
ELEN E4810: Digital Signal Processing

Topic 6:

Filters - Introduction

1. Simple Filters
2. Ideal Filters
3. Linear Phase and FIR filter types



1. Simple Filters

- **Filter** = system for altering signal in some 'useful' way
- LSI systems:
 - are characterized by $H(z)$ (or $h[n]$)
 - have different gains (& phase shifts) at different frequencies
 - can be designed systematically for specific filtering tasks



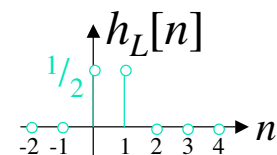
FIR & IIR

- FIR = **finite impulse response**
 - ⇔ no feedback in block diagram
 - ⇔ no poles (only zeros)
- IIR = **infinite impulse response**
 - ⇔ poles (and perhaps zeros)

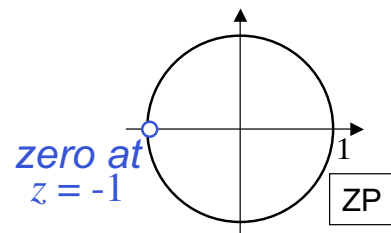


Simple FIR Lowpass

- $h_L[n] = \{\frac{1}{2} \frac{1}{2}\}$
(2 pt moving avg.)

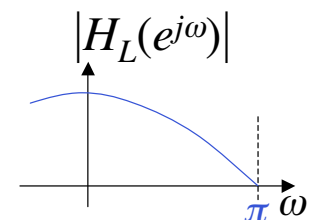


$$H_L(z) = \frac{1}{2} (1 + z^{-1}) = \frac{z+1}{2z}$$



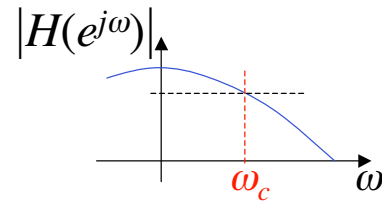
$$\Rightarrow H_L(e^{j\omega}) = e^{-j\omega/2} \cos(\omega/2)$$

Annotations:
- A green arrow points from $e^{-j\omega/2}$ to the text " $1/2$ sample delay".
- A green arrow points from $\cos(\omega/2)$ to the expression $e^{j\omega/2} + e^{-j\omega/2}$.



Simple FIR Lowpass

- Filters are often characterized by their **cutoff frequency ω_c** :



- Cutoff frequency is most often defined as the **half-power point** i.e.

$$\left| H(e^{j\omega_c}) \right|^2 = \frac{1}{2} \max \left\{ \left| H(e^{j\omega}) \right|^2 \right\} \Rightarrow H = \frac{1}{\sqrt{2}} H_{\max}$$

- If $\left| H(e^{j\omega}) \right| = \cos(\omega/2)$

$$\text{then } \omega_c = 2 \cos^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{2}$$



decibels

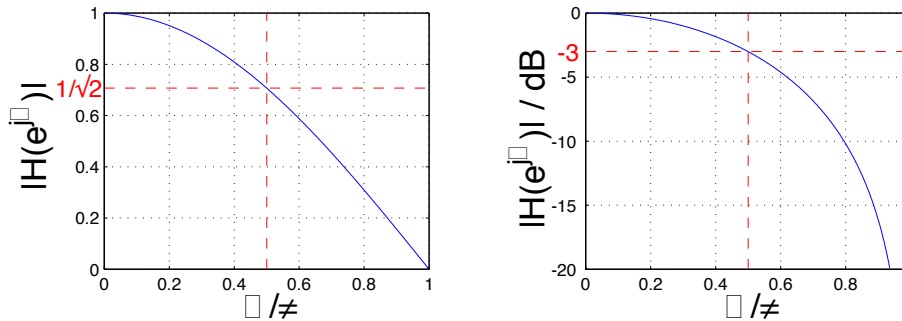
- Filter magnitude responses are very often described in decibels (dB)
- dB is simply a scaled log value:
$$dB = 20 \log_{10}(\text{level}) = 10 \log_{10}(\text{power})$$
 power = level²
- Half-power also known as **3dB point**:

$$\begin{aligned} |H|_{\text{cutoff}} &= \frac{1}{\sqrt{2}} |H|_{\max} \\ dB\{ |H|_{\text{cutoff}} \} &= dB\{ |H|_{\max} \} + 20 \log_{10} \left(\frac{1}{\sqrt{2}} \right) \\ &= dB\{ |H|_{\max} \} - 3.01 \end{aligned}$$



deciBels

- We often plot magnitudes in dB:



