Robust Late Fusion with Rank Minimization
Supplementary Material

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Theorem 1. *Given a set of n skew-symmetric matrices T_i, the SVT solver employed by Algorithm 1 produces a skew-symmetry matrix $\hat{T}$ if the spectrums between the dominant singular values are separated.*

Proof. Following the proof in [1], to prove the above theorem, we first introduce some properties of skew-symmetric matrices. Assuming T is a skew-symmetric matrix with the property $T = -T^\top$, then the eigenvalues of T are pure-imaginary and come in complex-conjugate pairs. Moreover, the rank of T is even [2]. Then the following lemma characterizes the Singular Value Decomposition (SVD) of a skew-symmetric matrix.

Lemma 1. *If T is an $m \times m$ skew-symmetric matrix with eigenvalues $i\lambda_1, -i\lambda_1, i\lambda_2, -i\lambda_2, \dots, i\lambda_j, -i\lambda_j$ where $\lambda_p > 0$, $p = 1, \dots, j$, $j = \lfloor (m/2) \rfloor$, and i denotes the imaginary unit. Then the SVD of T is given by $T = UDV^\top$ where*

$$D = \begin{pmatrix} \lambda_1 & & & & & & & & \\ & \lambda_1 & & & & & & & \\ & & \lambda_2 & & & & & & \\ & & & \lambda_2 & & & & & \\ & & & & . & & & & \\ & & & & & . & & & \\ & & & & & & . & & \\ & & & & & & & \lambda_j & \\ & & & & & & & & \lambda_j \end{pmatrix}$$

and the forms of U, V are given in the proof.

Proof. Based on Murnaghan-Wintner form of a real-valued matrix [3], the matrix T can be decomposed as

$$T = XZX^\top, \tag{1}$$

for a real-valued orthogonal matrix X and a real-valued block-upper-triangular matrix Z with 2-by-2 blocks along the diagonal.

Since T is skew-symmetric, the decomposed component Z is also skew-symmetric. Then Z has a block-diagonal form:

$$Z = \begin{pmatrix} 0 & \lambda_1 & & & & & & & \\ -\lambda_1 & 0 & & & & & & & \\ & & 0 & \lambda_2 & & & & & \\ & & -\lambda_2 & 0 & & & & & \\ & & & & . & & & & \\ & & & & & . & & & \\ & & & & & & . & & \\ & & & & & & & 0 & \lambda_j \\ & & & & & & & -\lambda_j & 0 \end{pmatrix}$$

Furthermore, the SVD of the block matrix $\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}$ is given by

$$\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \times \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Assume matrix A and matrix B are defined as follows:

$$A = \begin{pmatrix} 0 & 1 & & & & & & & \\ 1 & 0 & & & & & & & \\ & & 0 & 1 & & & & & \\ & & 1 & 0 & & & & & \\ & & & & . & & & & \\ & & & & & . & & & \\ & & & & & & . & & \\ & & & & & & & 0 & 1 \\ & & & & & & & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 0 & & & & & & & \\ 0 & 1 & & & & & & & \\ & & -1 & 0 & & & & & \\ & & 0 & 1 & & & & & \\ & & & & . & & & & \\ & & & & & . & & & \\ & & & & & & . & & \\ & & & & & & & -1 & 0 \\ & & & & & & & 0 & 1 \end{pmatrix}$$

Then, the real-valued matrix T has the following decomposition form: $T = XADBX^\top$. We construct U and V such that $U = XA$ and $V^\top = BX^\top$, which are real and orthogonal. We thus complete the lemma which constructs the SVD of T. □

Next, we use the following lemma to illustrate that the low rank approximation to a skew-symmetric matrix T generated by singular value thresholding is also skew-symmetric.

Lemma 2. *Let T be an $m \times m$ skew-symmetric matrix, and let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_j > \lambda_{j+1}$ be the magnitudes of the singular value pairs. (Recall that the previous lemma showed that the singular values come in pairs, e.g., ($i\lambda_p$, $-i\lambda_p$), $p = 1, 2, \ldots, j$.). Then the low rank approximation of T generated by singular value thresholding in an orthogonally invariant norm is also skew-symmetric.*

Proof. Since the number of singular values of T is even, and there is a gap between the kth and the $(k+1)$th singular value, we can always get the even number of singular values if we truncate the singular values based on a threshold. Therefore, based on the SVD form from Lemma 1, we naturally obtain a skew-symmetric matrix. □

Finally, we use the above lemma to prove that, given a set of n skew-symmetric matrices T_i, our ALM-based algorithm preserves a skew-symmetry matrix $\hat{T}^{(l)}$ in each iteration, where l denotes the iterative number.

Clearly, from Algorithm 1, $\hat{T}^{(0)}$, $E_i^{(0)}$, $Y_i^{(0)}$, $i = 1, \ldots, n$ are all skew-symmetric. In step 4 of our algorithm, we compute the SVD of a skew-symmetric matrix $\frac{1}{n\mu}\Sigma_{i=1}^n Y_i^{(0)} + \frac{1}{n}\Sigma_{i=1}^n T_i - \frac{1}{n}\Sigma_{i=1}^n E_i^{(0)}$ and truncate the singular values below the threshold, then the obtained $\hat{T}^{(1)}$ is skew-symmetric based on Lemma 2 and condition of our theorem. In step 5, the obtained $E_i^{(1)}$ is skew-symmetric due to the fact that $T_i + \frac{Y_i^{(0)}}{\mu} - \hat{T}^{(1)}$ is skew-symmetric. Similarly, in step 6, we can obtain $Y_i^{(1)}$ which is also skew-symmetric. As the iteration proceeds, we can obtain skew-symmetric matrices $\hat{T}^{(l)}$, $E_i^{(l)}$, $Y_i^{(l)}$ in each iteration. Therefore, we arrive at a skew-symmetric matrix $\hat{T}$ when Algorithm 1 converges, which completes the proof. □

References

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