COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Graph Transduction via Alternating Minimization

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Outline of the presentation

- Brief introduction and related work
- Problems with Graph Labeling
  - Imbalanced labels
  - Weak or uninformative labels
  - Noisy and non-separable data
- Proposed Method
  - Graph transduction via Alternating minimization (GTAM)
  - A *bivariate* optimization over graph function and graph labels
  - Label regularizer terms for handling imbalances
- Experiments

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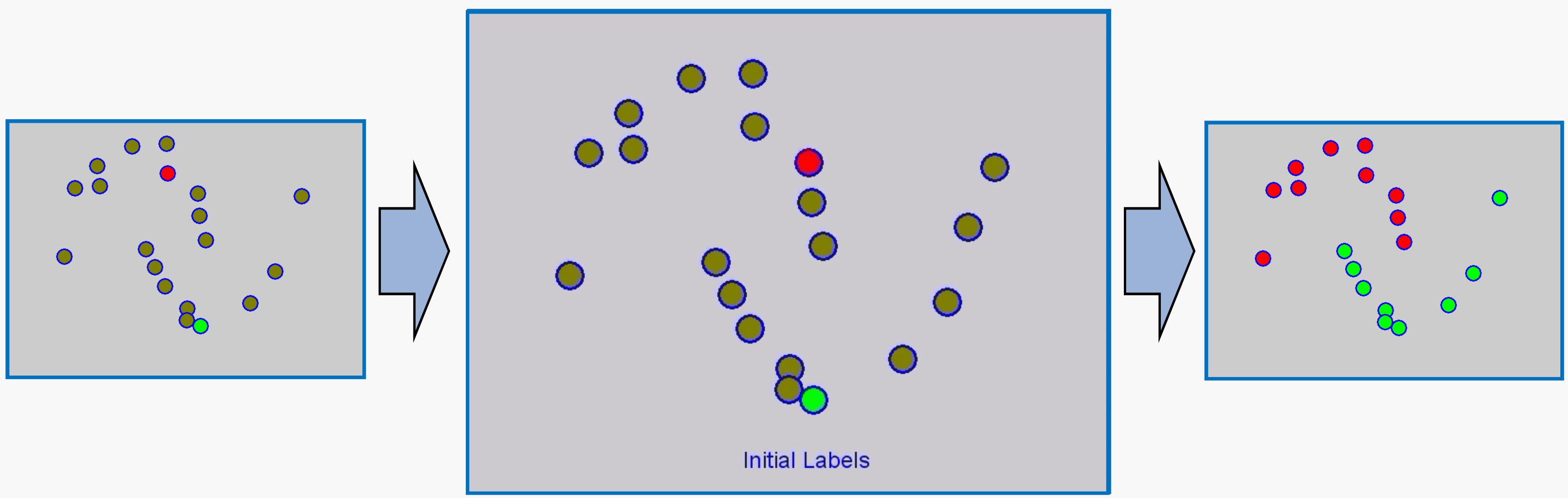
Graph Transduction –Review

- Label propagation on graphs



Input samples with sparse labels   Label propagation on graph   Label inference results

Unlabeled   Positive   Negative   Positive   Negative

Graph Transduction - Review

- Given a dataset $\mathcal{X} = (\mathcal{X}_l, \mathcal{X}_u)$ of labeled samples $\mathcal{X}_l$, and unlabeled samples $\mathcal{X}_u$
- Graph transduction here uses an *undirected* graph $\mathcal{G} = \{\mathcal{X}, \mathcal{E}\}$ of samples $\mathcal{X}$ as nodes and edges $\mathcal{E}$ weighted by sample similarity $w_{ij} = k(\mathbf{x}_i, \mathbf{x}_j)$;
- Define weight matrix $\mathbf{W} = \{w_{ij}\}$,
Node degree $\mathbf{D} = diag([d_1, \cdots, d_n])$,
graph Laplacian $\mathbf{\Delta} = \mathbf{D} - \mathbf{W}$,
and normalized Laplacian $\mathbf{L} = \mathbf{D}^{-1/2}\mathbf{\Delta}\mathbf{D}^{-1/2}$
label matrix $\mathbf{Y}$,

positive negative

$$\mathbf{Y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Graph Transduction – Review