- A gain of 0 corresponds to $-\infty$ dB



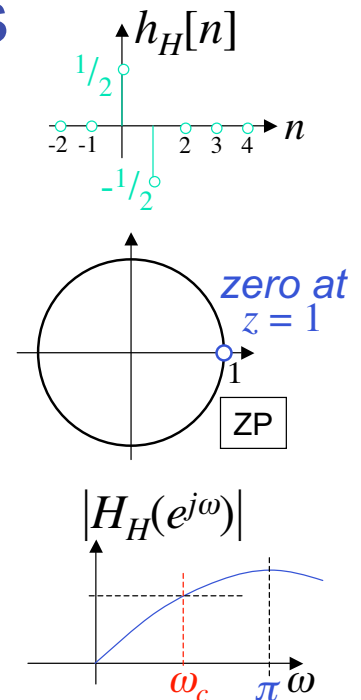
Simple FIR Highpass

- $h_H[n] = \{1/2, -1/2\}$

$$H_H(z) = \frac{1}{2}(1 - z^{-1}) = \frac{z-1}{2z}$$

$$\Rightarrow H_H(e^{j\omega}) = je^{-j\omega/2} \sin(\omega/2)$$

- 3dB point $\omega_c = \pi/2$ (again)



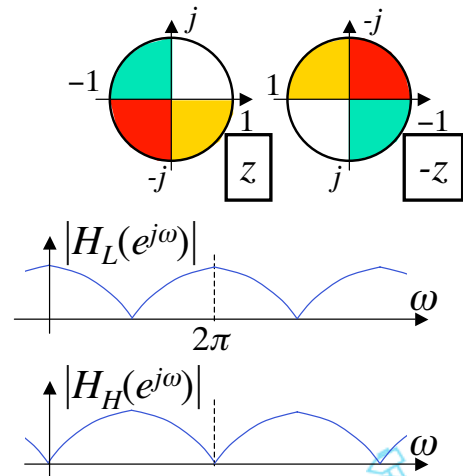
FIR Lowpass and Highpass

- Note:

$$h_L[n] = \{1/2 \ 1/2\} \quad h_H[n] = \{1/2 \ -1/2\}$$

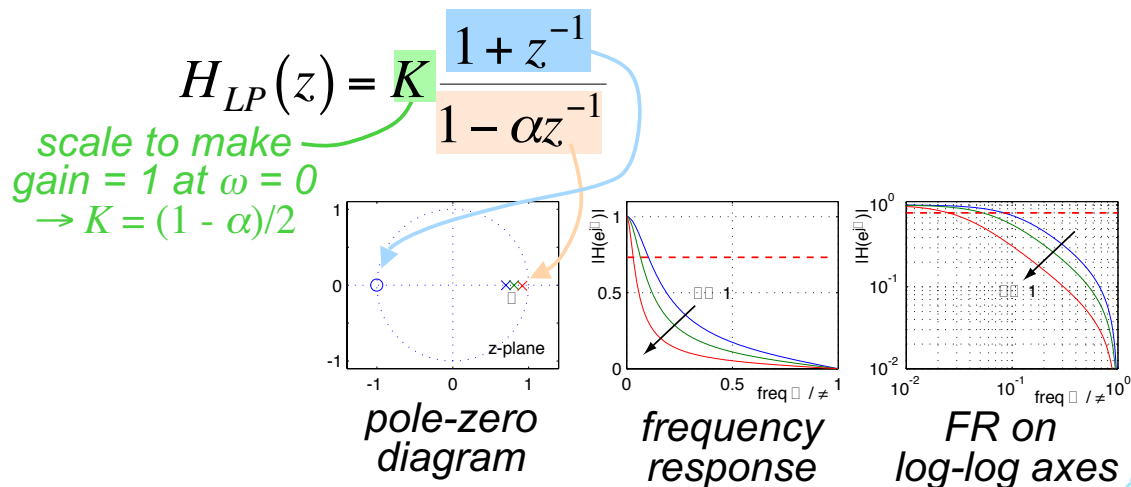
- i.e. $h_H[n] = (-1)^n h_L[n]$
 $\Rightarrow H_H(z) = H_L(-z)$

- i.e. **180° rotation** of the z-plane,
 $\Rightarrow$ **π shift** of frequency response



Simple IIR Lowpass

IIR $\rightarrow$ feedback, zeros **and** poles,
 conditional stability, $h[n]$ less useful



Simple IIR Lowpass

$$H_{LP}(z) = K \frac{1 + z^{-1}}{1 - \alpha z^{-1}}$$

*max = 1
using $K = (1 - \alpha)/2$*

■ Cutoff freq. ω_c from $|H_{LP}(e^{j\omega_c})|^2 = \frac{\max}{2}$

$$\Rightarrow \frac{(1 - \alpha)^2}{4} \frac{(1 + e^{-j\omega_c})(1 + e^{j\omega_c})}{(1 - \alpha e^{-j\omega_c})(1 - \alpha e^{j\omega_c})} = \frac{1}{2}$$

$$\Rightarrow \cos \omega_c = \frac{2\alpha}{1 + \alpha^2} \Rightarrow \alpha = \frac{1 - \sin \omega_c}{\cos \omega_c}$$

