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ELEN E4810: Digital Signal Processing  
Topic 8:  
Filter Design: IIR

1. Filter Design Specifications
2. Analog Filter Design
3. Digital Filters from Analog Prototypes

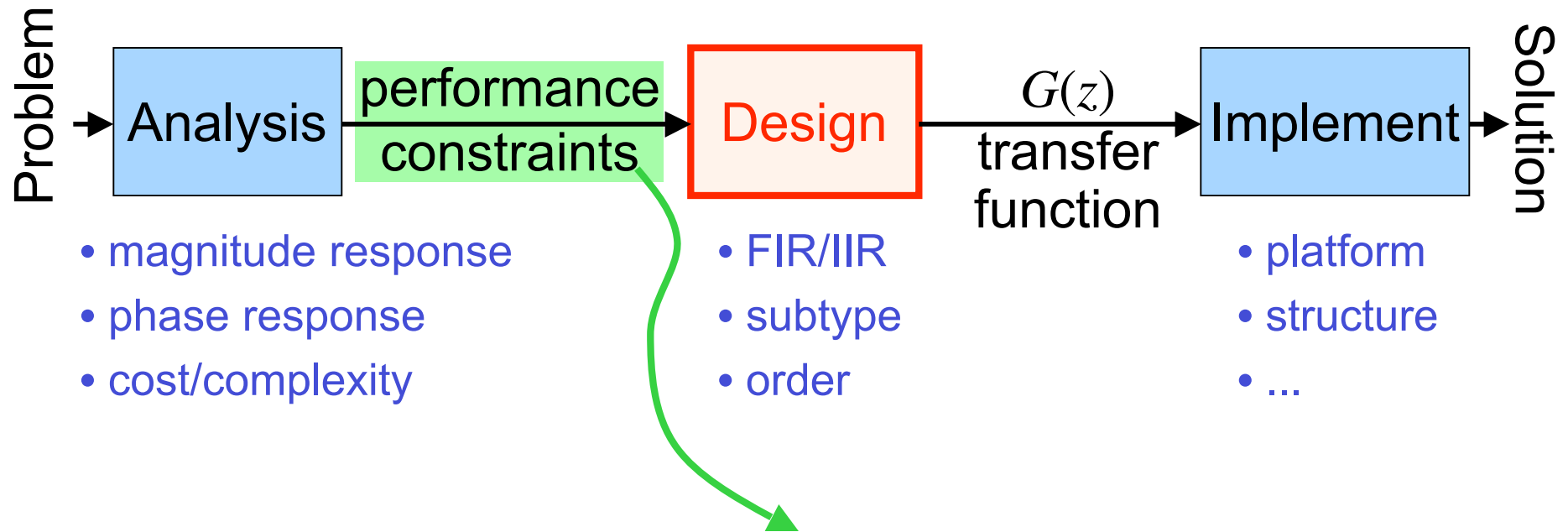


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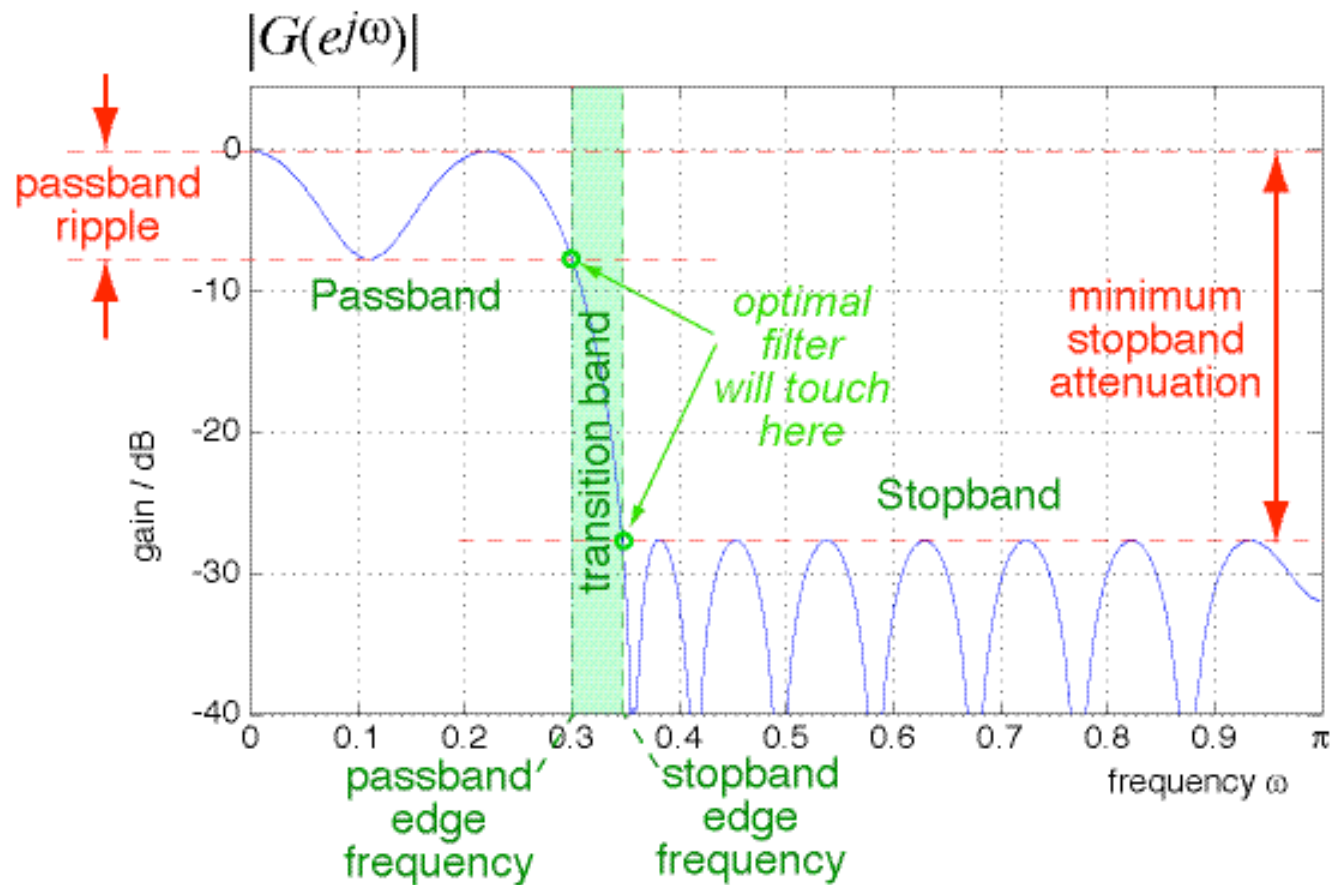
# 1. Filter Design Specifications

- The filter design process:



# Performance Constraints

- .. in terms of magnitude response:

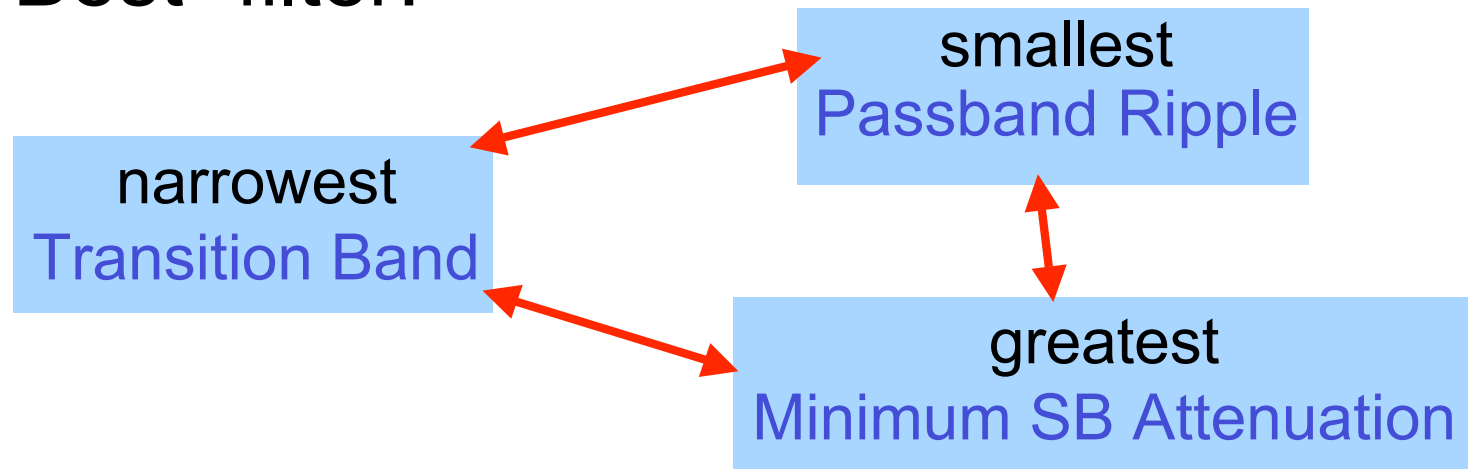


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# Performance Constraints

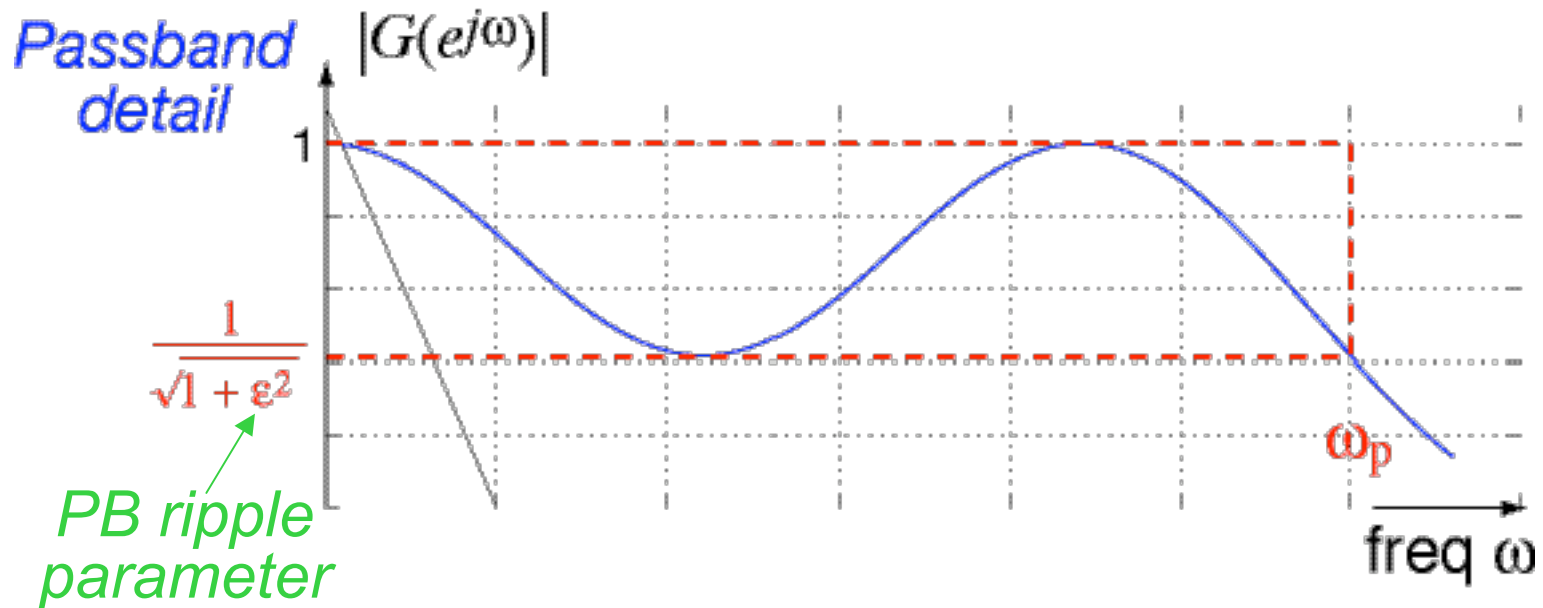
- “Best” filter:



- improving one usually worsens others
- But: increasing **filter order** (i.e. cost) improves all three measures



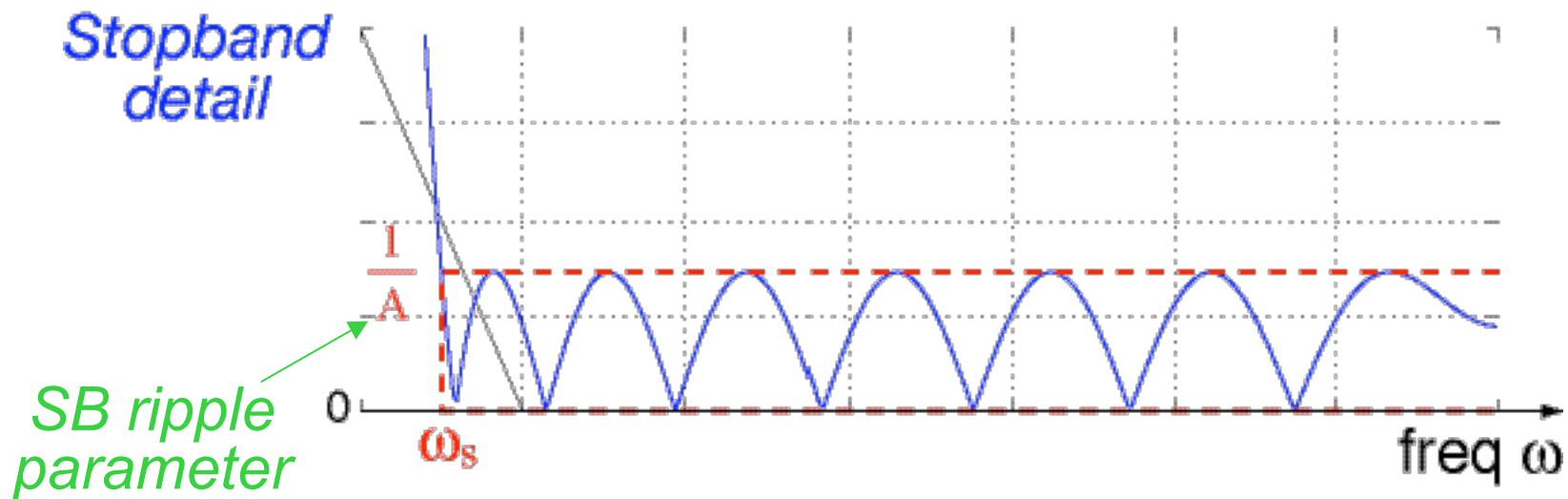
# Passband Ripple



- Assume peak passband gain = 1  
then *minimum* passband gain =  $\frac{1}{\sqrt{1+\epsilon^2}}$
- Or, **ripple**  $\alpha_{\max} = 20 \log_{10} \sqrt{1+\epsilon^2}$  dB



# Stopband Ripple



- Peak passband gain is  $A \times$  larger than peak stopband gain
- Hence, **minimum stopband attenuation**

$$\alpha_s = -20 \log_{10} \frac{1}{A} = 20 \log_{10} A \quad \text{dB}$$



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# Filter Type Choice: FIR vs. IIR

## FIR

- No feedback (just **zeros**)
- Always **stable**
- Can be **linear phase**
- BUT** ■ **High order** (20-2000)
- Unrelated to continuous-time filtering

## IIR

- Feedback (**poles** & zeros)
- May be **unstable**
- **Difficult** to control phase
- Typ. < **1/10th order** of FIR (4-20)
- Derive from ***analog prototype***



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# FIR vs. IIR

- If you care about **computational cost**  
→ use low-complexity **IIR**  
(computation no object → Lin Phs FIR)
- If you care about **phase response**  
→ use linear-phase **FIR**  
(phase unimportant → go with simple IIR)



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# IIR Filter Design

- IIR filters are directly related to analog filters (**continuous time**)
    - via a mapping of  $H(s)$  (**CT**) to  $H(z)$  (**DT**) that preserves many properties
  - Analog filter design is sophisticated
    - signal processing research since 1940s
- Design IIR filters via *analog prototype*
- hence, need to learn some **CT filter design**



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## 2. Analog Filter Design

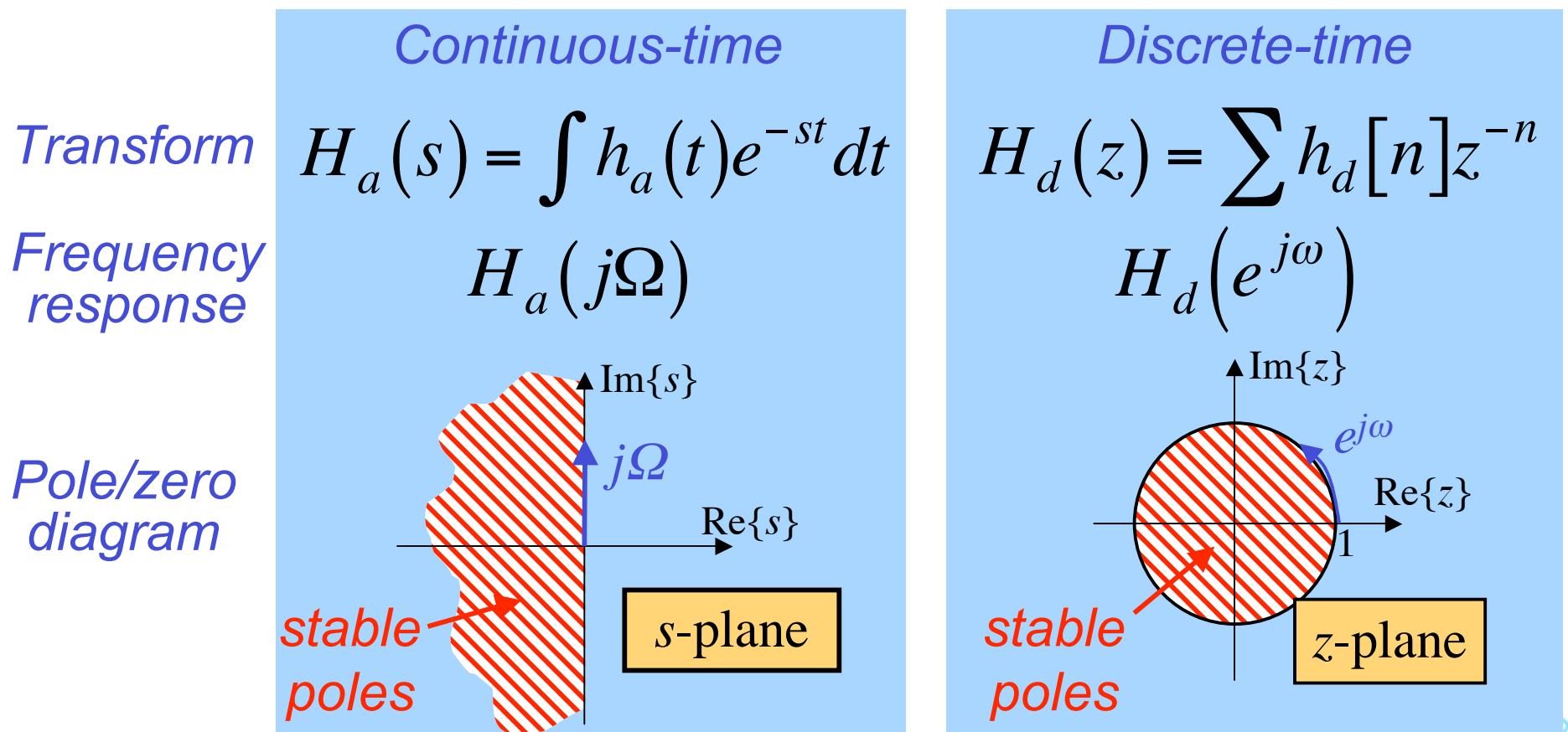
- Decades of analysis of transistor-based filters – sophisticated, well understood
- Basic choices:
  - ripples vs. flatness in stop and/or passband
  - more ripples → narrower transition band

