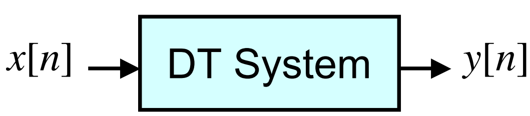
ELEN E4810: Digital Signal Processing
Topic 2: Time domain

1. Discrete-time systems
2. Convolution
3. Linear Constant-Coefficient Difference Equations (LCCDEs)
4. Correlation

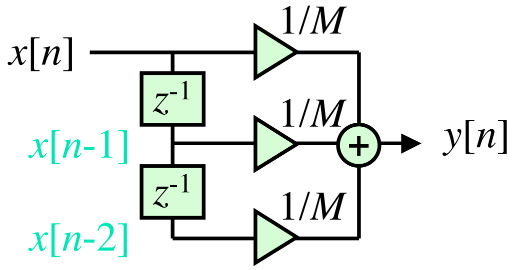
1. Discrete-time systems

- A **system** converts input to output:



$$y[n] = f(x[n])$$

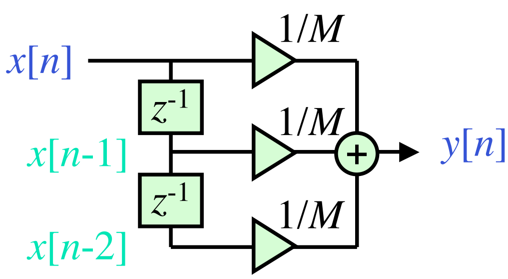
- E.g. Moving Average (MA):

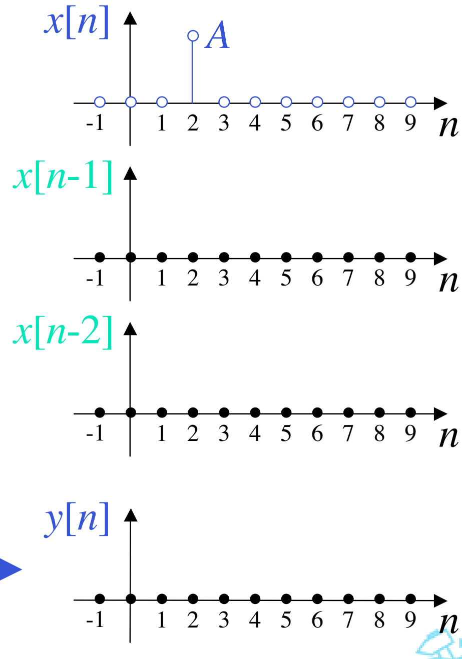


$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

$$(M = 3)$$

Moving Average (MA)





$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k] \rightarrow$$

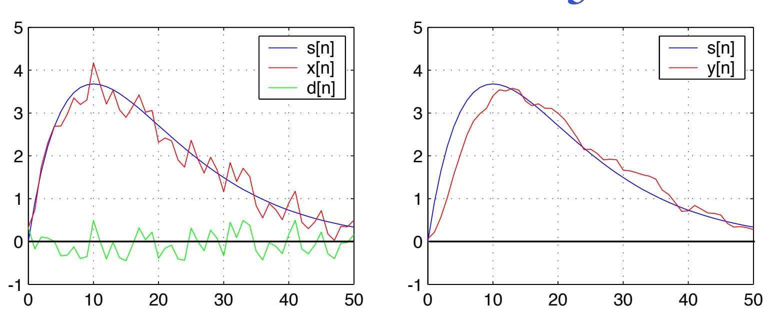
MA Smoother

- MA smooths out rapid variations (e.g. “12 month moving average”)
- e.g. *signal* *noise*

  $x[n] = s[n] + d[n]$

$$y[n] = \frac{1}{5}\sum_{k=0}^{4} x[n-k]$$

5-pt moving average



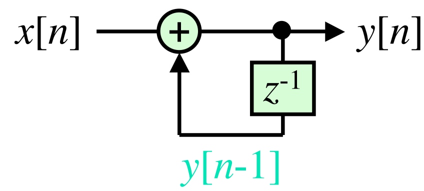
Accumulator

- Output accumulates all past inputs:

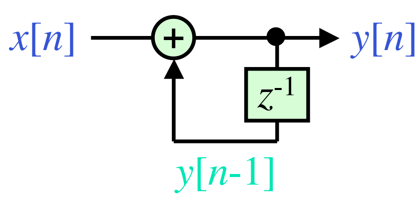
$$y[n] = \sum_{\ell=-\infty}^{n} x[\ell]$$