Design Equation



Simple IIR Highpass

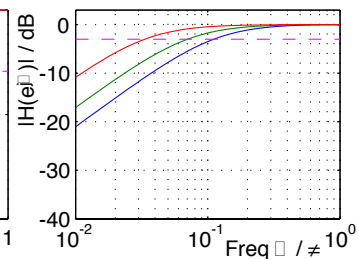
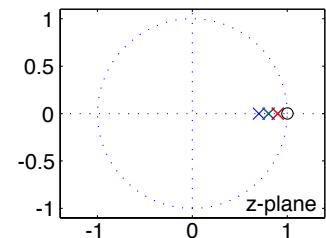
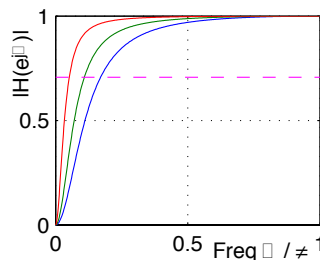
$$H_{HP}(z) = K \frac{1 - z^{-1}}{1 - \alpha z^{-1}}$$

*Pass $\omega = \pi \rightarrow H_{HP}(-1) = 1$
 $\rightarrow K = (1 + \alpha)/2$*

Design Equation:

$$\alpha = \frac{1 - \sin \omega_c}{\cos \omega_c}$$

(again)



Highpass and Lowpass

- Consider lowpass filter:

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & \omega \approx 0 \\ \sim 0 & \text{large } \omega \end{cases}$$

- Then:

$$1 - H_{LP}(e^{j\omega}) = \begin{cases} 0 & \omega \approx 0 \\ \sim 1 & \text{large } \omega \end{cases} \quad \begin{array}{l} \bullet \text{ Highpass} \\ \bullet c/w (-1)^n h[n] \end{array}$$

just another z poly

- However, $|1 - H_{LP}(z)| \neq 1 - |H_{LP}(z)|$
(unless $H(e^{j\omega})$ is pure real - not for IIR)



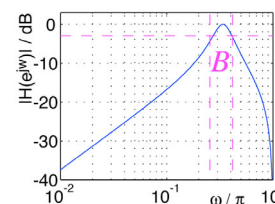
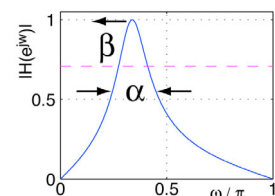
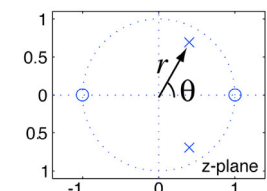
Simple IIR Bandpass

$$H_{BP}(z) = \frac{1-\alpha}{2} \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$$

$$= K \frac{(1+z^{-1})(1-z^{-1})}{1-2r\cos\theta \cdot z^{-1}+r^2 z^{-2}}$$

where $r = \sqrt{\alpha}$ $\cos\theta = \frac{\beta(1+\alpha)}{2\sqrt{\alpha}}$

Design $\left\{ \begin{array}{l} \text{Center freq } \omega_c = \cos^{-1} \beta \\ \text{3dB bandwidth } B = \cos^{-1} \left(\frac{2\alpha}{1+\alpha^2} \right) \end{array} \right.$



Simple Filter Example

- Design a second-order IIR bandpass filter with $\omega_c = 0.4\pi$, 3dB b/w of 0.1π

$$\omega_c = 0.4\pi \Rightarrow \beta = \cos \omega_c = 0.3090$$

$$B = 0.1\pi \Rightarrow \frac{2\alpha}{1 + \alpha^2} = \cos(0.1\pi) \Rightarrow \alpha = 0.7265$$

$$\Rightarrow H_{BP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}}$$

$$= \frac{0.1367(1 - z^{-2})}{1 - 0.5335z^{-1} + 0.7265z^{-2}}$$

sensitive..