- Function estimation through optimization
  - A continuous valued classification function is estimated by minimizing a cost $\mathcal{Q}$

$$\mathbf{F}^* = \arg\min_{\mathbf{F}} \mathcal{Q}(\mathbf{F}) = \arg\min_{\mathbf{F}} \left\{ \mathcal{Q}_{smooth}(\mathbf{F}) + \mathcal{Q}_{fit}(\mathbf{F}) \right\}$$

  - Trades off smoothness over graph with fitness on given labels

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Graph Transduction – Review

- Previous choices for cost: $\mathcal{Q}$

➢ Gaussian fields and Harmonic functions *GFHF*

(Zhu, Ghahramani, and Lafferty ICML03)

$$\mathcal{Q}(F) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\|F_{i\cdot} - F_{j\cdot}\|^2$$

➢ Local and global consistency *LGC*

(Zhou, Bousquet, Lal, Weston, and Scholkopf NIPS04)

$$\mathcal{Q}(F) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\left\|\frac{F_{i\cdot}}{\sqrt{D_{ii}}} - \frac{F_{j\cdot}}{\sqrt{D_{ii}}}\right\|^2 + \mu\sum_{i=1}^{n}\|F_{i\cdot} - Y_{i\cdot}\|^2$$

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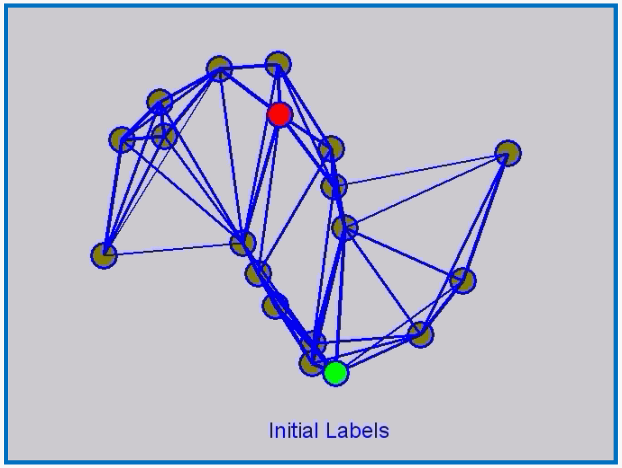
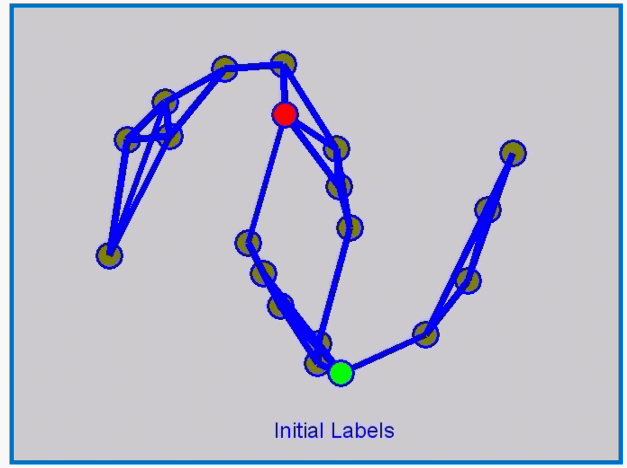
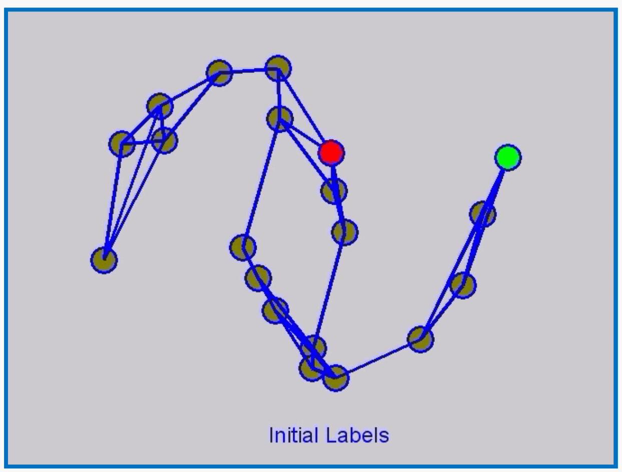
Graph Transduction: Problemistic Cases