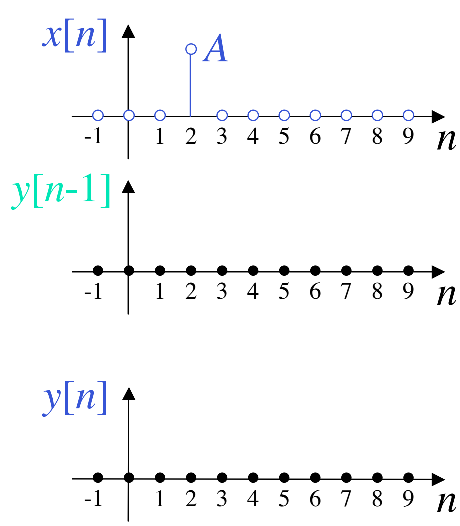
$$= \sum_{\ell=-\infty}^{n-1} x[\ell] + x[n]$$

$$= y[n-1] + x[n]$$



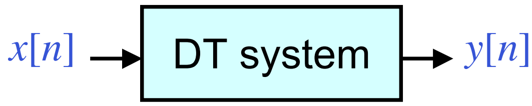
Accumulator





Classes of DT systems

- **Linear** systems obey superposition:



  - if input $x_1[n] \rightarrow$ output $y_1[n]$, $x_2 \rightarrow y_2$ ...
  - given a linear combination of inputs:

$$x[n] = \alpha\, x_1[n] + \beta\, x_2[n]$$

  - then output $\quad y[n] = \alpha\, y_1[n] + \beta\, y_2[n]$
    for *all* α, β, x_1, x_2
    i.e. same linear combination of outputs

Linearity: Example 1

- Accumulator: $\quad y[n] = \sum_{\ell=-\infty}^{n} x[\ell]$

$$x[n] = \alpha \cdot x_1[n] + \beta \cdot x_2[n]$$

$$\rightarrow \quad y[n] = \sum_{\ell=-\infty}^{n} \left(\alpha x_1[\ell] + \beta x_2[\ell]\right)$$

$$= \sum \left(\alpha x_1[\ell]\right) + \sum \left(\beta x_2[\ell]\right)$$

$$= \alpha \sum x_1[\ell] + \beta \sum x_2[\ell]$$

$$= \alpha \cdot y_1[n] + \beta \cdot y_2[n]$$

Linear

Linearity Example 2:

- "Energy operator": $y[n] = x^2[n] - x[n-1] \cdot x[n+1]$

$$x[n] = \alpha \cdot x_1[n] + \beta \cdot x_2[n]$$

$$\rightarrow y[n] = (\alpha x_1[n] + \beta x_2[n])^2 - (\alpha x_1[n-1] + \beta x_2[n-1]) \cdot (\alpha x_1[n+1] + \beta x_2[n+1])$$

$$\neq \alpha \cdot y_1[n] + \beta \cdot y_2[n]$$

✗ ***Nonlinear***

Linearity Example 3:

- 'Offset' accumulator: $y[n] = C + \sum_{\ell=-\infty}^{n} x[\ell]$

$$\Rightarrow y_1[n] = C + \sum_{\ell=-\infty}^{n} x_1[\ell]$$

but $y[n] = C + \sum_{\ell=-\infty}^{n} (\alpha x_1[\ell] + \beta x_2[\ell])$

$$\neq \alpha y_1[n] + \beta y_2[n]$$

✗ ***Nonlinear***

.. unless $C = 0$

Property: Shift (time) invariance

- **Time-shift** of input
  causes same shift in output
- i.e. if $x_1[n] \rightarrow y_1[n]$
  then $x[n] = x_1[n - n_0]$
  $\Rightarrow y[n] = y_1[n - n_0]$
- i.e. process doesn't depend on absolute value of n

Shift-invariance counterexample

- Upsampler: $x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow y[n]$

$$y[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \cdots \\ 0, & \text{otherwise} \end{cases}$$

$$y_1[n] = x_1[n/L] \quad (n = r \cdot L)$$
$$x[n] = x_1[n - n_0]$$
$$\Rightarrow y[n] = x_1[n/L - n_0]$$
$$= x_1\left[\frac{n - L \cdot n_0}{L}\right] = y_1[n - L \cdot n_0] \neq y_1[n - n_0]$$

Not shift invariant

Another counterexample

$$y[n] = n \cdot x[n]$$ *scaling by time index*

- Hence $y_1[n-n_0] = (n-n_0)\cdot x_1[n-n_0]$
- If $x[n] = x_1[n-n_0]$
  then $y[n] = n \cdot x_1[n-n_0]$ $\neq$
- Not shift invariant - parameters depend on n

Linear Shift Invariant (LSI)