<i>Family</i>	<i>PB</i>	<i>SB</i>
Butterworth	flat	flat
Chebyshev I	ripples	flat
Chebyshev II	flat	ripples
Elliptical	ripples	ripples



# CT Transfer Functions

- Analog systems:  $s$ -transform (Laplace)



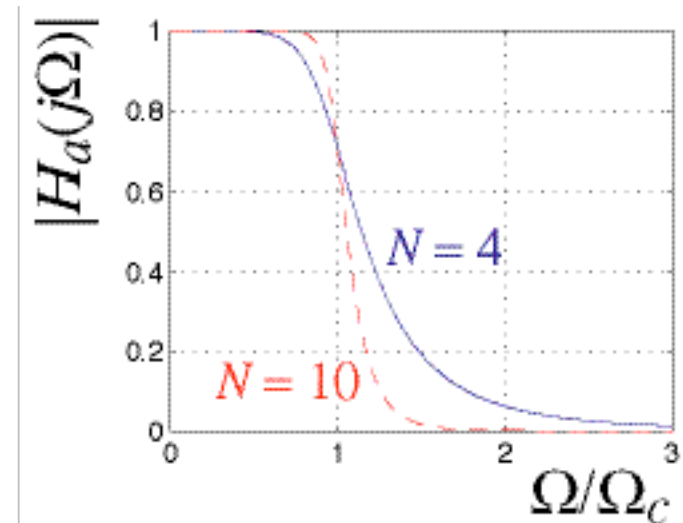
# Butterworth Filters

## Maximally flat in pass and stop bands

- Magnitude response (LP):  $|H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$  *filter order N*

- $\Omega \ll \Omega_c$ ,  
 $|H_a(j\Omega)|^2 \rightarrow 1$
- $\Omega = \Omega_c$ ,  
 $|H_a(j\Omega)|^2 = 1/2$

*3dB point*

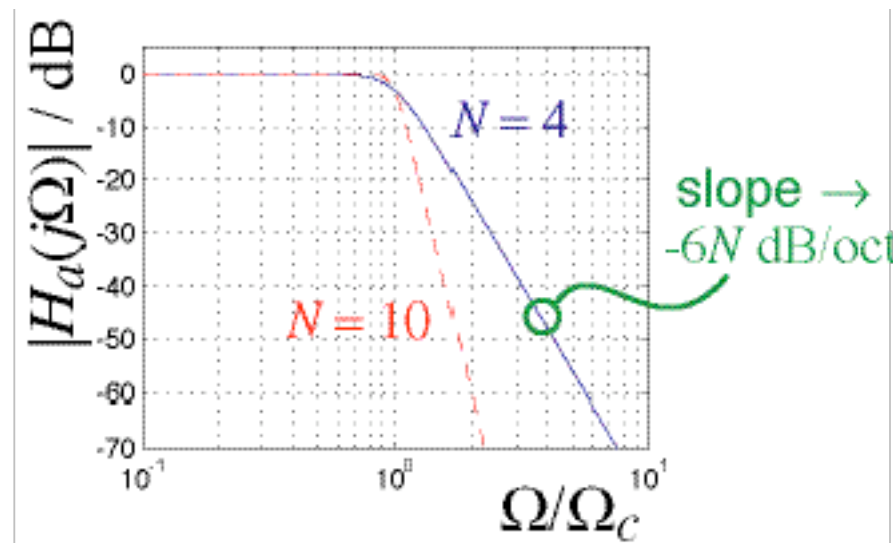


# Butterworth Filters

- $\Omega \gg \Omega_c, |H_a(j\Omega)|^2 \rightarrow (\Omega_c/\Omega)^{2N}$

6N dB/oct  
roll-off

Log-log  
magnitude  
response



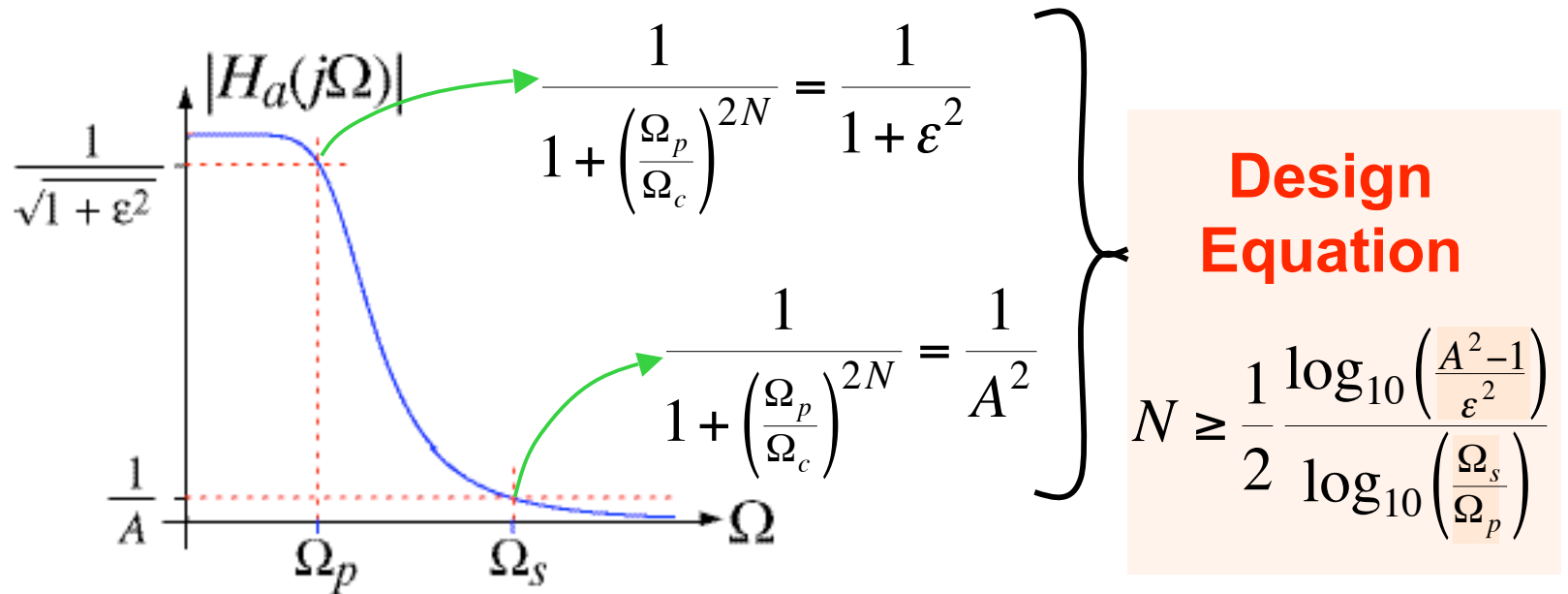
- *flat*  $\rightarrow \frac{d^n}{d\Omega^n} |H_a(j\Omega)|^2 = 0$

@  $\Omega = 0$  for  $n = 1 \dots 2N-1$



# Butterworth Filters

- How to meet design specifications?



- $$k_1 = \frac{\epsilon}{\sqrt{A^2 - 1}}$$

= "discrimination",  $\ll 1$

- $$k = \frac{\Omega_p}{\Omega_s}$$

= "selectivity",  $< 1$

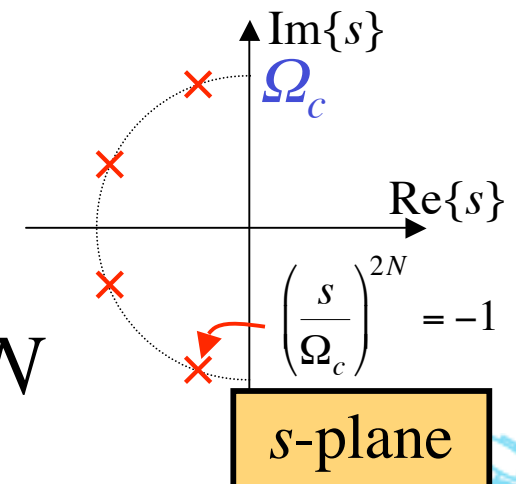


# Butterworth Filters

- $|H_a(j\Omega)|^2 = \frac{1}{1 + (\frac{\Omega}{\Omega_c})^{2N}}$  but what is  $H_a(s)$ ?
- Traditionally, look it up in a table
  - calculate  $N \rightarrow$  normalized filter with  $\Omega_c = 1$
  - **scale** all coefficients for desired  $\Omega_c$