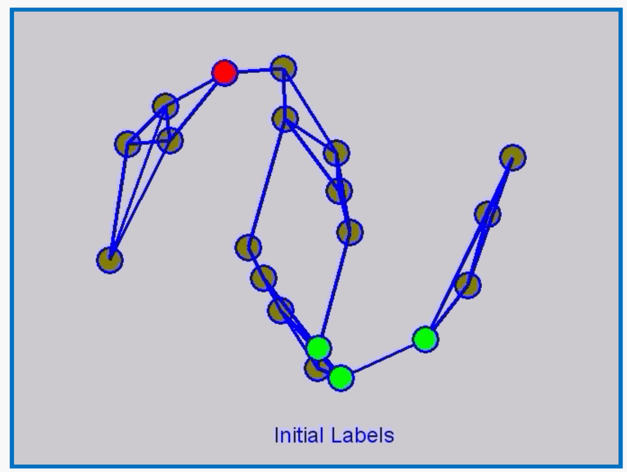
Over connected graph



Improper edge weighting



Difficult label location



Imbalance labels

Methodology – Our Choice for $\mathcal{Q}$

1) Start with *LGC*'s Cost

$$\mathcal{Q}(\mathbf{F}) = \frac{1}{2}\mathrm{tr}\left\{\mathbf{F}^T\mathbf{L}\mathbf{F} + \mu(\mathbf{F}-\mathbf{Y})^T(\mathbf{F}-\mathbf{Y})\right\}$$

2) Make into a bivariate optimization over $\mathbf{F}$ and $\mathbf{Y}$

$$\mathcal{Q}(\mathbf{F},\mathbf{Y}) = \frac{1}{2}\mathrm{tr}\left\{\mathbf{F}^T\mathbf{L}\mathbf{F} + \mu(\mathbf{F}-\mathbf{V}\mathbf{Y})^T(\mathbf{F}-\mathbf{V}\mathbf{Y})\right\}$$

3) Introduce label regularizer terms

$$\mathbf{v} = \sum_{j=1}^{c} \frac{\mathbf{Y}_{\cdot j} \odot \mathbf{D}\vec{1}}{\mathbf{Y}_{\cdot j}^T \mathbf{D}\vec{1}} \qquad \mathbf{V} = diag\{\mathbf{v}\}$$

Methodology– Label Regularizer

- Normalize labels among classes to handle imbalance
- Weight labels based on the degrees;

Example:

positive negative

$$\mathbf{Y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \Rightarrow \mathbf{V} = \begin{bmatrix} \frac{1}{1+3} & 0 & 0 & 0 \\ 0 & \frac{2}{2} & 0 & 0 \\ 0 & 0 & \frac{3}{1+3} & 0 \\ 0 & 0 & 0 & \frac{0}{4} \end{bmatrix}$$

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Methodology – Optimize $\mathbf{F}$

- Minimizing $\mathcal{Q}(\mathbf{F}, \mathbf{Y})$ is mixed integer program

$$\mathcal{Q}(\mathbf{F}, \mathbf{Y}) = \frac{1}{2}\text{tr}\left\{\mathbf{F}^T\mathbf{L}\mathbf{F} + \mu(\mathbf{F} - \mathbf{V}\mathbf{Y})^T(\mathbf{F} - \mathbf{V}\mathbf{Y})\right\}$$

- Try greedy solution via alternating minimization
- Solve for continuous valued $\mathbf{F}$: $\mathbf{P} = (\mathbf{L}/\mu + \mathbf{I})^{-1}$

$$\frac{\partial \mathcal{Q}}{\partial \mathbf{F}^*} = 0 \Rightarrow \mathbf{F}^* = (\mathbf{L}/\mu + \mathbf{I})^{-1}\mathbf{V}\mathbf{Y} = \mathbf{P}\mathbf{V}\mathbf{Y}$$

- Insert the solution gives NP hard problem for $\mathbf{Y}$ (slightly nonlinear MAXCUT)

$$\mathcal{Q}(\mathbf{Y}) = \frac{1}{2}\text{tr}\left(\mathbf{Y}^T\mathbf{V}^T\left[\mathbf{P}^T\mathbf{L}\mathbf{P} + \mu(\mathbf{P}^T - \mathbf{I})(\mathbf{P} - \mathbf{I})\right]\mathbf{V}\mathbf{Y}\right)$$