- Systems which are both linear and shift invariant are easily manipulated mathematically
- This is still a wide and useful class of systems
- If discrete index corresponds to time, called Linear Time Invariant (LTI)

Causality

- If output depends only on past and current inputs (not future), system is called **causal**
- Formally, if $x_1[n] \rightarrow y_1[n]$ & $x_2[n] \rightarrow y_2[n]$

Causal → $x_1[n] = x_2[n] \quad \forall n < N$
$\Leftrightarrow y_1[n] = y_2[n] \quad \forall n < N$

Causality example

- Moving average: $y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$

  $y[n]$ depends on $x[n\text{-}k]$, $k \geq 0$ → causal
- 'Centered' moving average

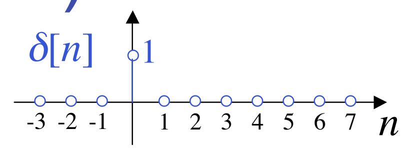
$$y[n] = \frac{1}{M}\left(x[n] + \sum_{k=1}^{(M-1)/2} x[n-k] + x[n+k]\right)$$

.. looks forward in time → noncausal

- Can sometimes make a non-causal system causal by delaying its output

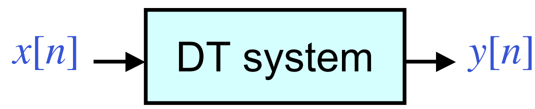
Impulse response (IR)

- Impulse $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$



  (unit sample sequence)

- Given a system:



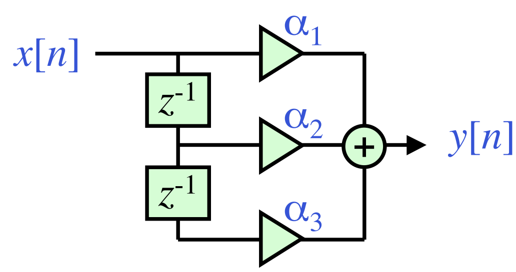
  if $x[n] = \delta[n]$ then $y[n] \triangleq h[n]$

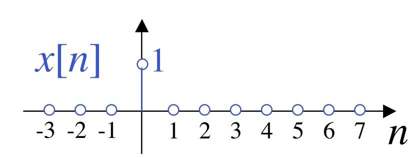
  "impulse response"

- LSI system completely specified by $h[n]$

Impulse response example

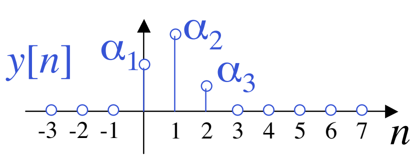
- Simple system:





$x[n] = \delta[n]$ impulse

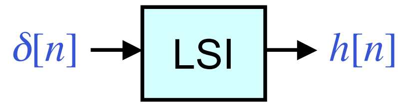
↓



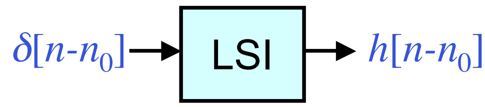
$y[n] = h[n]$ impulse response

2. Convolution

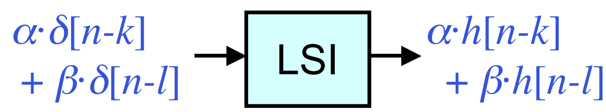
- Impulse response:



- Shift invariance:



- + Linearity:



- Can express any sequence with δs:
  $x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2]$..

Convolution sum

- Hence, since $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
- For LSI, $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ **Convolution sum**

  written as $y[n] = x[n] \circledast h[n]$
- Summation is symmetric in x and h
  i.e. $l = n - k \rightarrow$

$$x[n] \circledast h[n] = \sum_{l=-\infty}^{\infty} x[n-l]h[l] = h[n] \circledast x[n]$$

Convolution properties

- **LSI System output $y[n]$ = input $x[n]$ convolved with impulse response $h[n]$ → $h[n]$ completely describes system**
- Commutative: $x[n] \circledast h[n] = h[n] \circledast x[n]$
- Associative:

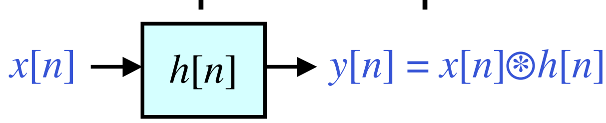
$$(x[n] \circledast h[n]) \circledast y[n] = x[n] \circledast (h[n] \circledast y[n])$$

- Distributive:

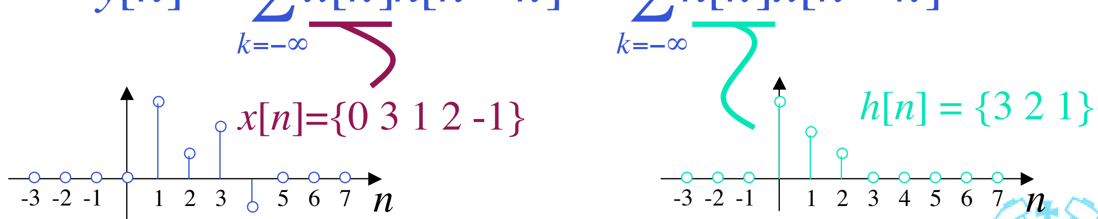
$$h[n] \circledast (x[n] + y[n]) = h[n] \circledast x[n] + h[n] \circledast y[n]$$

Interpreting convolution

- Passing a signal through a (LSI) system is equivalent to convolving it with the system's impulse response



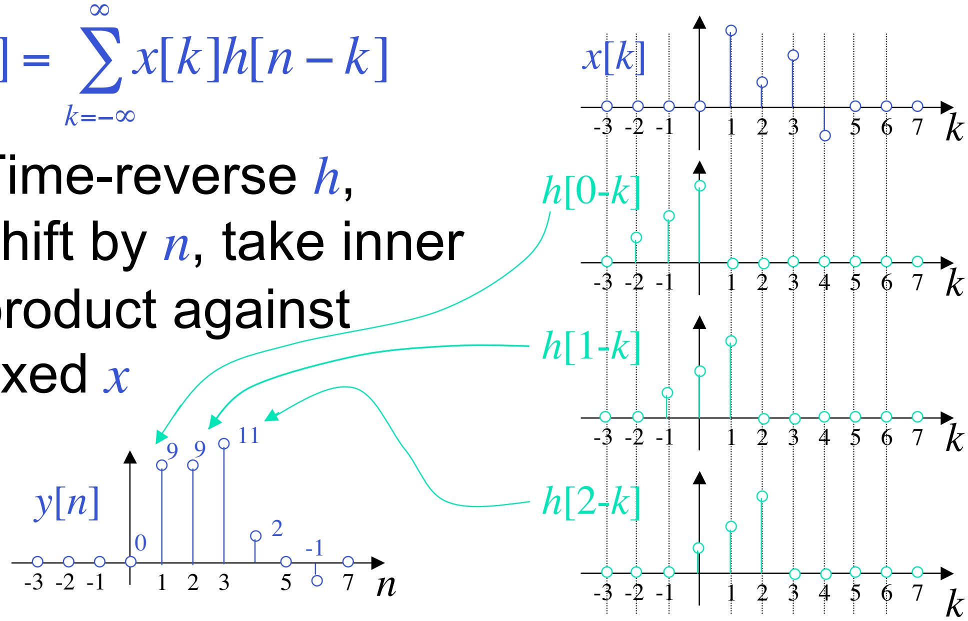
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$



Convolution interpretation 1

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

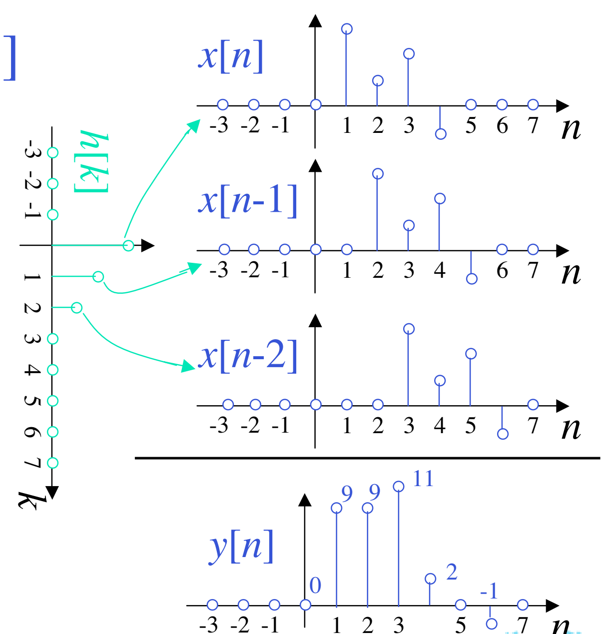
- Time-reverse h, shift by n, take inner product against fixed x



Convolution interpretation 2

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- Shifted x's weighted by points in h
- Conversely, weighted, delayed versions of h ...