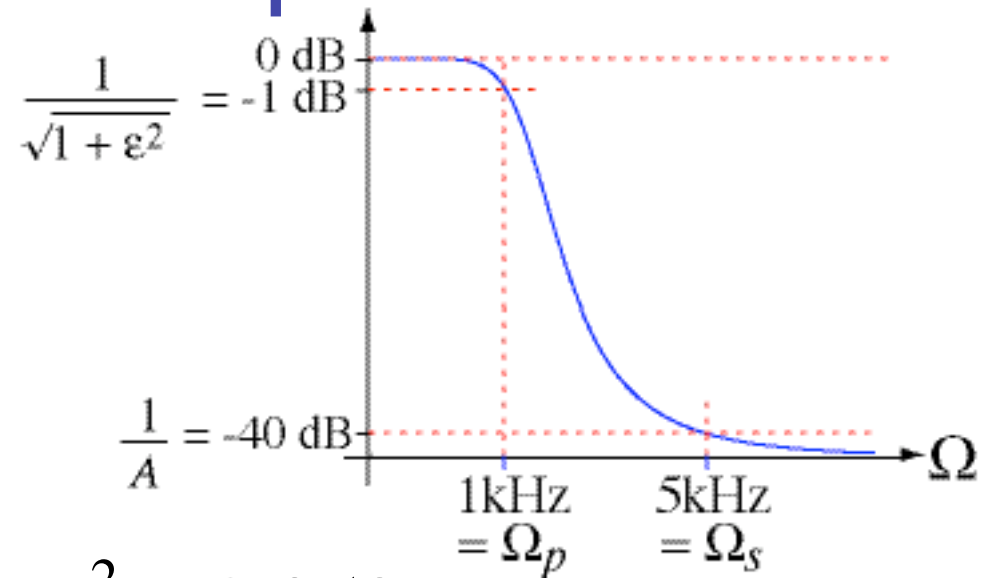
- In fact, 
$$H_a(s) = \frac{1}{\prod_i (s - p_i)}$$

where 
$$p_i = \Omega_c e^{j\pi \frac{N+2i-1}{2N}} \quad i = 1..N$$



# Butterworth Example

Design a Butterworth filter with 1 dB cutoff at 1kHz and a minimum attenuation of 40 dB at 5 kHz



$$-1\text{dB} = 20 \log_{10} \frac{1}{\sqrt{1+\epsilon^2}} \Rightarrow \epsilon^2 = 0.259$$

$$-40\text{dB} = 20 \log_{10} \frac{1}{A} \Rightarrow A = 100$$

$$\frac{\Omega_s}{\Omega_p} = 5$$

$$N \geq \frac{1}{2} \frac{\log_{10} \frac{9999}{0.259}}{\log_{10} 5}$$

$$\Rightarrow N = 4 \geq 3.28$$



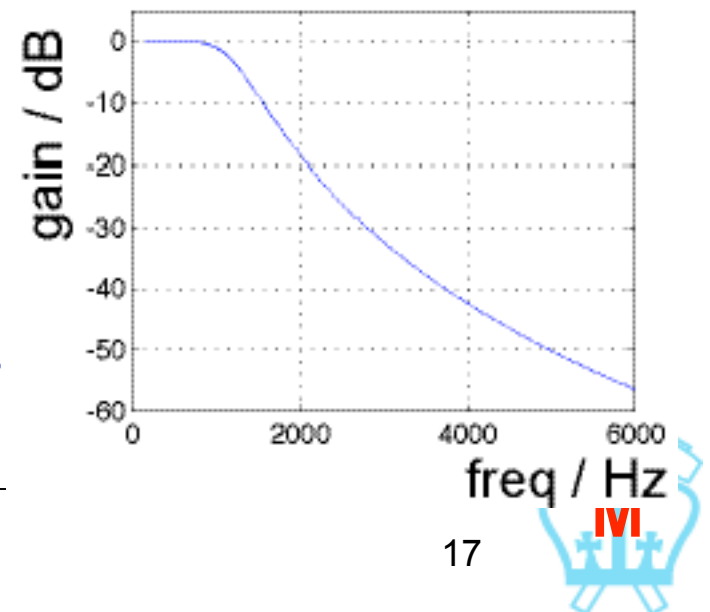
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# Butterworth Example

- Order  $N = 4$  will satisfy constraints;  
What are  $\Omega_c$  and filter coefficients?
  - from a table,  $\Omega_{-1\text{dB}} = 0.845$  when  $\Omega_c = 1$   
 $\Rightarrow \Omega_c = 1000/0.845 = 1.184$  kHz
  - from a table, get normalized coefficients for  
 $N = 4$ , scale by 1184
- Or, use Matlab:

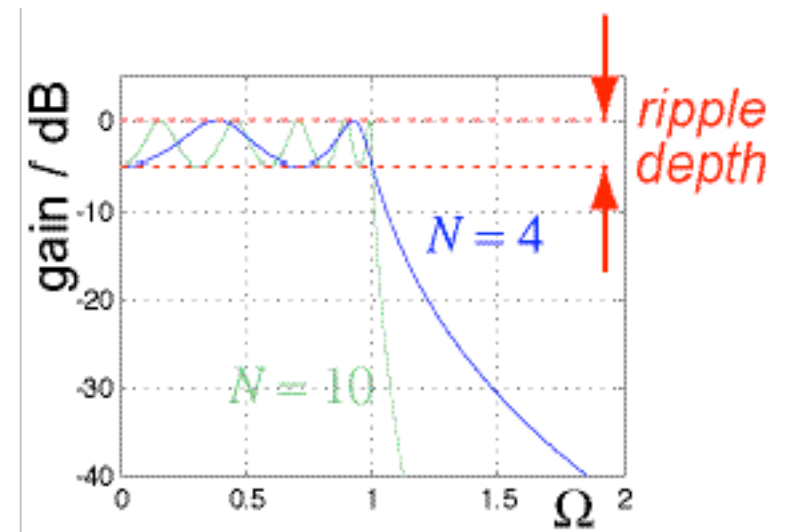
```
[b, a] =  
butter(N, Wc, 's');
```



# Chebyshev I Filter

- **Equiripple** in passband (flat in stopband)  
→ minimize **maximum** error

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$



*Chebyshev polynomial of order N* →  $T_N(\Omega) = \begin{cases} \cos(N \cos^{-1} \Omega) & |\Omega| \leq 1 \\ \cosh(N \cosh^{-1} \Omega) & |\Omega| > 1 \end{cases}$



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# Chebyshev I Filter

- Design procedure:
  - desired passband ripple  $\rightarrow \varepsilon$
  - min. stopband atten.,  $\Omega_p, \Omega_s \rightarrow N$  :

$$\frac{1}{A^2} = \frac{1}{1 + \varepsilon^2 T_N^2\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{1}{1 + \varepsilon^2 \left[ \cosh\left(N \cosh^{-1} \frac{\Omega_s}{\Omega_p}\right) \right]^2}$$

$$\Rightarrow N \geq \frac{\cosh^{-1}\left(\frac{\sqrt{A^2 - 1}}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$\leftarrow 1/k_1, \text{ discrimination}$

$\leftarrow 1/k, \text{ selectivity}$

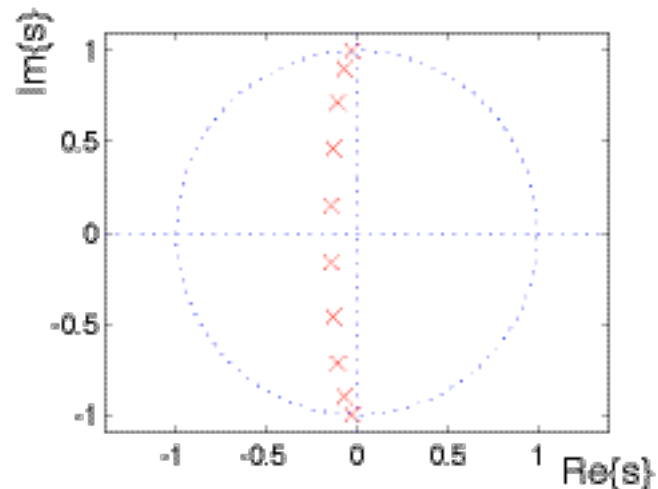


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# Chebyshev I Filter

- What is  $H_a(s)$ ?
  - complicated, get from a table
  - .. or from Matlab `cheby1(N,r,Wp,'s')`
  - all-pole; can inspect them:



..like squashed-in Butterworth

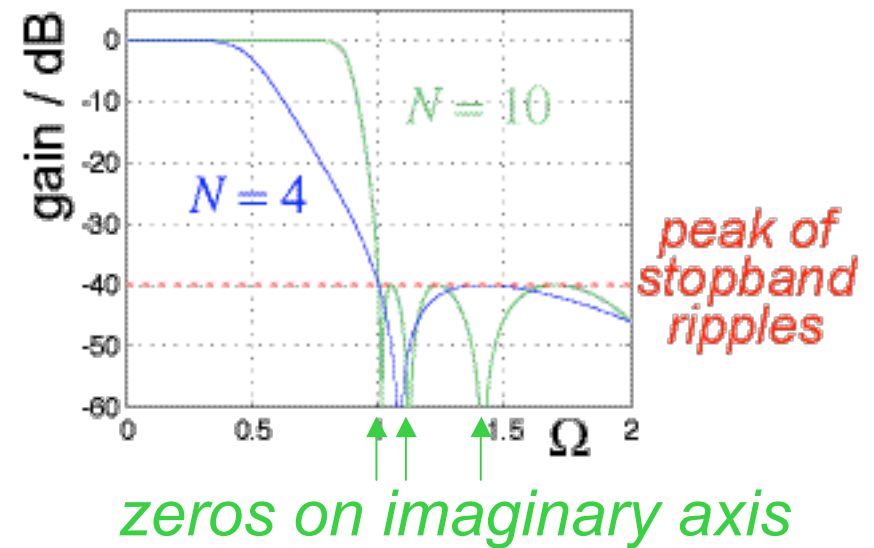


# Chebyshev II Filter

- Flat in **passband**, equiripple in **stopband**

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 \left( \frac{T_N\left(\frac{\Omega_s}{\Omega_p}\right)}{T_N\left(\frac{\Omega_s}{\Omega}\right)} \right)^2}$$