Simple IIR Bandstop

$$H_{BS}(z) = \frac{1 + \alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}}$$

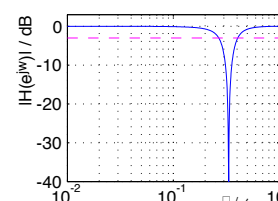
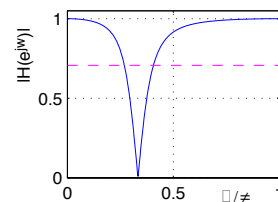
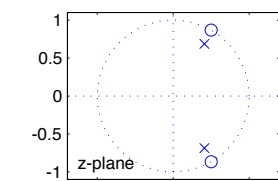
zeros at ω_c
same poles as H_{BP}

- Design eqns:

$$\omega_c = \cos^{-1} \beta \Rightarrow \beta = \cos \omega_c$$

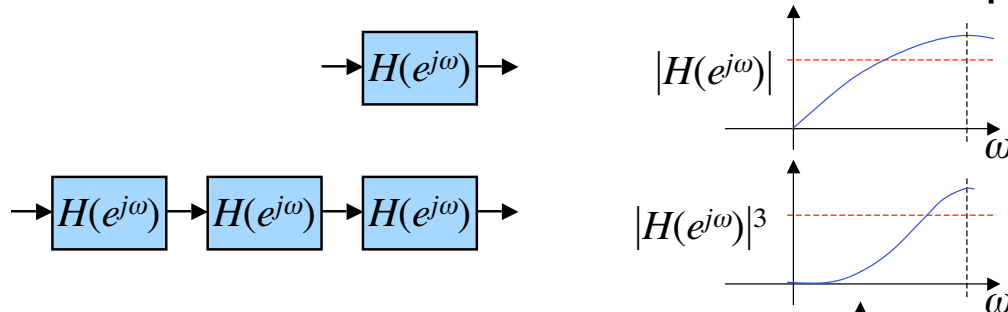
$$B = \cos^{-1} \left(\frac{2\alpha}{1 + \alpha^2} \right)$$

$$\Rightarrow \alpha = \frac{1}{\cos B} - \sqrt{\frac{1}{\cos^2 B} - 1}$$

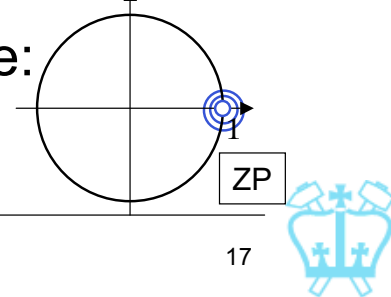


Cascading Filters

- Repeating a filter (**cascade** connection) makes its characteristics more abrupt:

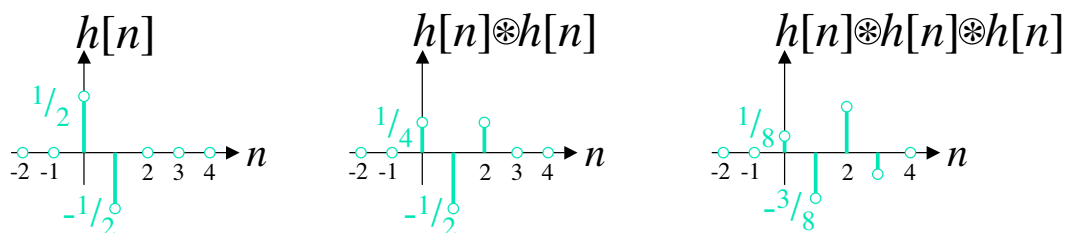


- Repeated roots in z-plane:



Cascading Filters

- Cascade systems are **higher order** e.g. longer (finite) impulse response:

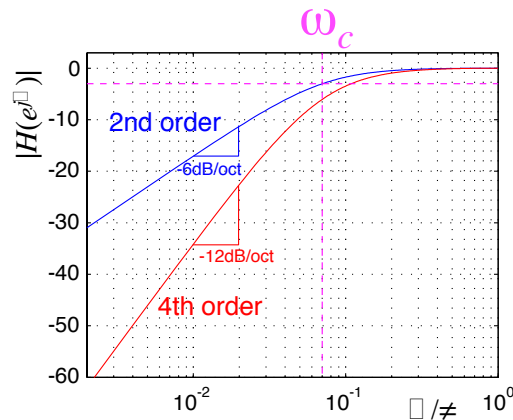


- In general, cascade filters will **not** be optimal (...) for a given order



Cascading Filters

- Cascading filters improves **rolloff** slope:

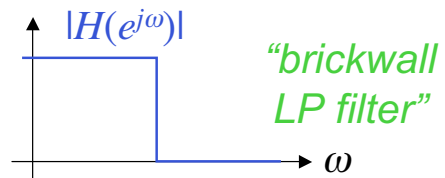


- But: 3dB cutoff frequency will change (gain at $\omega_c \rightarrow 3N$ dB)



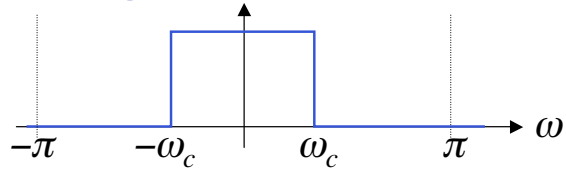
2. Ideal filters

- Typical filter requirements:
 - gain = 1 for wanted parts (**pass band**)
 - gain = 0 for unwanted parts (**stop band**)
- “Ideal” characteristics would be like:
 - no phase distortion etc.
- What is this filter?
 - can calculate IR $h[n]$ as IDFT of ideal response...



