ELEN E4810: Digital Signal Processing
Topic 4: The Z Transform

1. The Z Transform
2. Inverse Z Transform

1. The Z Transform

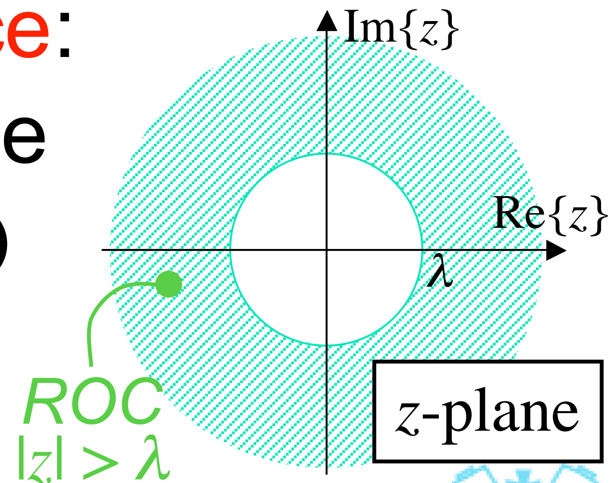
- Powerful tool for analyzing & designing DT systems
- Generalization of the DTFT:

$$G(z) = \mathcal{Z}\{g[n]\} = \sum_{n=-\infty}^{\infty} g[n] z^{-n}$$ *Z Transform*

- z is complex...
  - $z = e^{j\omega} \rightarrow$ DTFT
  - $z = r \cdot e^{j\omega} \rightarrow \sum_n g[n] r^{-n} e^{-j\omega n}$ *DTFT of* $r^{-n} \cdot g[n]$

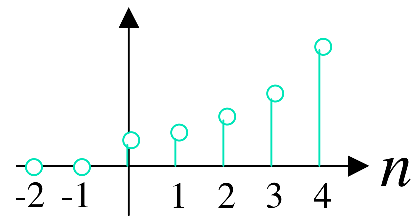
Region of Convergence (ROC)

- Critical question:
  Does summation $G(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ *converge* (to a finite value)?
- In general, depends on the value of z
  $\rightarrow$ Region of Convergence:
  Portion of complex z-plane for which a *particular* $G(z)$ will converge



ROC Example

- e.g. $x[n] = \lambda^n \mu[n]$

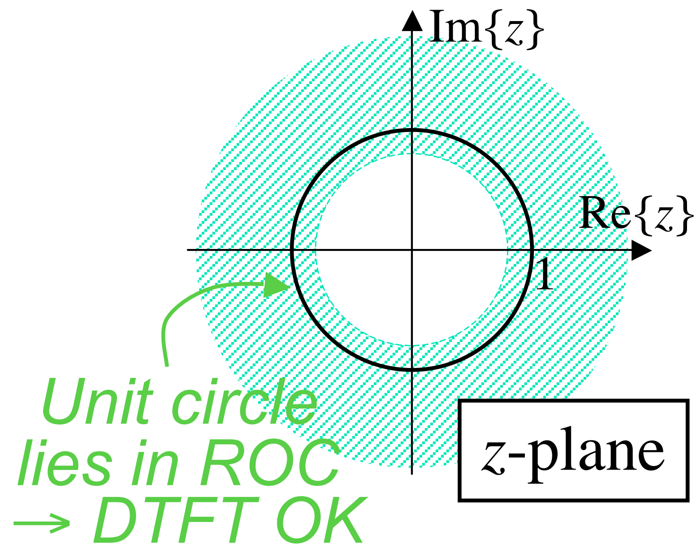


$$\Rightarrow X(z) = \sum_{n=0}^{\infty} \lambda^n z^{-n} = \frac{1}{1 - \lambda z^{-1}}$$

- Σ converges for $|\lambda z^{-1}| < 1$
  i.e. ROC is $|z| > |\lambda|$ *(see previous slide)*
- $|\lambda| < 1$ (e.g. 0.8) - finite energy sequence
- $|\lambda| > 1$ (e.g. 1.2) - divergent sequence, infinite energy, DTFT does not exist but still has ZT when $|z| > 1.2$ (ROC)