*constant* →  $T_N\left(\frac{\Omega_s}{\Omega_p}\right)$   
 $\sim 1/T_N(1/\Omega)$  →  $T_N\left(\frac{\Omega_s}{\Omega}\right)$



- Filter has poles **and zeros** (some)
- Complicated pole/zero pattern



# Elliptical (Cauer) Filters

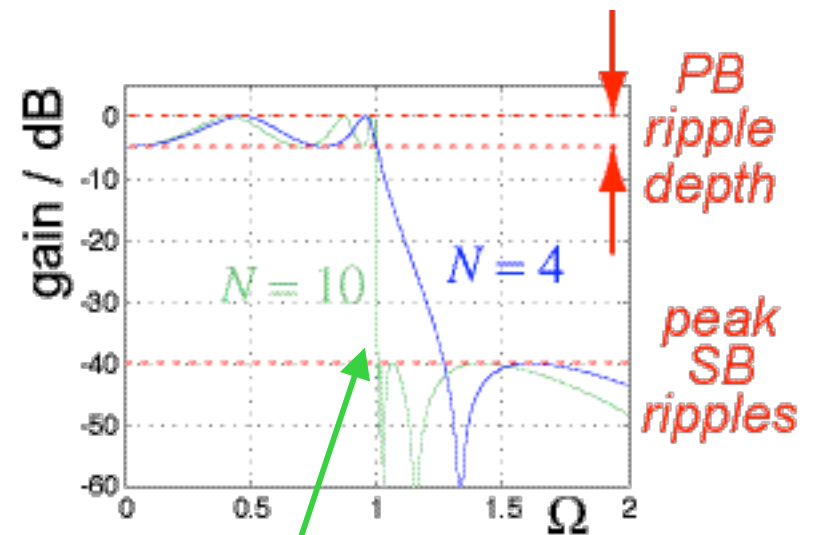
- Ripples in **both** passband and stopband

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 R_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$

*function; satisfies*

$$R_N(\Omega^{-1}) = R_N(\Omega)^{-1}$$

*zeros for  $\Omega > 1 \rightarrow$  poles for  $\Omega < 1$*

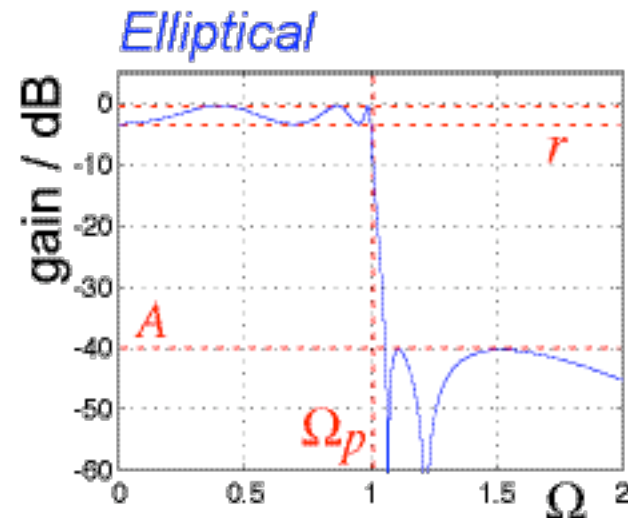
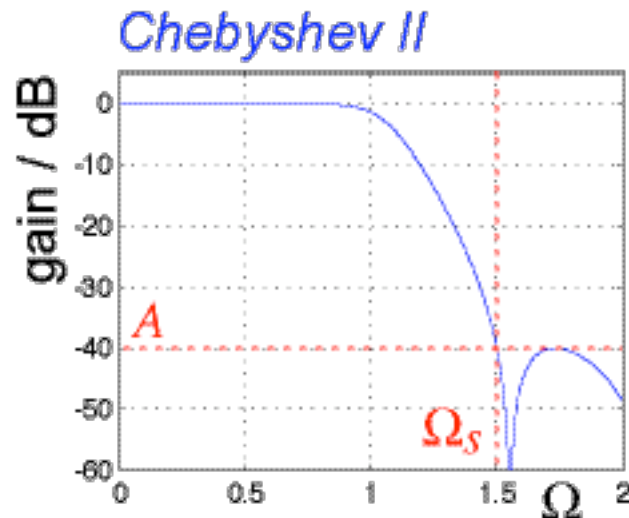
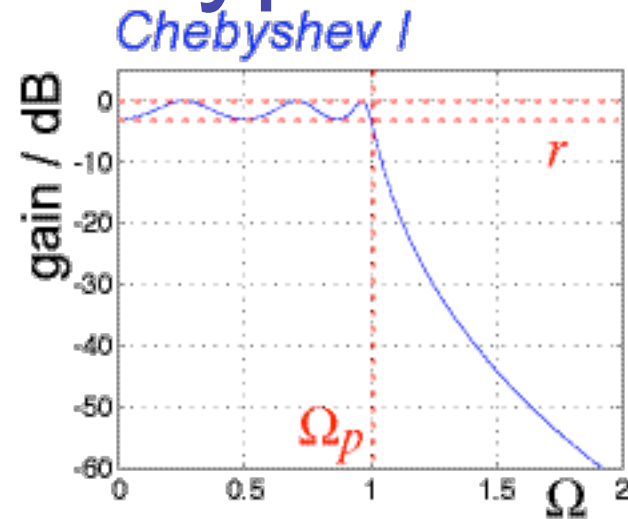
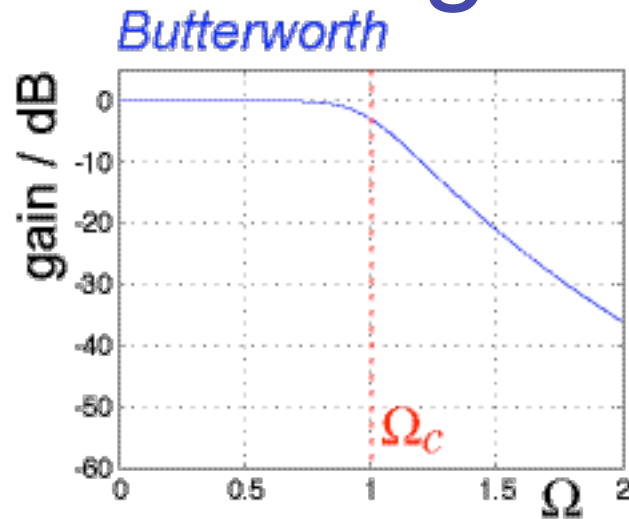


*very narrow transition band*

- Complicated; not even closed form for  $N$



# Analog Filter Types Summary



$$N = 6$$

$$r = 3 \text{ dB}$$

$$A = 40 \text{ dB}$$



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# Analog Filter Transformations

- All filter types shown as **lowpass**; other types (highpass, bandpass..) derived via **transformations**

- i.e.  $\hat{s} = F^{-1}(s)$   
*lowpass prototype*  $H_{LP}(s) \rightarrow H_D(\hat{s})$  *Desired alternate response; still a rational polynomial*

- General mapping of  $s$ -plane  
*BUT* keep  $j\Omega \rightarrow j\hat{\Omega}$ ;  
frequency response just ‘shuffled’



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# Lowpass-to-Highpass

- Example transformation:

$$H_{HP}(\hat{s}) = H_{LP}(s) \Big|_{s = \frac{\Omega_p \hat{\Omega}_p}{\hat{s}}}$$

- take prototype  $H_{LP}(s)$  polynomial
- replace  $s$  with  $\frac{\Omega_p \hat{\Omega}_p}{\hat{s}}$
- simplify and rearrange  
→ new polynomial  $H_{HP}(\hat{s})$



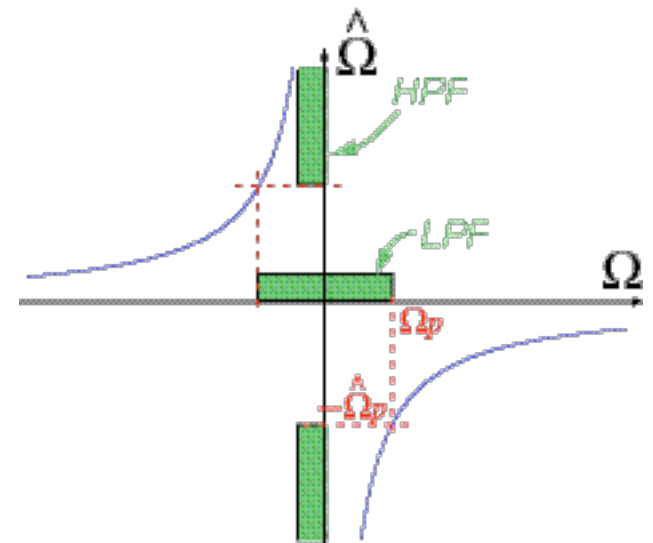
# Lowpass-to-Highpass

- What happens to frequency response?

