Homework Set no. 9

1. **Mutual information for correlated normals (EIT 9.3).** Find the mutual information $I(X; Y)$, where

$$
\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2 \left( 0, \begin{bmatrix} \sigma^2 & \rho \sigma^2 \\ \rho \sigma^2 & \sigma^2 \end{bmatrix} \right).
$$

Evaluate $I(X; Y)$ for $\rho = 1, \rho = 0$, and $\rho = -1$, and comment.

*Hint: this is a mechanical problem.*

The purpose of the problem is to learn how to compute the mutual information between two continuous random variables, and to get a feel for what the information looks like for a very common case.

2. **Mutual information** We know that the mutual information $I(X, Y)$ between two discrete random variables can be written as $H(Y) - H(Y \mid X)$. Let the conditional distribution of $Y$ given $X$, $P(Y \mid X)$, be fixed, and assume that we can vary only the distribution of $X$, $P(X)$. Is it true that a distribution $P_{\text{max}}(X)$ (not necessarily unique) that maximizes $H(Y)$ also maximizes $I(X, Y)$? Prove your answer formally (that is, if your answer is “yes”, you must provide a proof, while, if your answer is “no”, you can provide either a proof or a counterexample.)

*This is a difficult problem if you try a brute-force approach. Hint: if the answer were “yes”, what would be the consequences? Where, in this course, could we apply this theorem? Ask yourself the same questions assuming that the answer is “no”.)*

3. **The Channel coding Theorem** At the end of the proof of the Channel Coding Theorem we used certain properties of random variables, which we did not prove in class (or maybe we did ?)

- Show that, if the average of a set of non-negative numbers is equal to $a$, then at least half of the numbers must be less than or equal to $2a$.

- If $X$ is a discrete random variable, prove that it takes a value less than or equal to $E(X)$ with probability greater than zero, where $E(X)$ is the expected value of $X$.

*(Note: this is not a difficult problem, but needs a bit of care.)*

4. **Rate distortion function with infinite distortion.** Find the rate distortion function $R(D) = \min I(X; \hat{X})$ for $X \sim \text{Bernoulli} \left( \frac{1}{2} \right)$ and distortion

$$
d(x, \hat{x}) = \begin{cases} 
0, & x = \hat{x}, \\
1, & x = 1, \hat{x} = 0, \\
\infty, & x = 0, \hat{x} = 1.
\end{cases}
$$