Methodology – Gradient Greedy (1)

- Optimization on binary valued $\mathbf{Y}$ with constraint
- Almost Max Cut problem where Greedy is 0.5 optimal

$$\mathcal{Q}(\mathbf{Y}) = \frac{1}{2}\text{tr}\left(\mathbf{Y}^T\mathbf{V}^T\left[\mathbf{P}^T\mathbf{L}\mathbf{P} + \mu(\mathbf{P}^T - \mathbf{I})(\mathbf{P} - \mathbf{I})\right]\mathbf{V}\mathbf{Y}\right)$$

$$\mathbf{Z} = \mathbf{V}\mathbf{Y} \qquad \mathbf{A} = \mathbf{P}^T\mathbf{L}\mathbf{P} + (\mathbf{P}^T - \mathbf{I})(\mathbf{P} - \mathbf{I})$$

$$\mathcal{Q}(\mathbf{Z}) = \frac{1}{2}\text{tr}\left(\mathbf{Z}^T\mathbf{A}\mathbf{Z}\right)$$

- We do 'gradient greedy' on our problem. Find which entry of $\mathbf{Y}$ will most reduce cost and select it for labeling

$$\frac{\partial\mathcal{Q}}{\partial\mathbf{Y}} = \frac{\partial\mathcal{Q}}{\partial\mathbf{Z}}\frac{\partial\mathbf{Z}}{\partial\mathbf{Y}} \qquad \frac{\partial\mathcal{Q}}{\partial\mathbf{Z}} = \mathbf{A}\mathbf{Z} = \mathbf{A}\mathbf{V}\mathbf{Y}$$

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Methodology – Gradient Greedy (2)

- Find location with the steepest descent of the value of the loss function

$$(i^*, j^*) = \arg\min_{\mathbf{x}_i \in \mathcal{X}_u, 1 \leq j \leq c} \nabla_{\mathbf{Z}_{ij}} \mathcal{Q}$$

- Label the corresponding node: $\mathbf{Y}_{i^* j^*} = 1$

$$\mathbf{Y}^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow[i^* = 3, j^* = 1]{\nabla_{\mathbf{Z}} \mathcal{Q}^t = \begin{bmatrix} * & * \\ * & * \\ -0.31 & 0.07 \\ -0.17 & -0.04 \end{bmatrix}} \mathbf{Y}^{t+1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- Iterative repeat the above procedure until all the nodes are labeled

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Final Algorithm

1) Calculate gradient matrix

$$\nabla_{\mathbf{Z}} \mathcal{Q}^t = \mathbf{A} diag(\mathbf{v}^t) \mathbf{Y}^t$$

2) Label the most beneficial node with largest cost reduction

$$(i^*, j^*) = \arg\min_{\mathbf{x}_i \in \mathcal{X}_u, 1 \le j \le c} \nabla_{\mathbf{Z}_{ij}} \mathcal{Q}^t$$
$$\mathbf{Y}^{t+1}_{i^* j^*} = 1$$

3) Update the label regularizer

$$\mathbf{v}^{t+1} = \sum_{j=1}^{c} \frac{\mathbf{Y}^{t+1}_{\cdot j} \odot \mathbf{D}\vec{1}}{{\mathbf{Y}^{t+1}_{\cdot j}}^T \mathbf{D}\vec{1}}$$

4) Update labeled and unlabeled sets

$$\mathcal{X}_l^{t+1} \leftarrow \mathcal{X}_l^t + \mathbf{x}_{i^*}; \quad \mathcal{X}_u^{t+1} \leftarrow \mathcal{X}_u^t - \mathbf{x}_{i^*}; \quad t \leftarrow t + 1$$

5) If $\mathcal{X}_u^{t+1} = \emptyset$, output the labels; else go to 1)

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Some Intuition

- Previous methods (e.g. *LGC* and *GFHF*) prematurely commit to an erroneous labeling;
- Our method iteratively infers labels with the current given labels and each step only assign label to the most beneficial node with highest cost reduction;
- Greedy MaxCut is not bad (0.5), best solution is SDP 0.878 (but too slow).

