$$= \frac{\sin 0.15\pi n}{\pi n}\left(0.54 + 0.46\cos\tfrac{2\pi n}{25}\right) \quad -12 \le n \le 12$$

3. Iterative FIR Filter Design

- Can derive filter coefficients by iterative optimization:



Filter coefs $h[n]$

Goodness of fit criterion → error ε

desired response $H(e^{j\omega})$

Estimate derivatives $\partial\varepsilon/\partial h[n]$

Update filter to reduce ε

- Gradient descent / nonlinear optimiz’n

Error Criteria

$$\varepsilon = \int_{\omega\in R} \left| W(\omega)\cdot\left[D\left(e^{j\omega}\right) - H\left(e^{j\omega}\right)\right]\right|^{p} d\omega$$

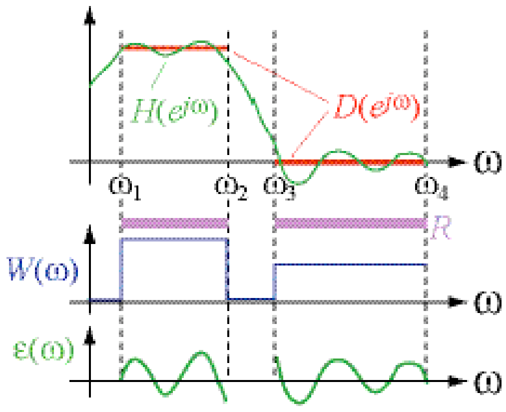
error measurement region

error weighting

desired response

actual response

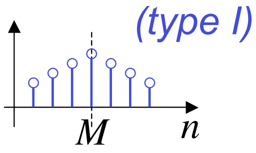
exponent:
2 → least sq’s
∞ → minimax



$= W(\omega)\cdot[D(e^{j\omega}) - H(e^{j\omega})]$

Minimax FIR Filters

- Iterative design of FIR filters with:
  - equiripple (minimax criterion)
  - linear-phase
    → *symmetric IR* $h[n] = (-)h[-n]$



- Recall, symmetric FIR filters have FR
  $H\left(e^{j\omega}\right) = e^{-j\omega M}\tilde{H}(\omega)$ with pure-real

$$\tilde{H}(\omega) = \sum_{k=0}^{M} a[k]\cos(k\omega)$$

$a[0] = h[M]$
$a[k] = 2h[M - k]$

i.e. combo of cosines of *multiples of* ω

Minimax FIR Filters

- Now, $\cos(k\omega)$ can be expressed as a polynomial in $\cos(\omega)^k$ and lower powers
  - e.g. $\cos(2\omega) = 2(\cos\omega)^2 - 1$
- Thus, we can find α's such that

$$\tilde{H}(\omega) = \sum_{k=0}^{M} \alpha[k](\cos\omega)^k$$

M^{th} order polynomial in $\cos\omega$

  - M^{th} order polynomial in $\cos\omega$
  - $\alpha[k]$s are simply related to $a[k]$s

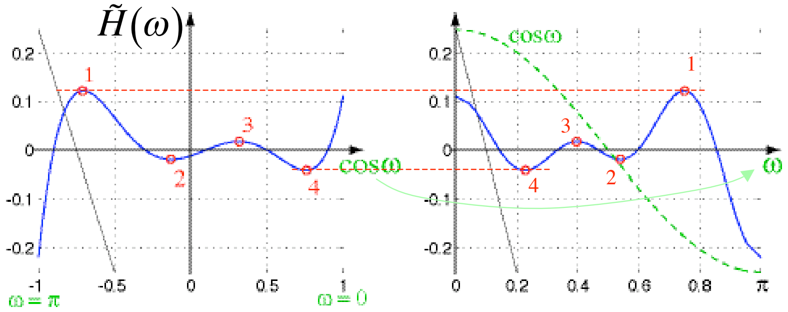
Minimax FIR Filters

$$\tilde{H}(\omega) = \sum_{k=0}^{M} \alpha[k](\cos\omega)^k$$

M^{th} order polynomial in $\cos\omega$

- An M^{th} order polynomial has at most M - 1 maxima and minima:

5th order polynomial in $\cos\omega$



$\Rightarrow \tilde{H}(\omega)$ has at most M-1 min/max (ripples)