About ROCs

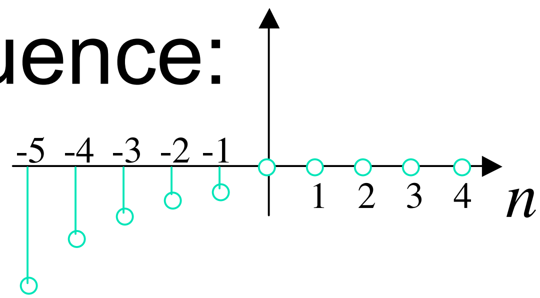
- ROCs always defined in terms of $|z|$
  $\rightarrow$ circular regions on z-plane
  (inside circles/outside circles/rings)
- If ROC includes unit circle ($|z| = 1$),
  $\rightarrow g[n]$ has a DTFT
  (finite energy sequence)



Another ROC example

- Anticausal (left-sided) sequence:

$$x[n] = -\lambda^n \mu[-n-1]$$



$$X(z) = \sum_n \left(-\lambda^n \mu[-n-1]\right) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} \lambda^n z^{-n} = -\sum_{m=1}^{\infty} \lambda^{-m} z^n$$

ROC: $|\lambda| > |z|$

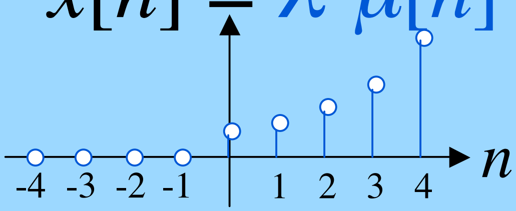
$$= -\lambda^{-1} z \frac{1}{1-\lambda^{-1}z} = \frac{1}{1-\lambda z^{-1}}$$

- Same ZT as $\lambda^n \mu[n]$, different sequence?

ROC is necessary!

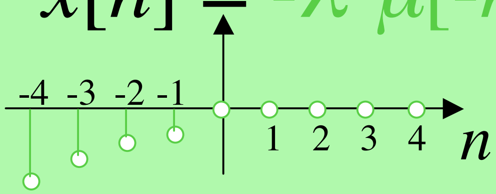
- To completely define a ZT, **you must specify the ROC**:

$$x[n] = \lambda^n \mu[n] \rightarrow X(z) = \frac{1}{1 - \lambda z^{-1}}$$

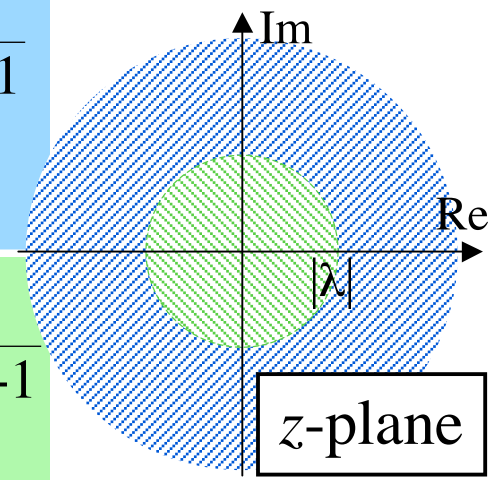


ROC $|z| > |\lambda|$

$$x[n] = -\lambda^n \mu[-n-1] \rightarrow X(z) = \frac{1}{1 - \lambda z^{-1}}$$



ROC $|z| < |\lambda|$



- A single $G(z)$ can describe several sequences with different ROCs

DTFTs?

Rational Z-transforms

- $G(z)$ can be any function;
  rational polynomials are important class:

$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \ldots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \ldots + d_{N-1} z^{-(N-1)} + d_N z^{-N}}$$

  - By convention, expressed in terms of z^{-1}
    – matches ZT definition
  - (Reminiscent of LCCDE expression...)