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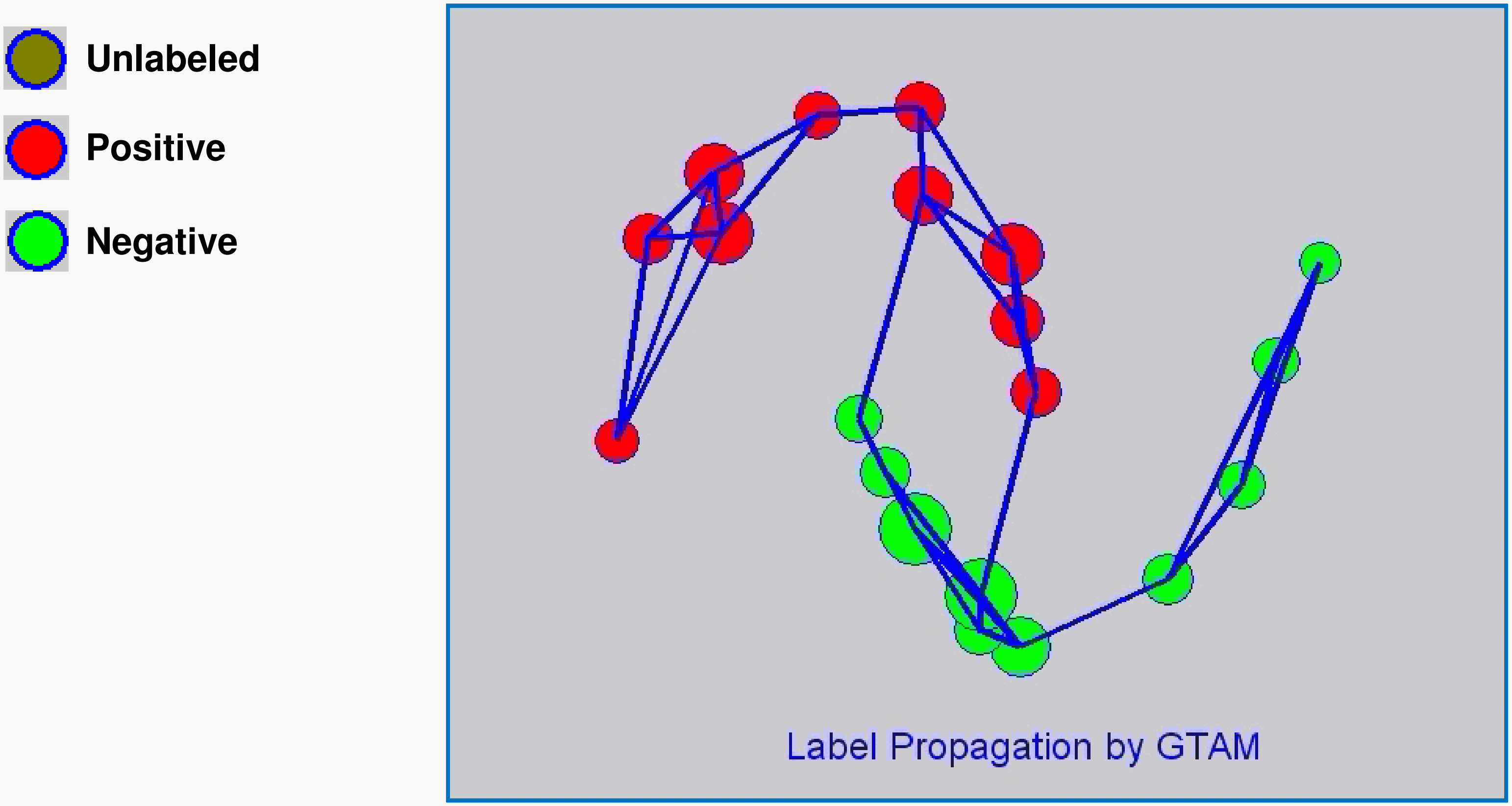
Intuition



Label propagation by GTAM

The scale of labeled nodes denotes the value of label regularizer

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Computation Efficiency

- Complexity is $\mathcal{O}(n^3)$
- Can run more efficiently
  - Applying superposition approach to achieve incremental updating (*Wang, Chang et al CVPR08*)
  - Can early-stop greedy algorithm after enough labeling
  - Can do multiple nodes labeling in each iteration