$$s = j\Omega \Rightarrow \hat{s} = \frac{\Omega_p \hat{\Omega}_p}{j\Omega} = j \left( \frac{-\Omega_p \hat{\Omega}_p}{\Omega} \right) \text{ imaginary axis stays on self...}$$

$$\Rightarrow \hat{\Omega} = \frac{-\Omega_p \hat{\Omega}_p}{\Omega} \quad \dots \text{freq.} \rightarrow \text{freq.}$$

- $\Omega = \Omega_p \rightarrow \hat{\Omega} = -\hat{\Omega}_p$   
 $\Omega < \Omega_p \rightarrow \hat{\Omega} < -\hat{\Omega}_p$   
*LP passband*      *HP passband*
- $\Omega > \Omega_p \rightarrow \hat{\Omega} > -\hat{\Omega}_p$   
*LP stopband*      *HP stopband*



- Frequency axes inverted



# Transformation Example

Design a Butterworth highpass filter with PB edge -0.1dB @ 4 kHz ( $\hat{\Omega}_p$ ) and SB edge -40 dB @ 1 kHz ( $\hat{\Omega}_s$ )

- Lowpass prototype: make  $\Omega_p = 1$

$$\Rightarrow \Omega_s = (-) \frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}_s} = (-)4$$

- Butterworth -0.1dB @  $\Omega_p=1$ , -40dB @  $\Omega_s=4$

$$N \geq \frac{1}{2} \frac{\log_{10}\left(\frac{A^2-1}{\epsilon^2}\right)}{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$\Omega_p$  @ -0.1dB  $\Rightarrow \frac{1}{1 + \left(\frac{\Omega_p}{\Omega_c}\right)^{10}} = 10^{\frac{-0.1}{10}}$

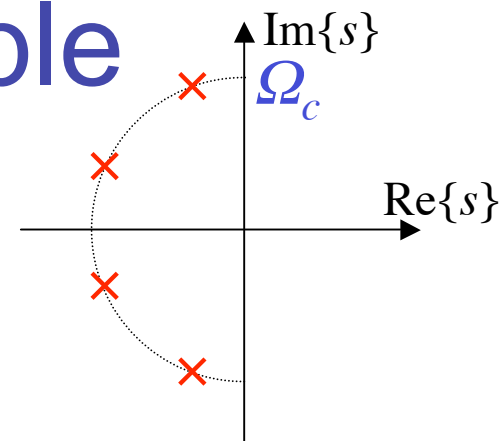
$$\rightarrow N = 5 \quad \rightarrow \Omega_c = \Omega_p / 0.6866 = 1.4564$$



# Transformation Example

- LPF proto has  $p_\ell = \Omega_c e^{j\pi \frac{N+2\ell-1}{2N}}$

$$\Rightarrow H_{LP}(s) = \frac{\Omega_c^N}{\prod_{i=1}^N (s - p_{i1})}$$



- Map to HPF:  $H_{HP}(\hat{s}) = H_{LP}(s) \Big|_{s = \frac{\Omega_p \hat{\Omega}_p}{\hat{s}}}$

$$\Rightarrow H_{HP}(\hat{s}) = \frac{\Omega_c^N}{\prod_{\ell=1}^N \left( \frac{\Omega_p \hat{\Omega}_p}{\hat{s}} - p_\ell \right)} = \frac{\Omega_c^N \hat{s}^N}{\prod_{\ell=1}^N \left( \Omega_p \hat{\Omega}_p - p_\ell \hat{s} \right)}$$

*N zeros @  $\hat{s} = 0$*

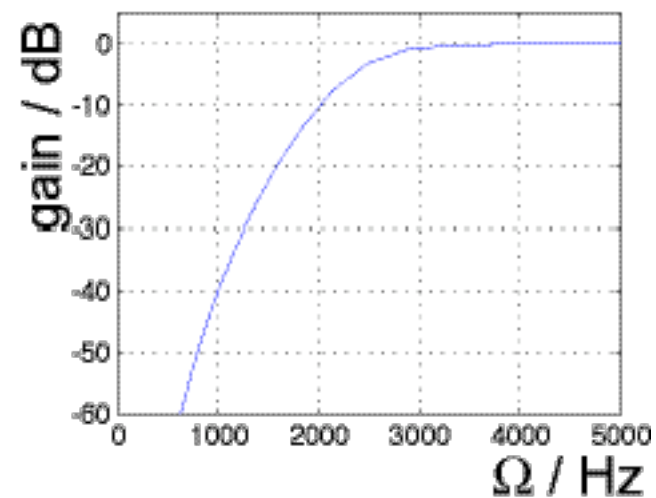
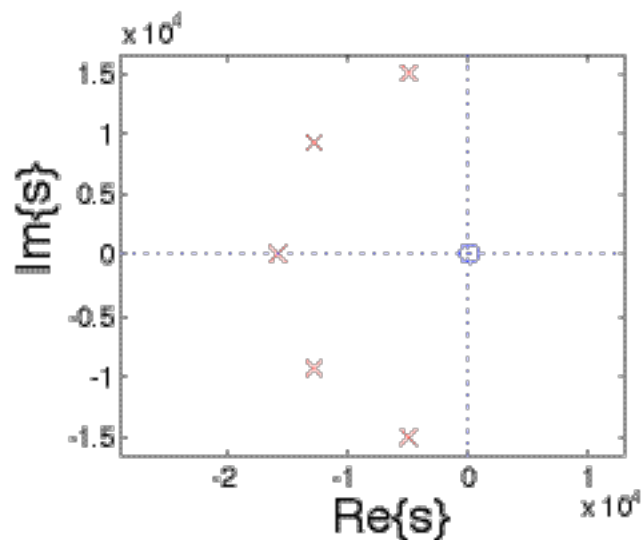
*new poles @  $\hat{s} = \Omega_p \hat{\Omega}_p / p_l$*



# Transformation Example

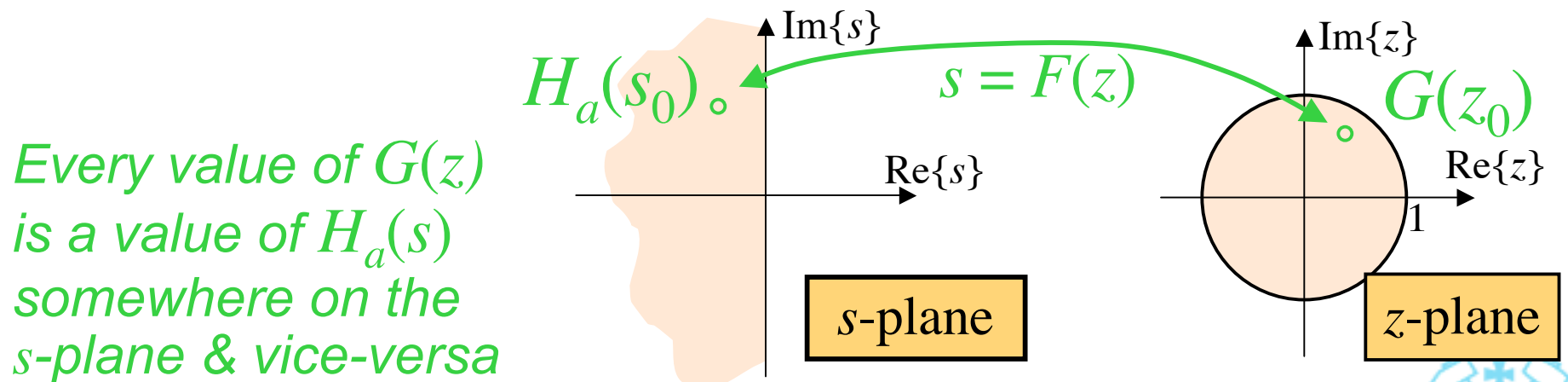
- In Matlab:

```
[N,Wc]=buttord(1,4,0.1,40,'s');  
[B,A]=butter(N,Wc,'s');  
[n,d]=lp2hp(B,A,2*pi*4000);
```



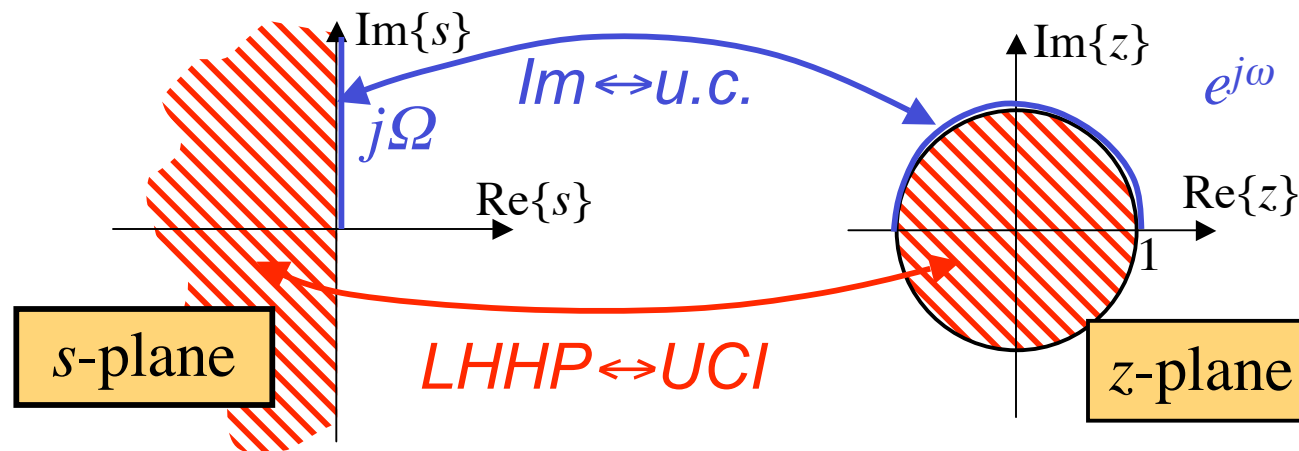
### 3. Analog Protos $\rightarrow$ IIR Filters