Factored rational ZTs

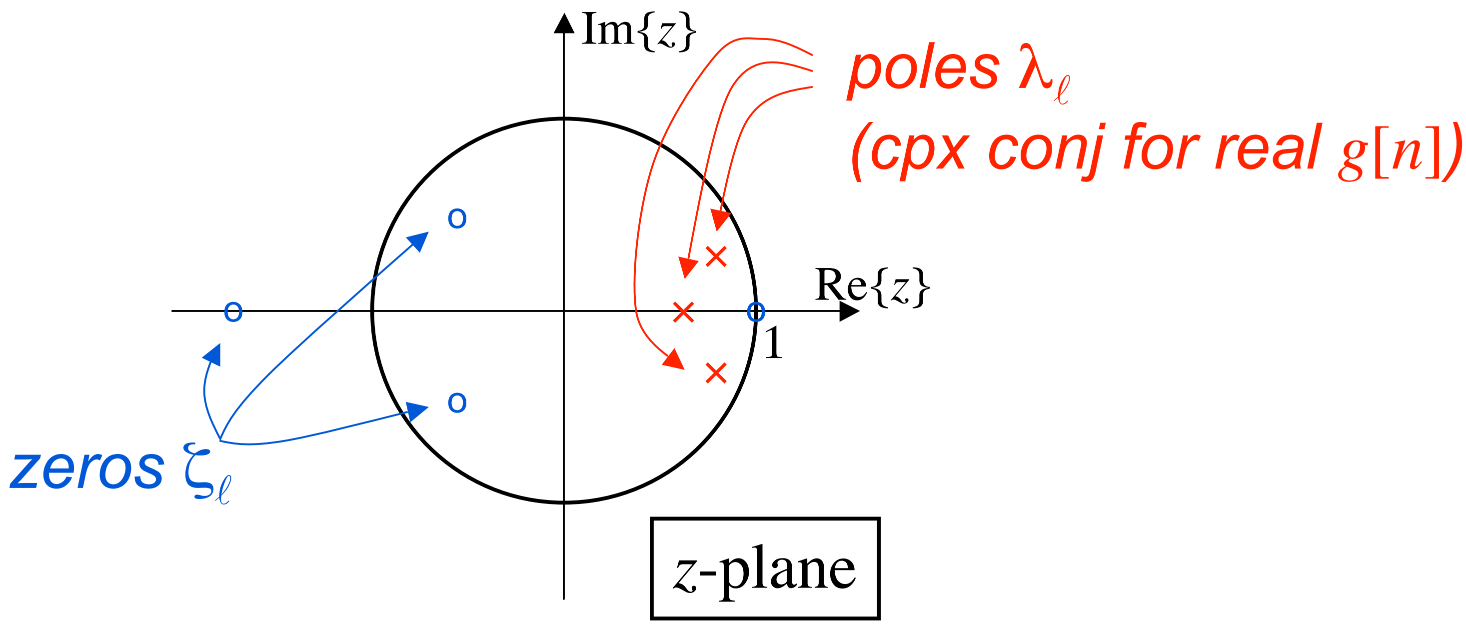
- Numerator, denominator can be factored:

$$G(z) = \frac{p_0 \prod_{\ell=1}^{M} \left(1 - \zeta_\ell z^{-1}\right)}{d_0 \prod_{\ell=1}^{N} \left(1 - \lambda_\ell z^{-1}\right)} = \frac{z^M p_0 \prod_{\ell=1}^{M} (z - \zeta_\ell)}{z^N d_0 \prod_{\ell=1}^{N} (z - \lambda_\ell)}$$

  - $\{\zeta_\ell\}$ are roots of *numerator* $\rightarrow G(z) = 0 \rightarrow \{\zeta_\ell\}$ **are the zeros of** $G(z)$
  - $\{\lambda_\ell\}$ are roots of *denominator* $\rightarrow G(z) = \infty \rightarrow \{\lambda_\ell\}$ **are the poles of** $G(z)$