Ideal Lowpass Filter

- Given ideal $H(e^{j\omega})$:
(assume $\theta(\omega) = 0$)



$$\begin{aligned}\Rightarrow h[n] &= IDTFT\{H(e^{j\omega})\} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega\end{aligned}$$

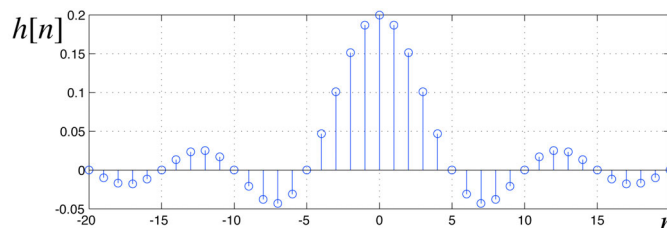
$$\Rightarrow h[n] = \frac{\sin \omega_c n}{\pi n}$$

Ideal lowpass filter



Ideal Lowpass Filter

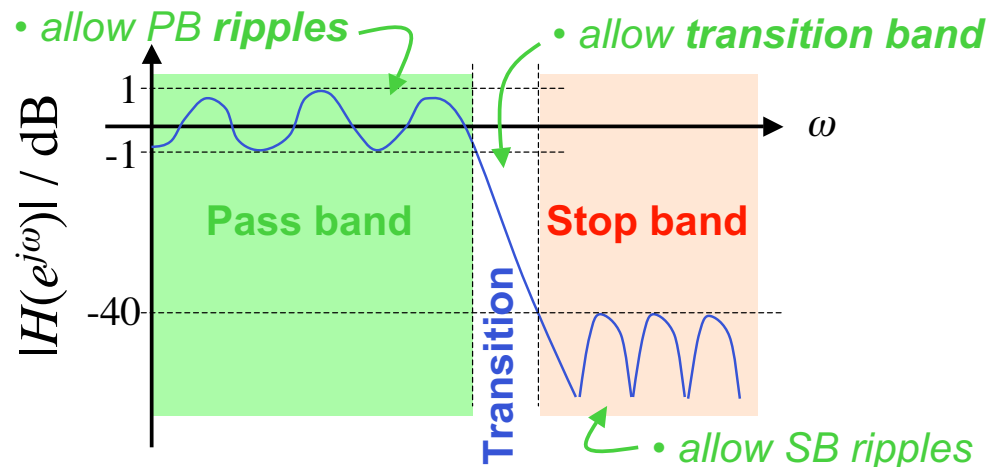
$$h[n] = \frac{\sin \omega_c n}{\pi n}$$



- Problems!
 - doubly infinite ($n = -\infty \dots \infty$)
 - no rational polynomial $\rightarrow$ very long FIR
 - excellent frequency-domain characteristics
 $\Leftrightarrow$ poor time-domain characteristics
(blurring, ringing – a general problem)



Practical filter specifications



- lower-order realization (less computation)
- better time-domain properties (less ringing)
- easier to design...

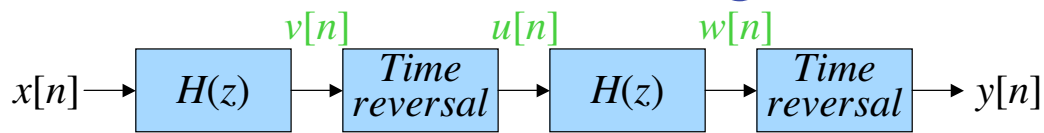


3. Linear-phase Filters

- $|H(e^{j\omega})|$ alone can hide *phase distortion*
 - differing delays for adjacent frequencies can **mangle** the signal
- Prefer filters with a **flat** phase response
e.g. $\theta(\omega) = 0$ **“zero phase filter”**
- A filter with **constant** delay $\tau_p = D$ at all freq's has $\theta(\omega) = -D\omega$ **“linear phase”**
 $\Rightarrow H(e^{j\omega}) = e^{-jD\omega} \tilde{H}(\omega)$ *pure-real (zero-phase) portion*
- Linear phase can ‘shift’ to zero phase