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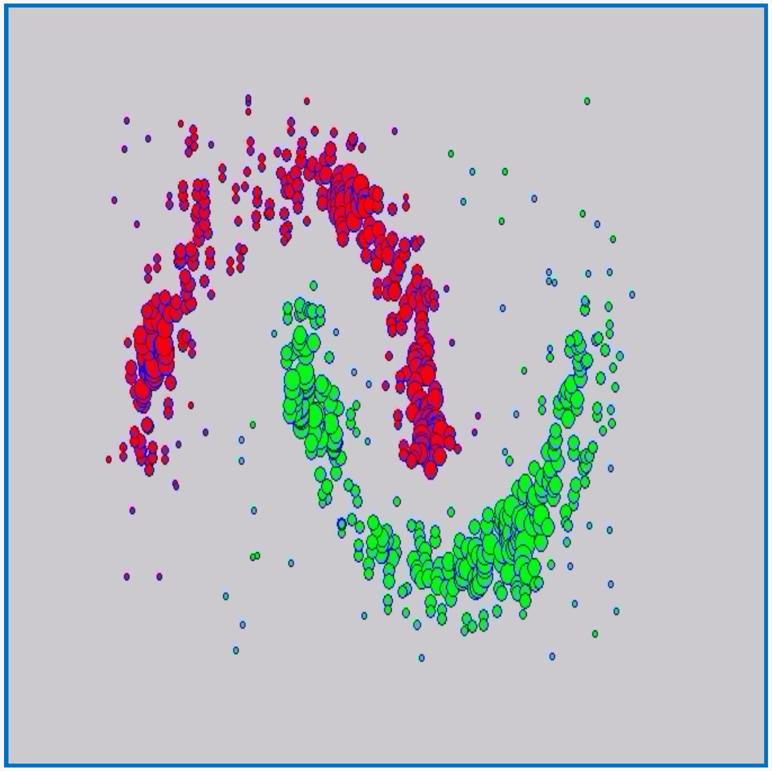
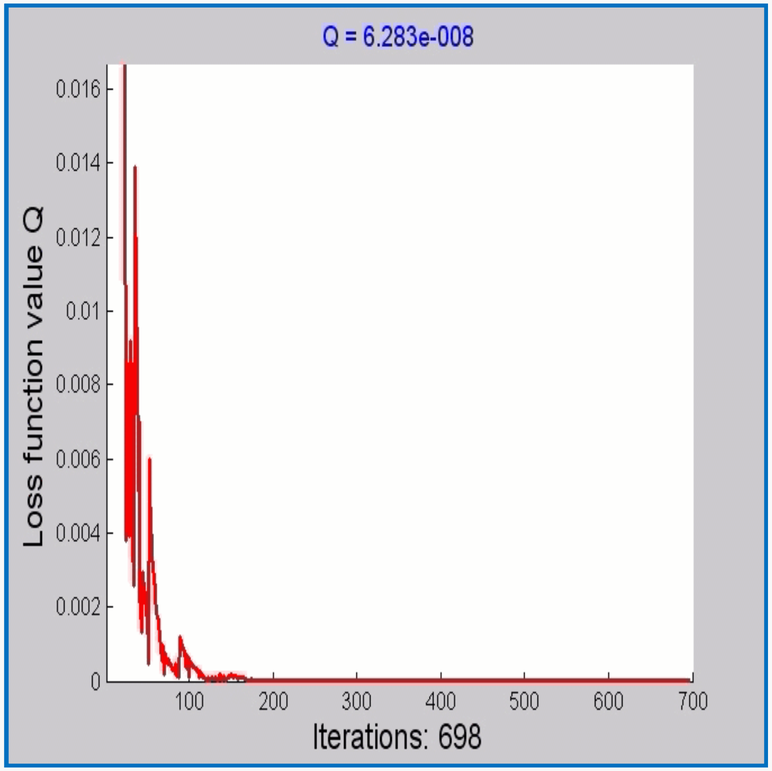
Experiments – Toy Data



Label propagation by GTAM



Convergence procedure

(non-monotonic due to gradient greedy discrete step size)