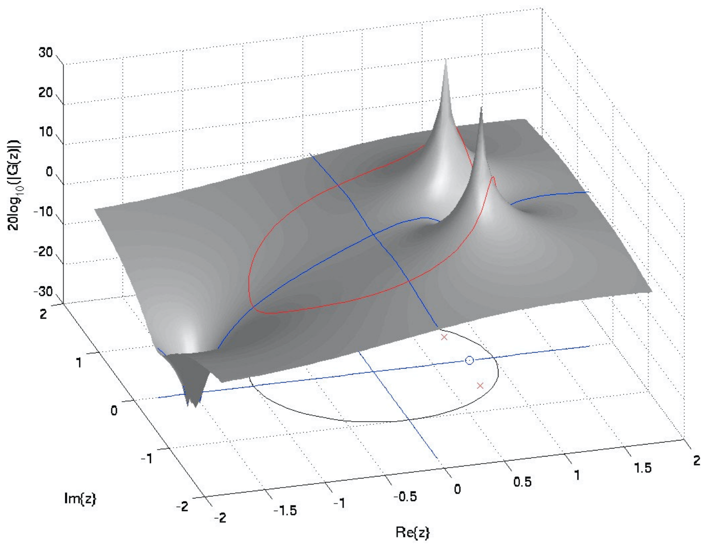
Pole-zero diagram

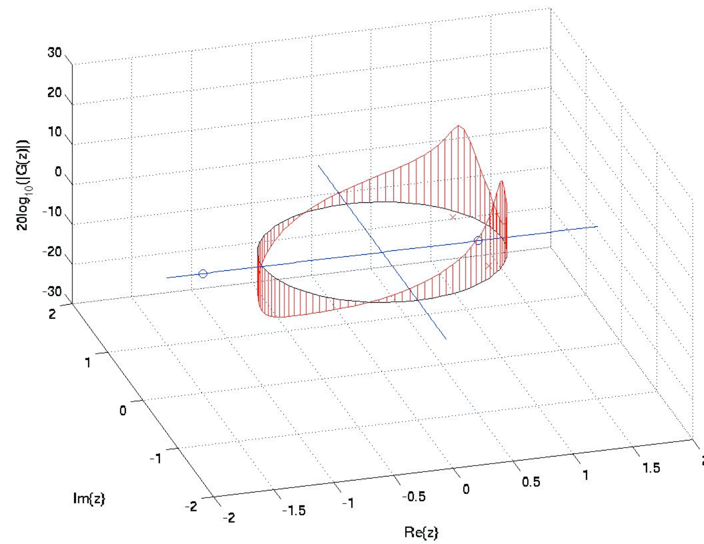
- Can plot poles and zeros on complex z-plane:



Z-plane surface

- $G(z)$: cpx *function* of a cpx *variable*
  - Can calculate value over entire z-plane

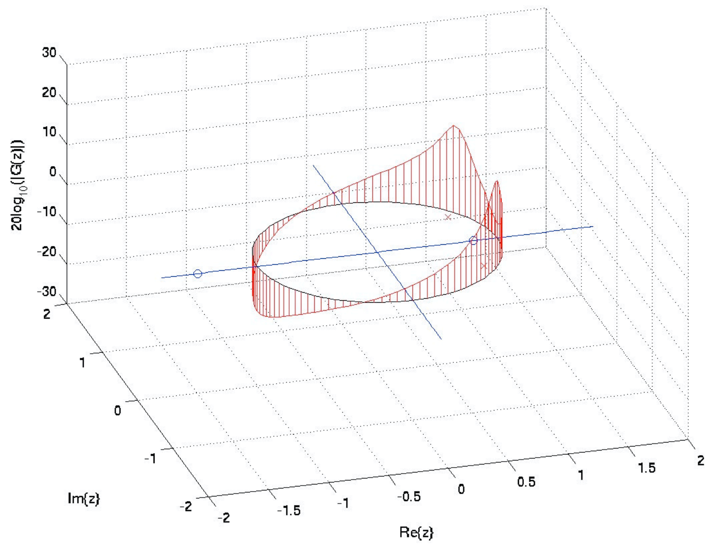


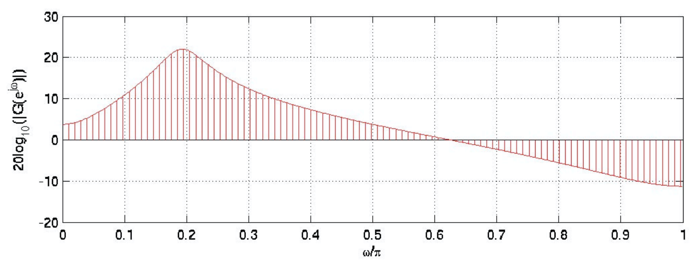


- Slice between surface and unit cylinder ($|z| = 1 \Rightarrow z = e^{j\omega}$) is $G(e^{j\omega})$, the DTFT

Z-plane and DTFT

- Unwrapping the cylindrical slice gives the DTFT:





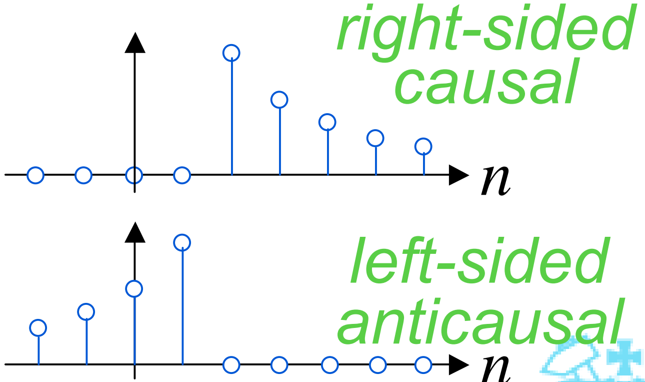
ROCs and sidedness

- LCCDEs have solutions of form:

$$y_c[n] = \alpha_i \lambda_i^{\,n} \mu[n] + \ldots$$

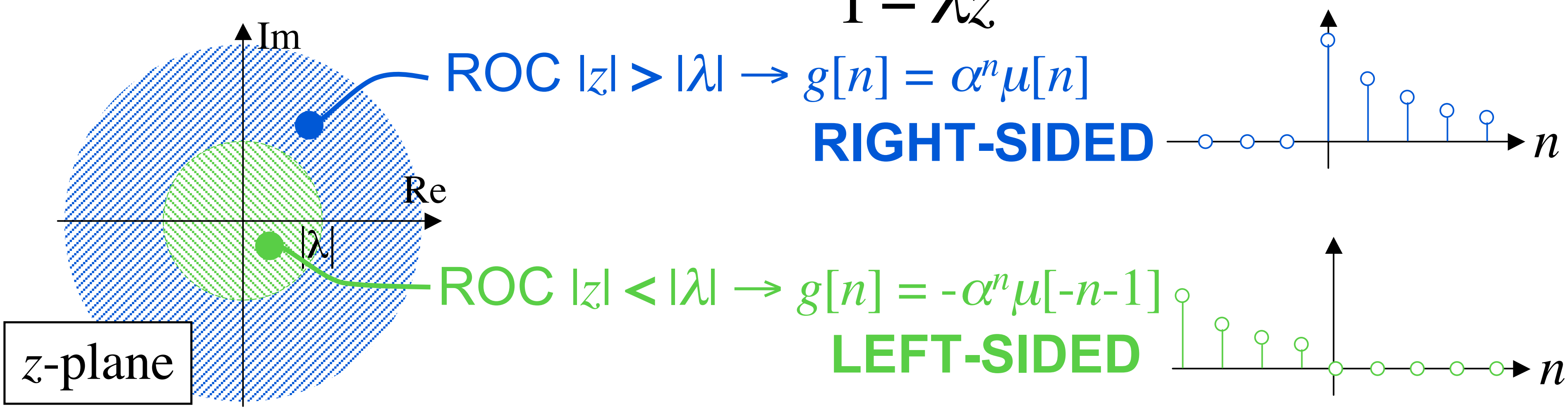
- Hence ZT with terms like $\dfrac{\alpha_i}{1-\lambda_i z^{-1}} + \cdots$ (same λs)
- Each pole λ_i of $G(z)$ corresponds to a term λ_i^n in $g[n]$
- but is it $\lambda_i^n\ \mu[n - n_0]$

  or $\lambda_i^n\ \mu[n_0 - n]$?



ROCs and sidedness

- We saw for $G(z) = \dfrac{1}{1 - \lambda z^{-1}}$:

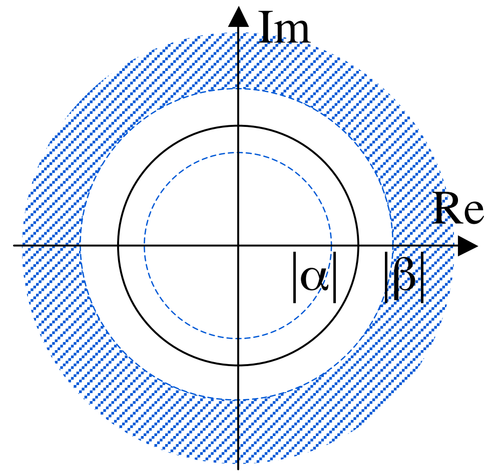


- Each pole → region in ROC outside or inside $|\lambda|$ for RH or LH sided term in $g[n]$
  - Overall ROC is intersection of each term's

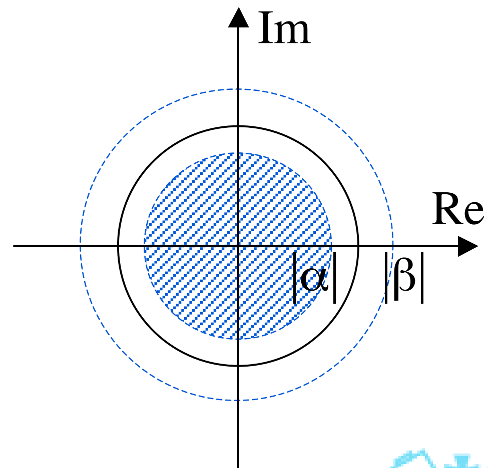
ROC intersections

2 poles, $|\alpha| < |\beta|$, give several possible ROCs:

1. $\alpha^n \mu[n] + \beta^n \mu[n]$
   both right-sided
   ROC $|z| > |\alpha|$, $|z| > |\beta|$



2. $\alpha^n \mu[-n] + \beta^n \mu[-n]$
   both left-sided
   ROC $|z| < |\alpha|$, $|z| < |\beta|$