- Can we map high-performance CT filters to DT domain?
- Approach: **transformation**  $H_a(s) \rightarrow G(z)$   
i.e.  $G(z) = H_a(s)|_{s=F(z)}$   
where  $s = F(z)$  maps  $s$ -plane  $\leftrightarrow$   $z$ -plane:



# CT to DT Transformation

- Desired properties for  $s = F(z)$ :
  - $s$ -plane  $j\Omega$  axis  $\leftrightarrow$   $z$ -plane unit circle  
→ preserves frequency response values
  - $s$ -plane LHP  $\leftrightarrow$   $z$ -plane unit circle interior  
→ preserves stability of poles



# Bilinear Transformation

- Solution:

$$s = \frac{1 - z^{-1}}{1 + z^{-1}} \quad \text{Bilinear Transform}$$

- Hence inverse:  $z = \frac{1 + s}{1 - s}$  *unique, 1:1 mapping*

- Freq. axis?  $s = j\Omega \rightarrow z = \frac{1 + j\Omega}{1 - j\Omega}$   *$|z| = 1$  i.e. on unit circle*

- Poles?  $s = \sigma + j\Omega \rightarrow z = \frac{(1 + \sigma) + j\Omega}{(1 - \sigma) - j\Omega}$

$$\Rightarrow |z|^2 = \frac{1 + 2\sigma + \sigma^2 + \Omega^2}{1 - 2\sigma + \sigma^2 + \Omega^2}$$

$$\sigma < 0$$

$$\Leftrightarrow |z| < 1$$

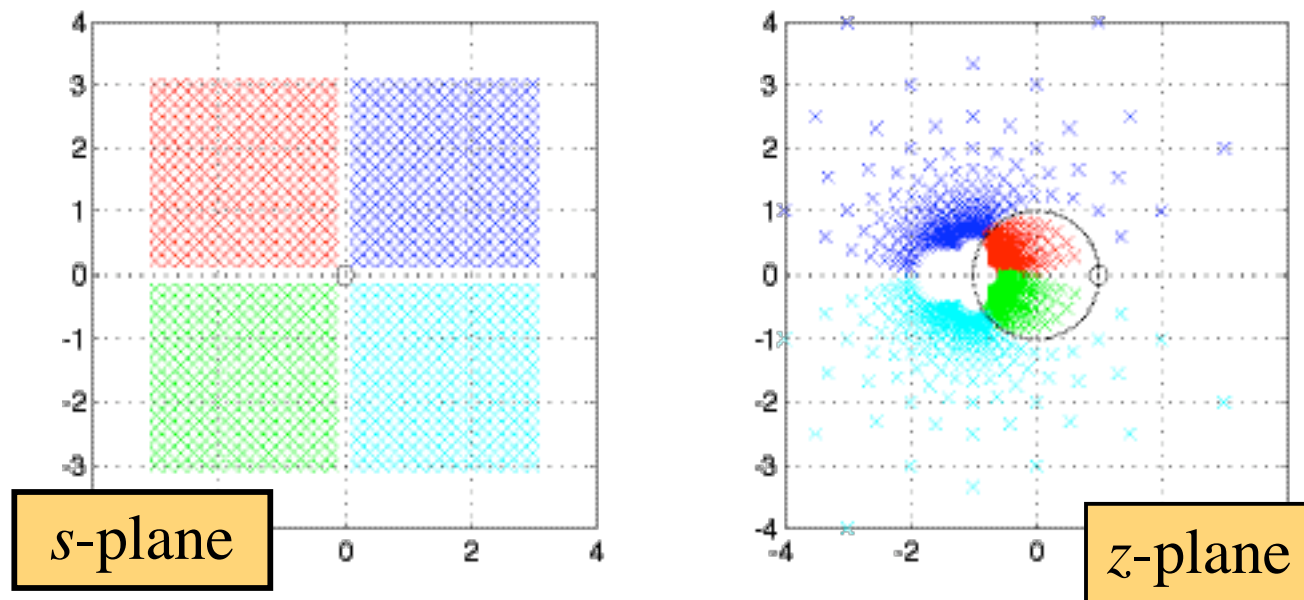


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# Bilinear Transformation

- How can entire half-plane fit inside u.c.?



- Highly nonuniform warping!



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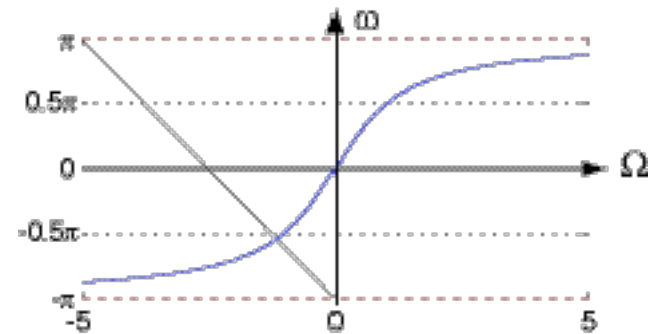
# Bilinear Transformation

- What is CT  $\leftrightarrow$  DT freq. relation  $\Omega \leftrightarrow \omega$  ?

$$z = e^{j\omega} \Rightarrow s = \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \frac{2j \sin \omega/2}{2 \cos \omega/2} = j \tan \frac{\omega}{2} \text{ im.axis}$$

*u.circle*

- i.e.  $\Omega = \tan\left(\frac{\omega}{2}\right)$   
 $\omega = 2 \tan^{-1} \Omega$



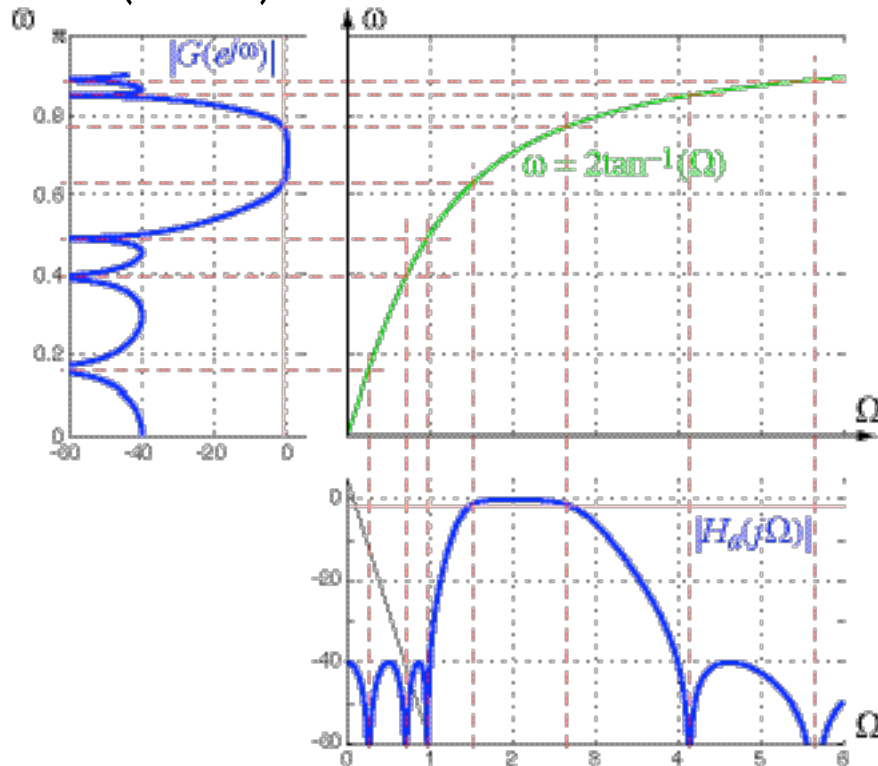
- *infinite* range of CT frequency  $-\infty < \Omega < \infty$   
maps to *finite* DT freq. range  $-\pi < \omega < \pi$
- nonlinear;  $\frac{d}{d\omega} \Omega \rightarrow \infty$  as  $\omega \rightarrow \pi$  *pack it all in!*



# Frequency Warping

- Bilinear transform makes

$$G(e^{j\omega}) = H_a(j\Omega) \Big|_{\omega=2 \tan^{-1} \Omega} \quad \text{for all } \omega, \Omega$$



- Same gain & phase ( $\varepsilon$ ,  $A\dots$ ), in same 'order', but with *warped* frequency axis



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# Design Procedure

- Obtain **DT** filter specs:
  - general form (LP, HP...),  $\omega_p, \omega_s, \frac{1}{\sqrt{1+\epsilon^2}}, \frac{1}{A}$
- ‘Warp’ frequencies to **CT**:
  - $\Omega_p = \tan \frac{\omega_p}{2} \quad \Omega_s = \tan \frac{\omega_s}{2}$

*Old-style*

- Design analog filter for  $\Omega_p, \Omega_s, \frac{1}{\sqrt{1+\epsilon^2}}, \frac{1}{A}$ 
  - $\rightarrow H_a(s)$ , **CT** filter polynomial

- Convert to **DT** domain:  $G(z) = H_a(s) \Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$ 
  - $\rightarrow G(z)$ , rational polynomial in  $z$