Alternation Theorum

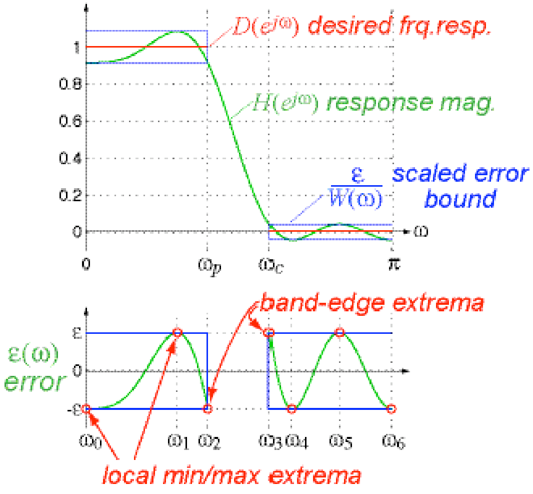
- Key ingredient to Parks-McClellan:

$\tilde{H}(\omega)$ is the unique, best, weighted-minimax order $2M$ approx. to $D(e^{j\omega})$

$\Leftrightarrow$
  - $\tilde{H}(\omega)$ has at least M+2 “extremal” freqs $\omega_0 < \omega_1 < \ldots < \omega_M < \omega_{M+1}$ over ω subset R
  - error magnitude is equal at each extremal: $|\varepsilon(\omega_i)| = \varepsilon \quad \forall i$
  - peak error alternates in sign: $\varepsilon(\omega_i) = -\varepsilon(\omega_{i+1})$

Alternation Theorum

- Hence, for a frequency response:



(10th order filter, M = 5)

- **If** $\varepsilon(\omega)$ reaches a peak error magnitude ε at some set of extremal frequences ω_i
- **And** the *sign* of the peak error alternates
- **And** we have at least M+2 of them
- **Then** **optimal minimax**

Alternation Theorum

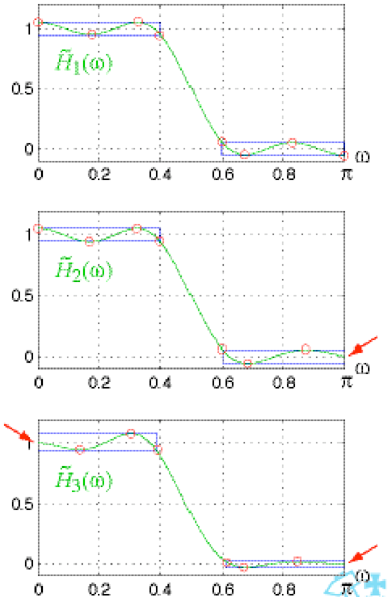
- By Alternation Theorum,
  M+2 extrema of alternating signs
  ⇒ optimal minimax filter
- **But** $\tilde{H}(\omega)$ has at most M-1 extrema

⇒ need at least **3** more from band edges

- 2 bands give **4** band edges
  ⇒ can afford to “miss” only **one**
- Alternation rules out transition band edges, thus have 1 or 2 outer edges

Alternation Theorum

- For $M = 5$ (10th order):
  - 8 extrema ($M+3$, 4 band edges) - **great!**
  - 7 extrema ($M+2$, 3 band edges) - **OK!**
  - 6 extrema ($M+1$, only 2 transition band edges) → **NOT OPTIMAL**



Parks-McClellan Algorithm

- To recap:
  - FIR CAD constraints
    $D(e^{j\omega}), W(\omega) \rightarrow \varepsilon(\omega)$
  - Zero-phase FIR
    $\tilde{H}(\omega) = \Sigma_k \alpha_k \cos^k\omega \rightarrow M\text{-}1$ min/max
  - Alternation theorum
    optimal → $\geq M+2$ pk errs, alter'ng sign
- Hence, can spot 'best' filter when we see it – but how to find it?

Parks-McClellan Algorithm

- Alternation → $[\tilde{H}(\omega)\text{-}\tilde{D}(\omega)]/W(\omega)$ must = $\pm\varepsilon$ at $M+2$ (unknown) frequencies $\{\omega_i\}$...
- Iteratively update $h[n]$ with Remez exchange algorithm:
  - estimate/guess $M+2$ extremals $\{\omega_i\}$
  - solve for $\alpha[n]$, ε ($\rightarrow h[n]$)
  - find actual min/max in $\varepsilon(\omega) \rightarrow$ new $\{\omega_i\}$
  - repeat until $|\varepsilon(\omega_i)|$ is constant
- Converges rapidly!

Parks-McClellan Algorithm

- In Matlab,

filter order (2M)

band edges ÷ π

```
>> h=remez(10, [0 0.4 0.6 1],
               [1 1 0 0],
               [1 2]);
```

desired magnitude at band edges

error weights per band