Matrix interpretation

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \cdots \end{bmatrix} = \begin{bmatrix} x[0] & x[-1] & x[-2] \\ x[1] & x[0] & x[-1] \\ x[2] & x[1] & x[0] \\ \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \end{bmatrix}$$

- Diagonals in $\mathbf{X}$ matrix are equal

Convolution notes

- Total nonzero length of convolving N and M point sequences is $N+M-1$
- Adding the indices of the terms within the summation gives n :

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \qquad k + (n-k) = n$$

i.e. summation indices move in opposite senses

Convolution in MATLAB

- The M-file `conv` implements the convolution sum of two finite-length sequences
- If `a = [0 3 1 2 -1]`

  `b = [3 2 1]`

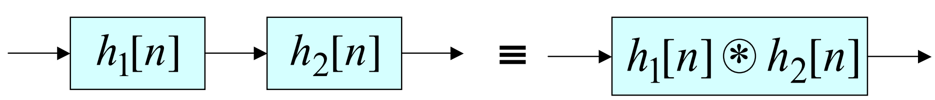
  then `conv(a,b)` yields

  `[0 9 9 11 2 0 -1]`

M

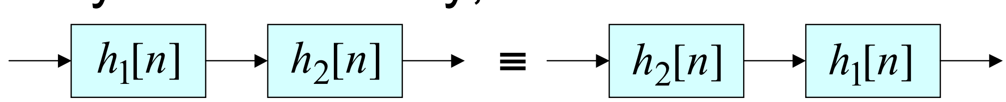
Connected systems

- **Cascade** connection:



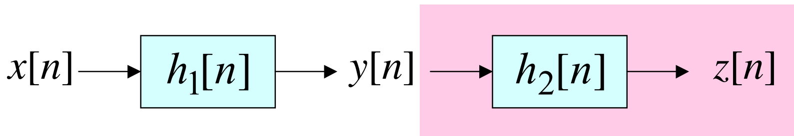
  Impulse response $h[n]$ of the cascade of two systems with impulse responses $h_1[n]$ and $h_2[n]$ is $h[n] = h_1[n] \circledast h_2[n]$

- By commutativity,



Inverse systems

- $\delta[n]$ is identity for convolution
  i.e. $x[n] \circledast \delta[n] = x[n]$
- Consider



$$z[n] = h_2[n] \circledast y[n] = h_2[n] \circledast h_1[n] \circledast x[n]$$

$$= x[n] \quad \textbf{if} \quad h_2[n] \circledast h_1[n] = \delta[n]$$

- $h_2[n]$ is the **inverse system** of $h_1[n]$

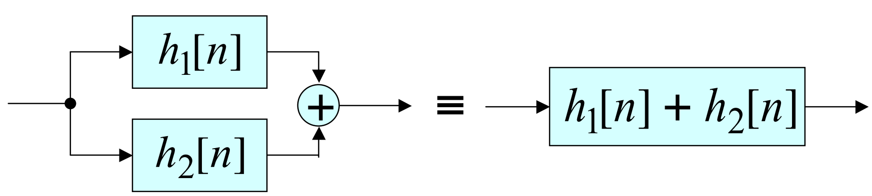
Inverse systems

- Use inverse system to recover input $x[n]$ from output $y[n]$ (e.g. to undo effects of transmission channel)
- Only sometimes possible - e.g. cannot 'invert' $h_1[n] = 0$
- In general, attempt to solve
  $h_2[n] \circledast h_1[n] = \delta[n]$

Inverse system example

- Accumulator:
  Impulse response $h_1[n] = \mu[n]$
- 'Backwards difference'
  $$h_2[n] = \delta[n] - \delta[n-1]$$
  .. has desired property:
  $$\mu[n] - \mu[n-1] = \delta[n]$$
- Thus, 'backwards difference' is inverse system of accumulator.

Parallel connection



- Impulse response of two parallel systems added together is:
  $$h[n] = h_1[n] + h_2[n]$$

3. Linear Constant-Coefficient Difference Equation (LCCDE)

- General spec. of DT, LSI, finite-dim sys:

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

  - defined by $\{d_k\}$, $\{p_k\}$
  - **order** = $\max(N,M)$
- Rearrange for $y[n]$ in causal form:

$$y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k]$$

- WLOG, always have $d_0 = 1$

Solving LCCDEs

- "Total solution"

$$y[n] = y_c[n] + y_p[n]$$

Complementary Solution

satisfies $\sum_{k=0}^{N} d_k y[n-k] = 0$

Particular Solution

for given forcing function $x[n]$

Complementary Solution

- General form of unforced oscillation i.e. system's 'natural modes'
- Assume y_c has form $y_c[n] = \lambda^n$