Time reversal filtering



- $v[n] = x[n] \otimes h[n] \rightarrow V(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$
- $u[n] = v[-n] \rightarrow U(e^{j\omega}) = V(e^{-j\omega}) = V^*(e^{j\omega})$ *if v real*
- $w[n] = u[n] \otimes h[n] \rightarrow W(e^{j\omega}) = H(e^{j\omega})U(e^{j\omega})$
- $y[n] = w[-n] \rightarrow Y(e^{j\omega}) = W^*(e^{j\omega})$

$$= (H(e^{j\omega})(H(e^{j\omega})X(e^{j\omega}))^*)^*$$

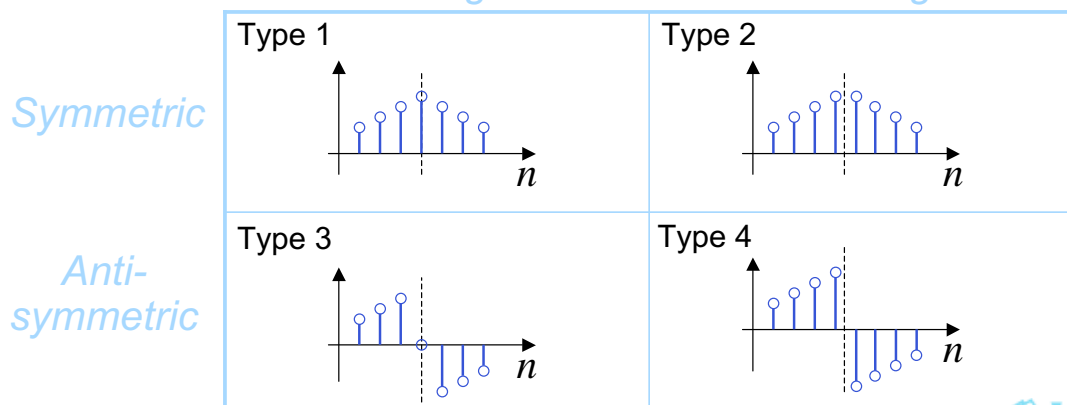
$$\rightarrow Y(e^{j\omega}) = X(e^{j\omega})|H(e^{j\omega})|^2$$

- Achieves zero-phase result
- Not causal! Need whole signal first



Linear Phase FIR filters

- (Anti)Symmetric FIR filters are almost the only way to get zero/linear phase
- 4 types: *Odd length* *Even length*



Linear Phase FIR: Type 1

■ Length L **odd** $\rightarrow$ order $N = L - 1$ **even**

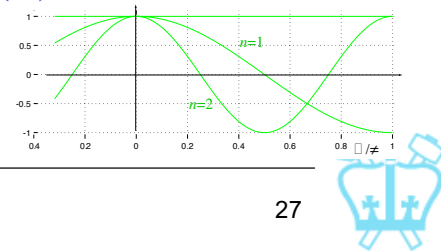
■ **Symmetric** $\rightarrow h[n] = h[N - n]$
($h[N/2]$ unique)

■ $H(e^{j\omega}) = \sum_{n=0}^N h[n] e^{-j\omega n}$

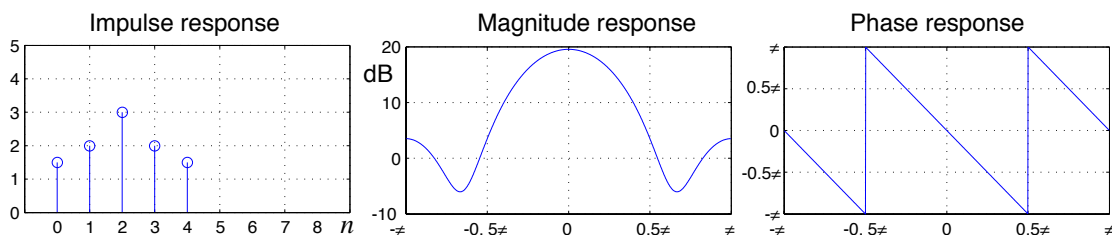
linear phase $\rightarrow$ $D = -\theta(\omega)/\omega = N/2$

$$= e^{-j\omega \frac{N}{2}} \left(h\left[\frac{N}{2}\right] + 2 \sum_{n=1}^{N/2} h\left[\frac{N}{2} - n\right] \cos \omega n \right)$$

pure-real $\tilde{H}(\omega)$ from cosine basis



Linear Phase FIR: Type 1



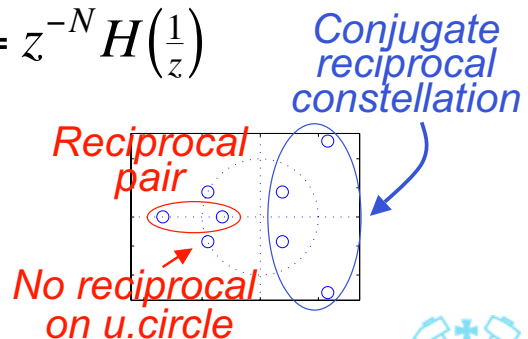
■ Where are the N zeros?