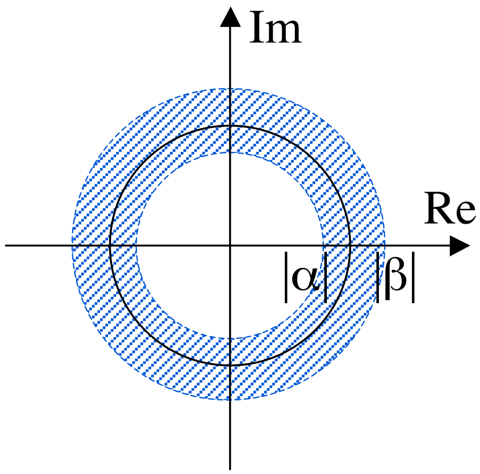


ROC intersections

3. $\alpha^n \mu[n] + \beta^n \mu[-n]$

   two-sided

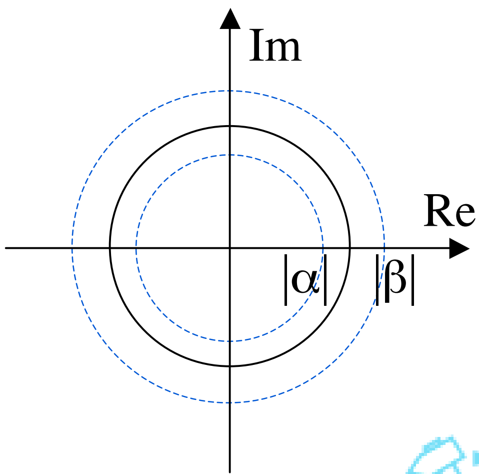
   ROC $|z| > |\alpha|$, $|z| < |\beta|$



4. $\alpha^n \mu[-n] + \beta^n \mu[n]$

   two-sided

   ROC $|z| < |\alpha|$, $|z| > |\beta|$

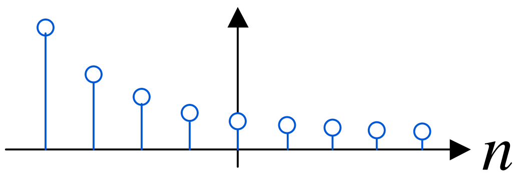


- no ROC means...

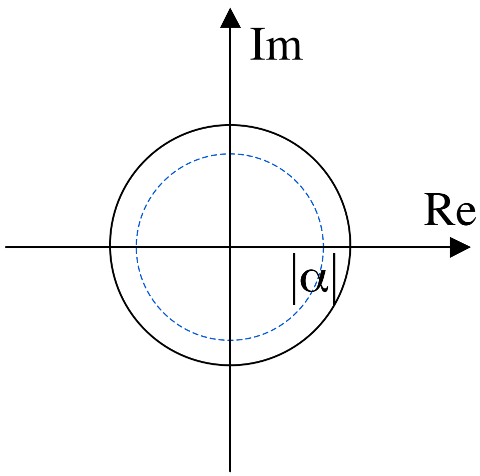
ROC intersections

- Note: Two-sided exponential

$$g[n] = \alpha^n \qquad -\infty < n < \infty$$



$$= \underbrace{\alpha^n \mu[n]}_{\substack{\text{ROC} \\ |z| > |\alpha|}} + \underbrace{\alpha^n \mu[-n-1]}_{\substack{\text{ROC} \\ |z| < |\alpha|}}$$



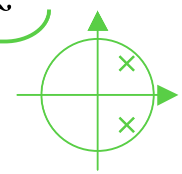
- **No** overlap in ROCs
  → ZT *does not exist*

Some common Z transforms

$g[n]$	$G(z)$	ROC
$\delta[n]$	1	$\forall z$
$\mu[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n \mu[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$r^n \cos(\omega_0 n)\mu[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > r$
$r^n \sin(\omega_0 n)\mu[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > r$

sum of $re^{j\omega_0 n} + re^{-j\omega_0 n}$

poles at $z = re^{\pm j\omega_0}$



"conjugate pole pair"

Z Transform properties

	$g[n]$	$\leftrightarrow$ $G(z)$	w/ROC $\mathcal{R}g$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}g$
Time reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}g$
Time shift	$g[n-n_0]$	$z^{-n_0}G(z)$	$\mathcal{R}g$ $(0/\infty?)$
Exp. scaling	$\alpha^n g[n]$	$G(z/\alpha)$	$\lvert\alpha\rvert\mathcal{R}g$
Diff. wrt z	$ng[n]$	$-z\dfrac{dG(z)}{dz}$	$\mathcal{R}g$ $(0/\infty?)$