$$\Rightarrow \sum_{k=0}^{N} d_k \lambda^{n-k} = 0$$

$$\Rightarrow \lambda^{n-N}\left(d_0\lambda^N + d_1\lambda^{N-1} + \ldots + d_{N-1}\lambda + d_N\right) = 0$$

$$\Rightarrow \sum_{k=0}^{N} d_k \lambda^{N-k} = 0$$

Characteristic polynomial of system - depends only on $\{d_k\}$

Complementary Solution

- $\sum_{k=0}^{N} d_k \lambda^{N-k} = 0$ factors into roots λ_i, i.e.
  $(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots = 0$
- Each/any λ_i satisfies eqn.
- Thus, complementary solution:
  $y_c[n] = \alpha_1\lambda_1{}^n + \alpha_2\lambda_2{}^n + \alpha_3\lambda_3{}^n + \ldots$
  Any linear combination will work
  $\rightarrow$ α_is are free to match initial conditions

Complementary Solution

- Repeated roots in chr. poly:

$$(\lambda - \lambda_1)^L (\lambda - \lambda_2) \ldots = 0$$
$$\Rightarrow y_c[n] = \alpha_1 \lambda_1^n + \alpha_2 n \lambda_1^n + \alpha_3 n^2 \lambda_1^n + \ldots + \alpha_L n^{L-1} \lambda_1^n + \ldots$$

- Complex λ_is → sinusoidal $y_c[n] = \alpha_i \lambda_i^n$

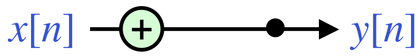
Particular Solution

- Recall: Total solution $y[n] = y_c[n] + y_p[n]$
- Particular solution reflects input
- 'Modes' usually decay away for large n leaving just $y_p[n]$
- Assume 'form' of $x[n]$, scaled by β:
  e.g. $x[n]$ constant → $y_p[n] = \beta$

$$x[n] = \lambda_0^n \rightarrow y_p[n] = \beta \cdot \lambda_0^n \quad (\lambda_0 \notin \lambda_i)$$
$$\text{or} = \beta\, n^{L+1}\, \lambda_0^n \quad (\lambda_0 \in \lambda_i)$$

LCCDE example

$$y[n] + y[n-1] - 6y[n-2] = x[n]$$



- Need input: $x[n] = 8\mu[n]$
- Need initial conditions: $y[-1] = 1,\ y[-2] = -1$

LCCDE example

- Complementary solution:

$$y[n] + y[n-1] - 6y[n-2] = 0; \quad y[n] = \lambda^n$$

$$\Rightarrow \lambda^{n-2}\left(\lambda^2 + \lambda - 6\right) = 0$$

$$\Rightarrow (\lambda + 3)(\lambda - 2) = 0 \rightarrow \text{roots } \lambda_1 = -3,\ \lambda_2 = 2$$

$$\Rightarrow y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n$$

- $\alpha_1,\ \alpha_2$ are unknown at this point

LCCDE example

- Particular solution:
- Input $x[n]$ is constant $= 8\mu[n]$
  assume $y_p[n] = \beta$, substitute in:
  $y[n] + y[n-1] - 6y[n-2] = x[n]$ ('large' n)
  $\Rightarrow \beta + \beta - 6\beta = 8\mu[n]$
  $\Rightarrow -4\beta = 8 \Rightarrow \beta = -2$

LCCDE example

- Total solution $y[n] = y_c[n] + y_p[n]$
  $= \alpha_1(-3)^n + \alpha_2(2)^n + \beta$
- Solve for unknown α_is by substituting *initial conditions* into DE at $n = 0, 1, \ldots$
  $y[n] + y[n-1] - 6y[n-2] = x[n]$
- $\underline{n = 0}$ $y[0] + y[-1] - 6y[-2] = x[0]$ (from ICs)
  $\Rightarrow \alpha_1 + \alpha_2 + \beta + 1 + 6 = 8$
  $\Rightarrow \alpha_1 + \alpha_2 = 3$

LCCDE example

- $\underline{n = 1}$ $\quad y[1] + y[0] - 6y[-1] = x[1]$
  $\Rightarrow \alpha_1(-3) + \alpha_2(2) + \beta + \alpha_1 + \alpha_2 + \beta - 6 = 8$
  $\Rightarrow -2\alpha_1 + 3\alpha_2 = 18$
- solve: $\alpha_1 = -1.8,\ \alpha_2 = 4.8$
- Hence, system output:
  $y[n] = -1.8(-3)^n + 4.8(2)^n - 2 \quad n \geq 0$
- *Don't* find α_is by solving with ICs at $n = -1,-2$ *(ICs may not reflect natural modes)*