$$h[n] = h[-n] \Rightarrow H(z) = z^{-N} H\left(\frac{1}{z}\right)$$

thus for a zero ζ

$$H(\zeta) = 0 \Rightarrow H\left(\frac{1}{\zeta}\right) = 0$$

*Reciprocal zeros
(as well as cplx conj)*



Linear Phase FIR: Type 2

- Length L **even** $\rightarrow$ order $N = L - 1$ **odd**

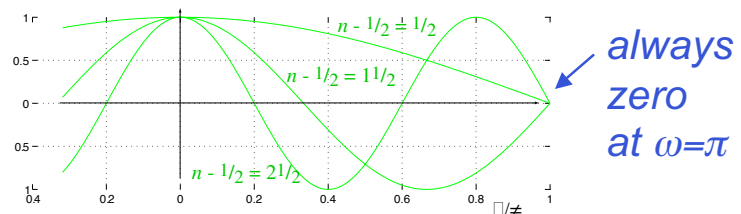
- Symmetric** $\rightarrow h[n] = h[N - n]$

(no unique point)

- $H(e^{j\omega}) = e^{-j\omega \frac{N}{2}} \sum_{n=1}^{(N+1)/2} h[\frac{N+1}{2} - n] \cos \omega(n - \frac{1}{2})$

Non-integer delay
of $N/2$ samples

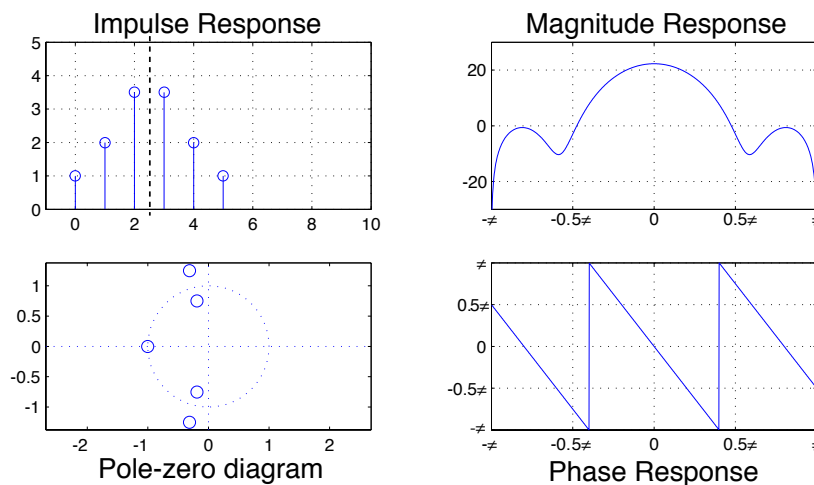
$\tilde{H}(\omega)$ from **double-length** cosine basis



always
zero
at $\omega = \pi$



Linear Phase FIR: Type 2



- Zeros: $H(z) = z^{-N} H(\frac{1}{z})$
at $z = -1$, $H(-1) = (-1)^N H(-1) \Rightarrow H(e^{j\pi}) = 0$
odd

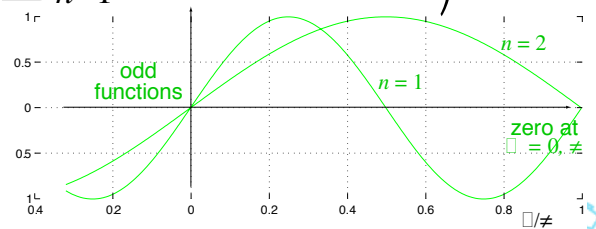
LPF-like



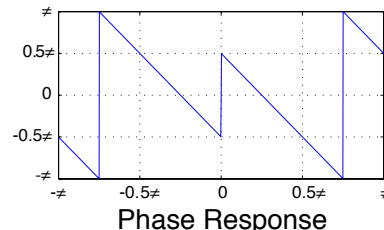
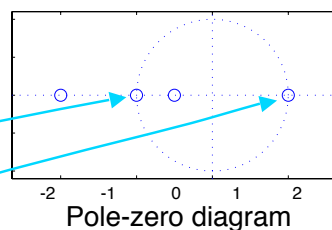
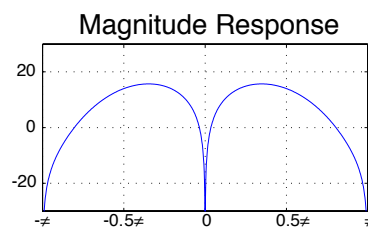
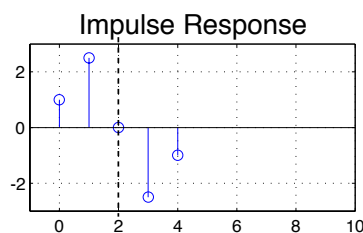
Linear Phase FIR: Type 3