Z Transform properties

	$g[n]$	$G(z)$	*ROC*
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	*at least* $\mathcal{R}g \cup \mathcal{R}h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j}\oint_C G(v)H\left(z/v\right)v^{-1}dv$	*at least* $\mathcal{R}g\,\mathcal{R}h$

Parseval: $$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j}\oint_C G(v)H^*\left(1/v\right)v^{-1}dv$$

ZT Example

- $x[n] = r^n \cos(\omega_0 n)\mu[n]$; can express as

$$\tfrac{1}{2}\mu[n]\left(\left(re^{j\omega_0}\right)^n + \left(re^{-j\omega_0}\right)^n\right) = v[n] + v^*[n]$$

$v[n] = {}^1/_2\mu[n]\alpha^n$; $\alpha = re^{j\omega_0}$
$\rightarrow V(z) = 1/(2(1 - re^{j\omega_0}z^{-1}))$
ROC: $|z| > r$

- Hence, $X(z) = V(z) + V^*(z^*)$

$$= \frac{1}{2}\left(\frac{1}{1-re^{j\omega_0}z^{-1}} + \frac{1}{1-re^{-j\omega_0}z^{-1}}\right)$$

$$= \frac{1 - r\cos(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$$

Another ZT example

$$y[n] = (n+1)\alpha^n \mu[n]$$

$$= x[n] + nx[n]$$ where $x[n] = \alpha^n \mu[n]$

$$X(z) = \frac{1}{1-\alpha z^{-1}} \qquad \leftrightarrow -z\frac{dX(z)}{dz}$$

$$= -z\frac{d}{dz}\left(\frac{1}{1-\alpha z^{-1}}\right) = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$$

$$\Rightarrow Y(z) = \frac{1}{1-\alpha z^{-1}} + \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} = \frac{1}{(1-\alpha z^{-1})^2}$$

repeated root - IZT

2. Inverse Z Transform (IZT)

- Forward z transform was defined as:

$$G(z) = \mathcal{Z}\{g[n]\} = \sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

- 3 approaches to inverting $G(z)$ to $g[n]$:
  - Generalization of inverse DTFT
  - Power series in z (long division)
  - Manipulate into recognizable pieces (partial fractions)

← *the useful one*

IZT #1: Generalize IDTFT

- If $z = re^{j\omega}$ then

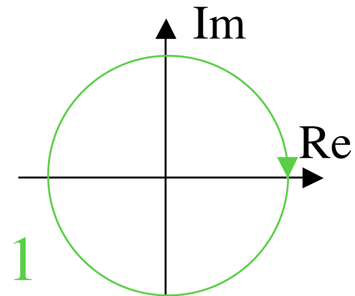
$$G(z) = G\left(re^{j\omega}\right) = \sum g[n]r^{-n}e^{-j\omega n} = \mathrm{DTFT}\left\{g[n]r^{-n}\right\}$$

- so $g[n]r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G\left(re^{j\omega}\right)e^{j\omega n}d\omega$ *IDTFT*

$z = re^{j\omega} \Rightarrow d\omega = dz/jz$

$$= \frac{1}{2\pi j}\oint_C G(z)z^{n-1}dz$$

Counterclockwise closed contour at $|z| = 1$



- Any closed contour around origin will do
- Cauchy: $g[n] = \Sigma[\text{residues of } G(z)z^{n-1}]$

IZT #2: Long division

- Since $G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n}$ if we could express $G(z)$ as a simple power series $G(z) = a + bz^{-1} + cz^{-2}$ ... then can just read off $g[n] = \{a, b, c, ...\}$
- Typically $G(z)$ is right-sided (causal) and a rational polynomial $G(z) = \frac{P(z)}{D(z)}$
- Can expand as power series through long division of polynomials

IZT #2: Long division

- Procedure:
  - Express numerator, denominator in descending powers of z (for a causal fn)
  - Find constant to cancel highest term $\rightarrow$ first term in result
  - Subtract & repeat $\rightarrow$ lower terms in result
- Just like long division for base-10 numbers