Unlabeled   Positive   Negative

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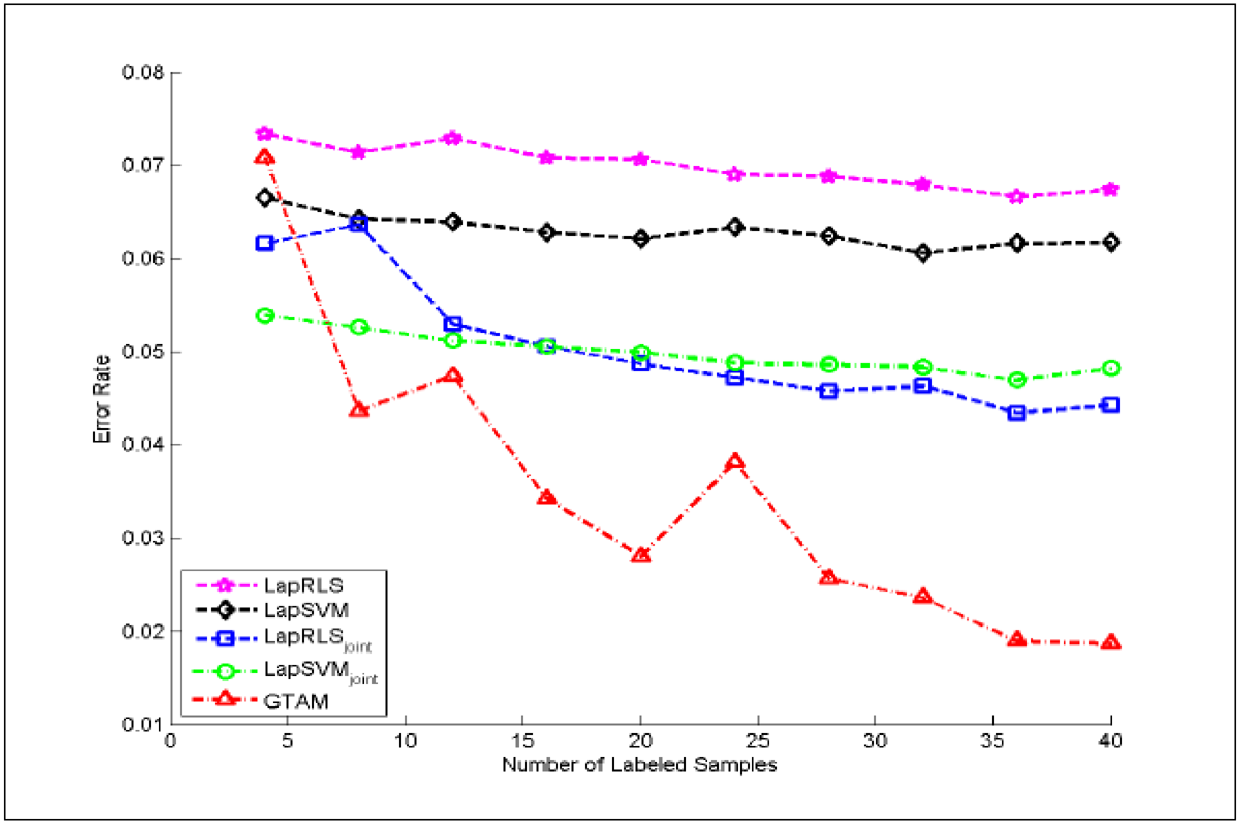
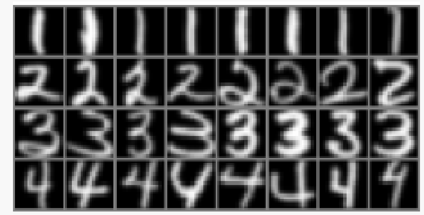
Experiments – WebKB Data