- Length L **odd** $\rightarrow$ order $N = L - 1$ **even**
- Antisymmetric** $\rightarrow h[n] = -h[N - n]$
 $\Rightarrow h[N/2] = -h[N/2] = 0$
- $H(e^{j\omega}) = \sum_{n=1}^{N/2} h[\frac{N}{2} - n] \left(e^{-j\omega(\frac{N}{2}-n)} - e^{-j\omega(\frac{N}{2}+n)} \right)$
 $= je^{-j\omega\frac{N}{2}} \left(2 \sum_{n=1}^{N/2} h[\frac{N}{2} - n] \sin \omega n \right)$

$\theta(\omega) = \pi/2 - \omega N/2$
Antisymmetric $\Rightarrow$
 $\pi/2$ phase shift in
 addition to linear phase



Linear Phase FIR: Type 3



- Zeros:** $H(z) = -z^{-N} H(\frac{1}{z})$
 $\Rightarrow H(1) = -H(1) = 0 ; H(-1) = -H(-1) = 0$



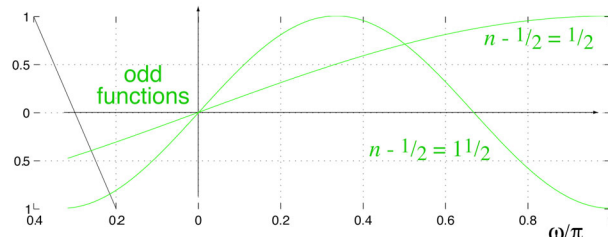
Linear Phase FIR: Type 4

- Length L **even** $\rightarrow$ order $N = L - 1$ **odd**
- Antisymmetric** $\rightarrow h[n] = -h[N - n]$
(no center point)
- $H(e^{j\omega}) = je^{-j\omega\frac{N}{2}} 2 \sum_{n=1}^{N/2} h[\frac{N+1}{2} - n] \sin \omega(n - \frac{1}{2})$

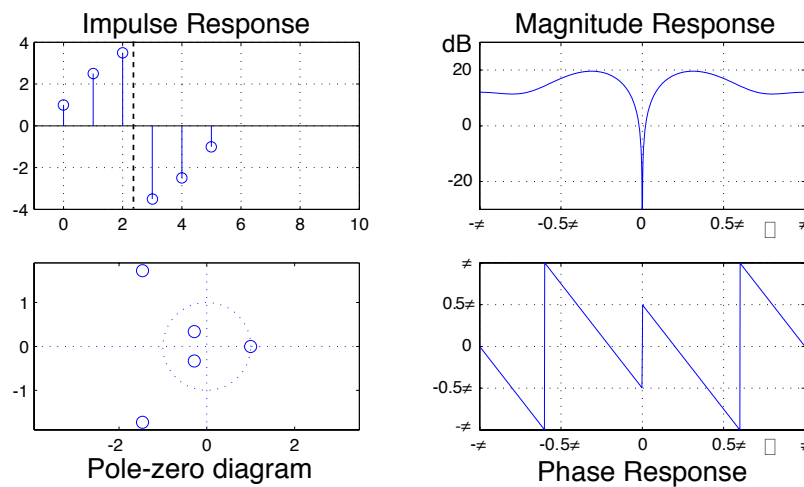
$\pi/2$ offset

fractional-sample delay

offset sine basis



Linear Phase FIR: Type 4



- Zeros: $H(1) = -H(1) = 0$
($H(-1)$ OK because N is odd)



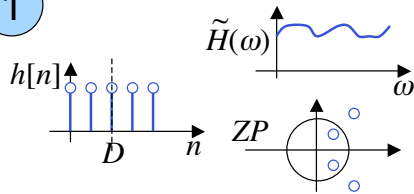
4 Linear Phase FIR Types

Odd length

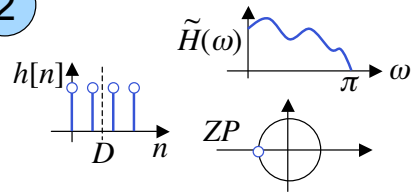
Even length

Symmetric

1

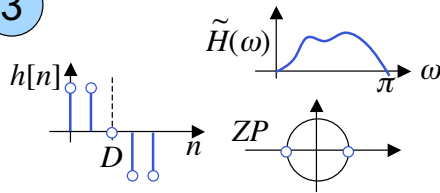


2



Antisymmetric

3



4