LCCDE solving summary

- Difference Equation (DE):
  $Ay[n] + By[n-1] + \ldots = Cx[n] + Dx[n-1] + \ldots$
  Initial Conditions (ICs): $y[-1] = \ldots$
- DE RHS = 0 with $y[n]=\lambda^n \rightarrow$ roots $\{\lambda_i\}$ gives complementary soln $y_c[n] = \sum \alpha_i \lambda_i^n$
- Particular soln: $y_p[n] \sim x[n]$
  solve for $\beta\lambda_0^n$ "at large n"
- α_is by substituting DE at $n = 0, 1, \ldots$
  ICs for $y[-1], y[-2]$; $y_t=y_c+y_p$ for $y[0], y[1]$

LCCDEs: zero input/zero state

- Alternative approach to solving LCCDEs is to solve two subproblems:
  - $y_{zi}[n]$, response with zero input (just ICs)
  - $y_{zs}[n]$, response with zero state (just $x[n]$)
- Because of linearity, $y[n] = y_{zi}[n]+y_{zs}[n]$
- Both subproblems are ‘real’
- But, have to solve for α_is twice (then sum them)

Impulse response of LCCDEs

- Impulse response:

  i.e. solve with $x[n] = \delta[n] \rightarrow y[n] = h[n]$ (zero ICs)
- With $x[n] = \delta[n]$, ‘form’ of $y_p[n] = 0$

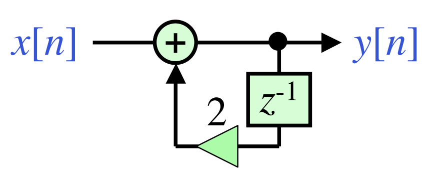
  $\rightarrow$ just solve $y_c[n]$ for $n = 0,1\ldots$ to find α_is

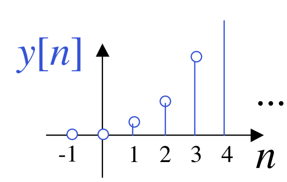
LCCDE IR example

- e.g. $y[n] + y[n-1] - 6y[n-2] = x[n]$
  (from before); $x[n] = \delta[n]$; $y[n] = 0$ for $n<0$
- $y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n$
- $\underline{n = 0:}$ $y[0] + \cancel{y[-1]} - 6\cancel{y[-2]} = \cancel{x[0]} \rightarrow 1$
  $\Rightarrow \alpha_1 + \alpha_2 = 1$
- $\underline{n = 1:}$ $\alpha_1(-3) + \alpha_2(2) + 1 = 0$
  $\Rightarrow \alpha_1 = 0.6, \quad \alpha_2 = 0.4$
- thus $h[n] = 0.6(-3)^n + 0.4(2)^n \quad n \geq 0$
  Infinite length

System property: **Stability**

- Certain systems can be unstable e.g.





Output grows without limit in some conditions

Stability

- Several definitions for stability; we use **Bounded-input, bounded-output (BIBO) stable**
- For *every* bounded input $|x[n]| < B_x \quad \forall n$ output is also subject to a finite bound, $|y[n]| < B_y \quad \forall n$

Stability example

- MA filter: $y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$

$$|y[n]| = \left|\frac{1}{M}\sum_{k=0}^{M-1} x[n-k]\right|$$

$$\leq \frac{1}{M}\sum_{k=0}^{M-1} |x[n-k]|$$

$$\leq \frac{1}{M} M \cdot B_x \leq B_y$$ → **BIBO Stable**

Stability & LCCDEs

- LCCDE output is of form:

$$y[n] = \alpha_1 \lambda_1^{\ n} + \alpha_2 \lambda_2^{\ n} + \ldots + \beta \lambda_0^{\ n} + \ldots$$

- αs and βs depend on input & ICs, *but* to be bounded for any input we need $|\lambda| < 1$

4. Correlation

- Correlation ~ identifies similarity between sequences:

Cross correlation of x against y

$$r_{xy}[\ell] = \sum_{n=-\infty}^{\infty} y[n]x[n-\ell]$$

"lag"

- Note: $r_{yx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]y[n-\ell]$ call $m = n - l$

$$= \sum_{m=-\infty}^{\infty} x[m+\ell]y[m] = r_{xy}[-\ell]$$

Correlation and convolution

- Correlation: $r_{xy}[n] = \sum_{k=-\infty}^{\infty} y[k]x[k-n]$
- Convolution: $x[n] \circledast y[n] = \sum_{k=-\infty}^{\infty} y[k]x[n-k]$
- Hence: $r_{xy}[n] = y[n] \circledast x[-n]$

  Correlation may be calculated by convolving with time-reversed sequence

Autocorrelation

- Autocorrelation (AC) is correlation of signal with itself:

$$r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]x[n-\ell] = r_{xx}[-\ell]$$