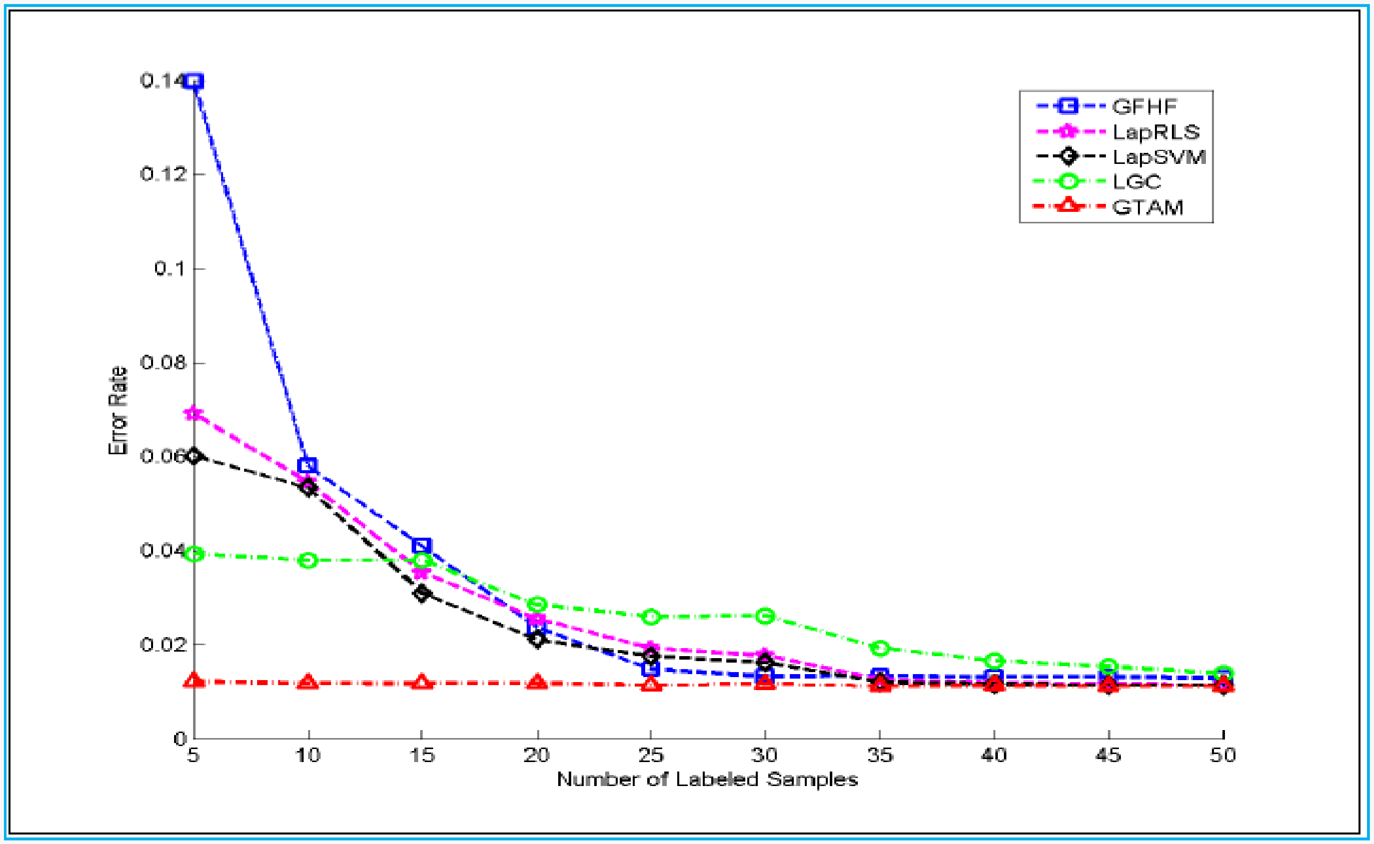
1051 documents (course & non-course) containing **1840** page + **3000** link attributes;
Comparing with approaches reported in *Sindhwani, Niyogi, and Belkin ICML 2005;*
100 random test , evaluation based average error rate;



Experiments – USPS Digits Data

3874 digit samples (16*16 image) containing four digits 1, 2, 3, 4;
Comparing with LGC, GFHF and LapSVM et al;
20 random test , evaluation based average error rate;





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Summary

- Cast graph transduction as cost over labels $\mathbf{Y}$ and graph functions $\mathbf{F}$
- Add label normalization terms
- Greedy alternating optimization of $\mathbf{F}$ and $\mathbf{Y}$ (reminiscent of MaxCut)
- This produces gradual propagation-style algorithm
- Fast and robust to labeling degeneracies
- Reduces error rate of existing approaches on WebKB and USPS digits by more than half
- Open questions ….