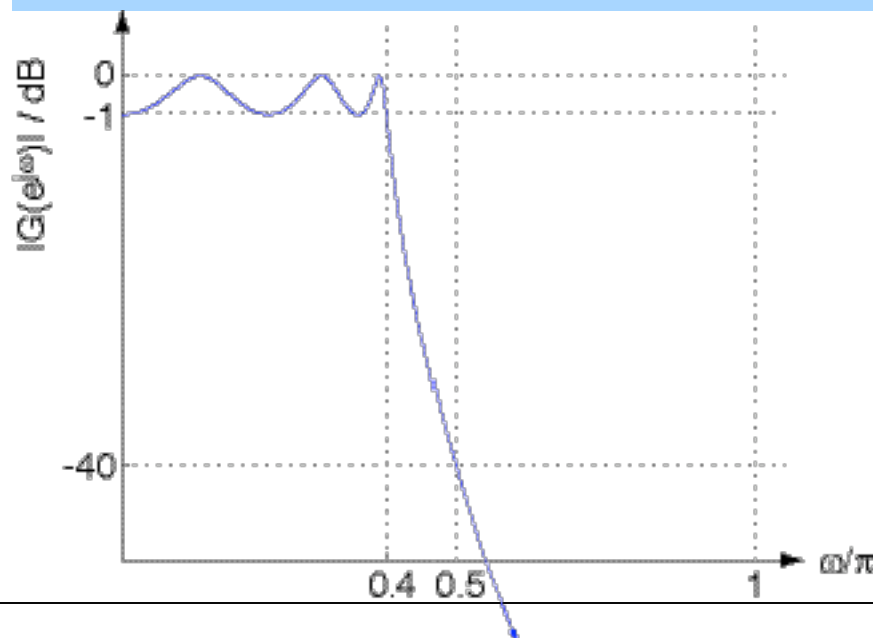
- Implement digital filter!



# Bilinear Transform Example

- DT domain requirements:

Lowpass, 1 dB ripple in PB,  $\omega_p = 0.4\pi$ ,  
SB attenuation  $\geq 40$  dB @  $\omega_s = 0.5\pi$ ,  
attenuation increases with frequency



- SB ripples,  
PB monotonic  
→ Chebyshev I



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# Bilinear Transform Example

- Warp to CT domain:

$$\Omega_p = \tan \frac{\omega_p}{2} = \tan 0.2\pi = 0.7265 \text{ rad/sec}$$

$$\Omega_s = \tan \frac{\omega_s}{2} = \tan 0.25\pi = 1.0 \text{ rad/sec}$$

- Magnitude specs:

1 dB PB ripple

$$\Rightarrow \frac{1}{\sqrt{1+\varepsilon^2}} = 10^{-1/20} = 0.8913 \Rightarrow \varepsilon = 0.5087$$

40 dB SB atten.

$$\Rightarrow \frac{1}{A} = 10^{-40/20} = 0.01 \Rightarrow A = 100$$



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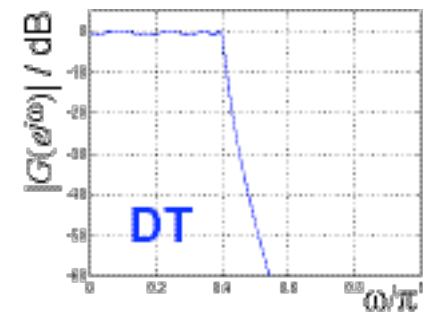
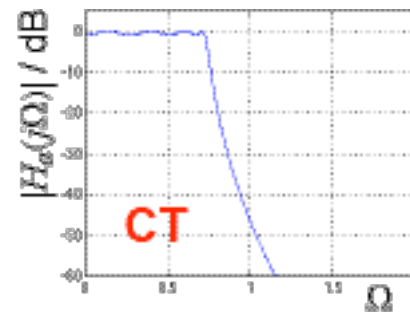
# Bilinear Transform Example

- Chebyshev I design criteria:

$$N \geq \frac{\cosh^{-1}\left(\frac{\sqrt{A^2-1}}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} = 7.09 \quad \text{i.e. need } N = 8$$

- Design analog filter, map to DT, check:

```
>> N=8;  
>> wp=0.7265;  
>> [B,A]=cheby1(N,1,wp,'s');  
>> [b,a] = bilinear(B,A,.5);
```

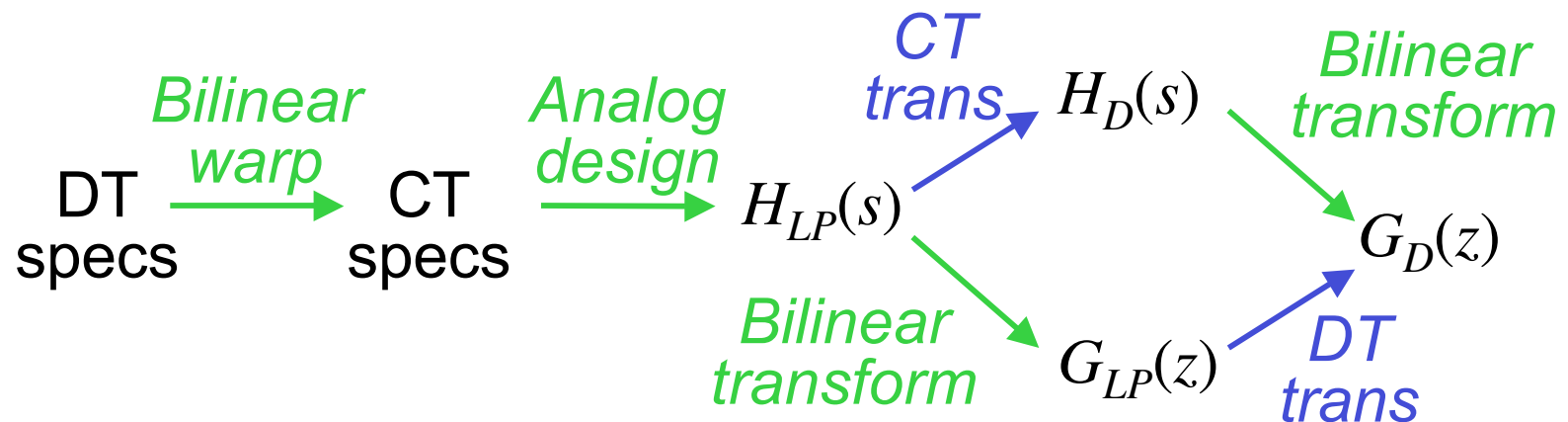


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# Other Filter Shapes

- Example was IIR LPF from LP prototype
- For other shapes (HPF, bandpass,...):



- Transform LP  $\rightarrow$  X in CT or DT domain...



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# DT Spectral Transformations

- Same idea as CT LPF→HPF mapping, but in  $z$ -domain:

$$G_D(\hat{z}) = G_L(z)|_{z=F(\hat{z})} = G_L(F(\hat{z}))$$

- To behave well,  $z = F(\hat{z})$  should:
  - map u.c.  $\rightarrow$  u.c. (preserve  $G(e^{j\omega})$  values)
  - map u.c. interior  $\rightarrow$  u.c. interior (stability)
- i.e.  $|F(\hat{z})| = 1 \Leftrightarrow |\hat{z}| = 1$      $|F(\hat{z})| < 1 \Leftrightarrow |\hat{z}| < 1$ 
  - in fact,  $F(\hat{z})$  matches the definition of an **allpass filter** ... replace delays with  $F(\hat{z})^{-1}$



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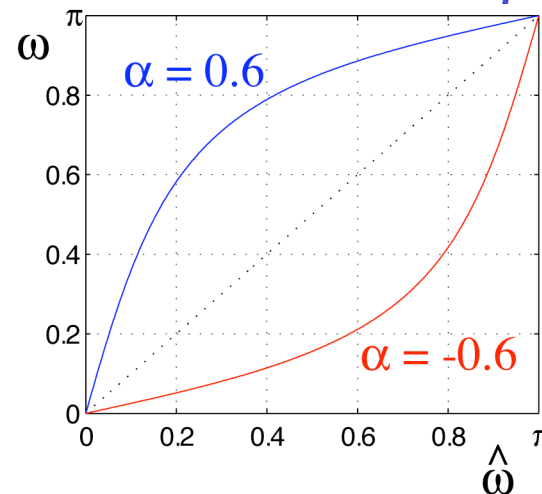
# DT Frequency Warping

- Simplest mapping  $z = F(\hat{z}) = \frac{\hat{z} - \alpha}{1 - \alpha\hat{z}}$   
has effect of warping frequency axis:

$$\hat{z} = e^{j\hat{\omega}} \Rightarrow z = e^{j\omega} = \frac{e^{j\hat{\omega}} - \alpha}{1 - \alpha e^{j\hat{\omega}}}$$

$\alpha > 0$  :  
expand HF

$$\Rightarrow \tan\left(\frac{\omega}{2}\right) = \frac{1+\alpha}{1-\alpha} \tan\left(\frac{\hat{\omega}}{2}\right)$$



$\alpha < 0$  :  
expand LF



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# Another Design Example

- Spec:
  - Bandpass, from 800-1600 Hz (SR = 8kHz)
  - Ripple = 1dB, min. stopband atten. = 60 dB
  - 8th order, best transition band
- Use **elliptical** for best performance
- Full design path:
  - design analog LPF prototype
  - analog LPF → BPF
  - CT BPF → DT BPF (Bilinear)



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# Another Design Example

- Or, do it all in one step in Matlab:

```
[b,a] = ellip(8,1,60,  
            [800 1600]/(8000/2));
```