- Note: $r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = \varepsilon_x$ **Energy** of sequence $x[n]$

Correlation maxima

- Note: $r_{xx}[\ell] \le r_{xx}[0] \Rightarrow \frac{r_{xx}[\ell]}{r_{xx}[0]} \le 1$
- Similarly: $r_{xy}[\ell] \le \sqrt{\varepsilon_x \varepsilon_y} \Rightarrow \frac{r_{xy}[\ell]}{\sqrt{r_{xx}[0] r_{yy}[0]}} \le 1$
- From geometry,

$$\sqrt{\sum x_i^2}$$

$$\langle \mathbf{xy} \rangle = \sum_i x_i y_i = |\mathbf{x}||\mathbf{y}| \cos\theta$$

angle between x and y

- when $\mathbf{x} // \mathbf{y}$, $\cos\theta = 1$, else $\cos\theta < 1$

AC of a periodic sequence

- Sequence of period N: $\tilde{x}[n] = \tilde{x}[n+N]$
- Calculate AC over a finite window:

$$r_{\tilde{x}\tilde{x}}[\ell] = \frac{1}{2M+1} \sum_{n=-M}^{M} \tilde{x}[n]\tilde{x}[n-\ell]$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n]\tilde{x}[n-\ell] \qquad \text{if } M >> N$$

AC of a periodic sequence

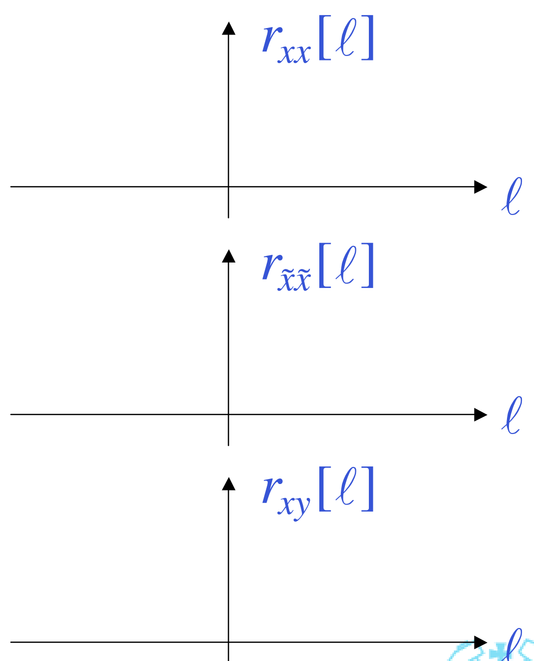
$$r_{\tilde{x}\tilde{x}}[0] = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}^2[n] = P_{\tilde{x}}$$ ← Average energy per sample or **Power** of x

$$r_{\tilde{x}\tilde{x}}[\ell + N] = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]\tilde{x}[n-\ell-N] = r_{\tilde{x}\tilde{x}}[\ell]$$

- i.e AC of periodic sequence is periodic

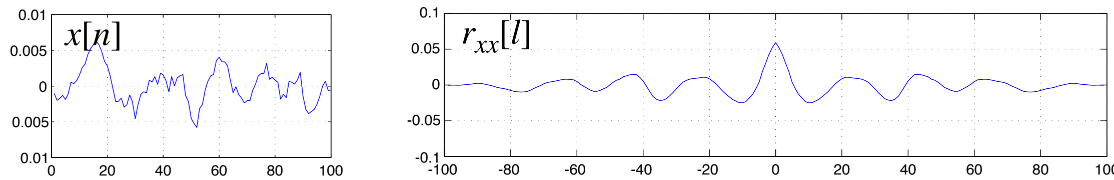
What correlations look like

- AC of any $x[n]$
- AC of periodic
- Cross correlation

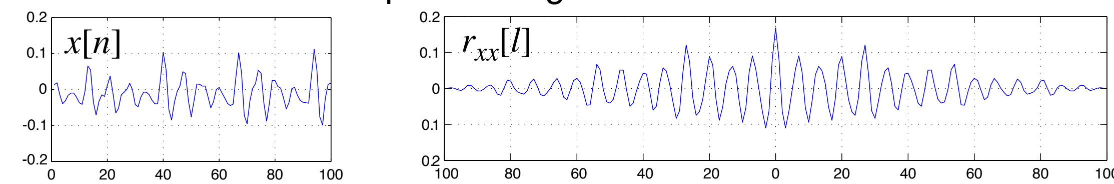


What correlation looks like

Autocorrelation of generic signal



Autocorrelation of near-periodic signal



Cross-correlation