IZT #2: Long division

- e.g. $H(z) = \dfrac{1+2z^{-1}}{1+0.4z^{-1}-0.12z^{-2}}$

Result

$$
\begin{array}{r l}
 & 1+1.6z^{-1}-0.52z^{-2}+0.4z^{-3}\ldots \\
1+0.4z^{-1}-0.12z^{-2} \;\big) & 1+2z^{-1} \\
 & \underline{1+0.4z^{-1}-0.12z^{-2}} \\
 & \quad 1.6z^{-1}+0.12z^{-2} \\
 & \quad \underline{1.6z^{-1}+0.64z^{-2}-0.192z^{-3}} \\
 & \qquad -0.52z^{-2}+0.192z^{-3} \\
 & \qquad \ldots
\end{array}
$$

IZT#3: Partial Fractions

- Basic idea: Rearrange $G(z)$ as sum of terms *recognized* as simple ZTs
  - especially $\dfrac{1}{1-\alpha z^{-1}} \quad \leftrightarrow \quad \alpha^n \mu[n]$
    or sin/cos forms
- i.e. given products $\dfrac{P(z)}{\left(1-\alpha z^{-1}\right)\left(1-\beta z^{-1}\right)\cdots}$

  rearrange to sums $\dfrac{A}{1-\alpha z^{-1}} + \dfrac{B}{1-\beta z^{-1}} + \cdots$

Partial Fractions

- Note that:

order 2 polynomial
$u + vz^{-1} + wz^{-2}$

$$\frac{A}{1-\alpha z^{-1}} + \frac{B}{1-\beta z^{-1}} + \frac{C}{1-\gamma z^{-1}} =$$

$$\frac{A\left(1-\beta z^{-1}\right)\left(1-\gamma z^{-1}\right) + B\left(1-\alpha z^{-1}\right)\left(1-\gamma z^{-1}\right) + C\left(1-\alpha z^{-1}\right)\left(1-\beta z^{-1}\right)}{\left(1-\alpha z^{-1}\right)\left(1-\beta z^{-1}\right)\left(1-\gamma z^{-1}\right)}$$

order 3 polynomial

- Can do the *reverse* i.e.

go from $\dfrac{P(z)}{\Pi_{\ell=1}^{N}(1-\lambda_\ell z^{-1})}$ to $\sum_{\ell=1}^{N} \dfrac{\rho_\ell}{1-\lambda_\ell z^{-1}}$

- if order of $P(z)$ is less than $D(z)$

else cancel w/ long div.

Partial Fractions

- Procedure:

order N-1

$$F(z) = \frac{P(z)}{\prod_{\ell=1}^{N}(1-\lambda_\ell z^{-1})} = \sum_{\ell=1}^{N}\frac{\rho_\ell}{1-\lambda_\ell z^{-1}}$$

no repeated poles!

$$\rightarrow f[n] = \sum_{\ell=1}^{N}\rho_\ell(\lambda_\ell)^n\mu[n]$$

- where $\rho_\ell = \left(1-\lambda_\ell z^{-1}\right)F(z)\Big|_{z=\lambda_\ell}$

  i.e. evaluate $F(z)$ *at the pole* but *multiplied* by the pole term
  → dominates = residue of pole

  (cancels term in denominator)

Partial Fractions Example

- Given $H(z) = \dfrac{1 + 2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}$ (again)

factor:

$$= \frac{1 + 2z^{-1}}{\left(1 + 0.6z^{-1}\right)\left(1 - 0.2z^{-1}\right)} = \frac{\rho_1}{1 + 0.6z^{-1}} + \frac{\rho_2}{1 - 0.2z^{-1}}$$

- where:

$$\rho_1 = \left(1 + 0.6z^{-1}\right)H(z)\Big|_{z=-0.6} = \frac{1 + 2z^{-1}}{1 - 0.2z^{-1}}\Bigg|_{z=-0.6} = -1.75$$

$$\rho_2 = \frac{1 + 2z^{-1}}{1 + 0.6z^{-1}}\Bigg|_{z=0.2} = 2.75$$

Partial Fractions Example

- Hence $H(z) = \dfrac{-1.75}{1+0.6z^{-1}} + \dfrac{2.75}{1-0.2z^{-1}}$
- If we know ROC $|z| > |\alpha|$ i.e. $h[n]$ causal:

$$\Rightarrow h[n] = (-1.75)(-0.6)^n \mu[n] + (2.75)(0.2)^n \mu[n]$$

$$= -1.75\{\ 1\ \ \text{-}0.6\ \ 0.36\ \ \text{-}0.216\ \ldots\}$$

$$+2.75\{\ 1\ \ 0.2\ \ 0.04\ \ 0.008\ \ldots\}$$

$$= \{1\ \ 1.6\ \ \text{-}0.52\ \ 0.4\ \ldots\}$$

same as long division!