Problem Set #2 Solutions

1 Remark: We work under the condition $\Phi_1(t) = X_1(t)/\sqrt{E_b}$, since $||\Phi(t)|| = 1$. (It is a minor typo in the problem that $\Phi_1(t) = X_1(t)/\sqrt{E}$).

a) $\Phi_1(t) = \frac{X_1(t)}{\sqrt{E_b}} = \sqrt{\frac{1}{T}}, 0 \leq t \leq T$

By Gram-Schmidt procedure, we can compute

$$c_{21} = \langle \Phi_1(t), X_2(t) \rangle = \sqrt{\frac{E_b}{2}},$$

and therefore,

$$\Phi_2(t) = \frac{X_2(t) - c_{21}\Phi_1(t)}{||X_2(t) - c_{21}\Phi_1(t)||} = \begin{cases} \sqrt{\frac{1}{T}} & 0 \leq t \leq T/2 \\ -\sqrt{\frac{1}{T}} & T/2 \leq t \leq T. \end{cases}$$

It is easy to compute

$$\rho = \frac{1}{E_b} \int_0^{T/2} \frac{E_b \sqrt{2}}{T} = \frac{\sqrt{2}}{2}.$$

b) Note that $X_1 = \sqrt{E_b}\Phi(t)$, and it is also easy to compute

$$X_2(t) = \sqrt{\frac{E_b}{2}}\Phi_1(t) + \sqrt{\frac{E_b}{2}}\Phi_2(t),$$

therefore, $b_1 = b_2 = \rho\sqrt{E_b}$.

2 Assume that $X(t) = \sum_{i=1}^{\infty} a_i \Phi_i(t), Y(t) = \sum_{i=1}^{\infty} b_i \Phi_i(t)$ and $a = (a_1, a_2, \cdots), b = (b_1, b_2, \cdots)$. We obtain, by using the orthonormality of $\{\Phi_i\}$,

$$d_{X,Y}^2 = \int_0^T (X(t) - Y(t))^2 dt$$

$$= \int_0^T \left[ \sum_{i=1}^{\infty} a_i \Phi_i(t) - \sum_{i=1}^{\infty} b_i \Phi_i(t) \right]^2 dt$$

$$= \sum_{i=1}^{\infty} (a_i - b_i)^2 \int_0^T \Phi_i^2(t) dt$$

$$= |a - b|^2.$$
3 Using the fact that \( s_1 \) and \( s_0 \) are two signals with equal unit energy, it is easy to see that

\[
\int_0^T [r(t) - \sqrt{E_1} s_1(t)]^2 dt < \int_0^T [r(t) - \sqrt{E_0} s_0(t)]^2 dt
\]

\(
\Leftrightarrow \int_0^T [r^2(t) - 2r(t)\sqrt{E_1} s_1(t) + E_1 s_1^2(t)] dt < \int_0^T [r^2(t) - 2r(t)\sqrt{E_0} s_0(t) + E_0 s_0^2(t)] dt
\)

\(
\Leftrightarrow \frac{E_1 - E_0}{2} < \int_0^T \sqrt{E_1} r(t) s_1(t) dt - \int_0^T \sqrt{E_0} r(t) s_0(t) dt,
\)

which is the output of the correlation receiver.

4 a) It is easy to check that \( s_1 \) and \( s_0 \) are two orthonormal basis functions. The optimum two-branch correlation receiver is

\[
\begin{align*}
\sqrt{E} s_1(t) & \quad \text{positive} \\
\sqrt{E} s_0(t) & \quad \text{negative}
\end{align*}
\]

b) Suppose that the received signal is \( r(t) = X(t) + n(t) \) where \( X(t) \) is one of the two signals shown in the problem and \( n(t) \) is the white noise. If \( X(t) \) is equal to \( \sqrt{E} s_1 \), then the output of the correlation receiver is

\[
\int_0^T [r(t) + n(t)]\sqrt{E} s_1(t) dt - \int_0^T [r(t) + n(t)]\sqrt{E} s_0(t) dt
\]

\[
= E + \int_0^T \sqrt{E} n(t) s_1(t) dt - \int_0^T \sqrt{E} n(t) s_0(t) dt
\]

\[
\text{define} \quad E + S_1 - S_0,
\]

where \( S_1 \) and \( S_0 \) are two independent Gaussian random variables (please check that the correlation of \( S_1 \) and \( S_0 \) is equal to zero by recalling that \( s_1 \) and \( s_0 \) are two orthonormal basis functions) satisfying

\[
\text{Var}(S_1) = \text{Var}(S_0) = E \int_0^T \int_0^T \mathbb{E}[n(u)n(v)] s_1(u)s_1(v) du dv
\]

\[
= E \int_0^T \int_0^T \mathbb{E}[n(u)n(v)] s_1(u)s_1(v) du dv
\]

\[
= E \int_0^T \int_0^T \frac{N_0}{2} \delta(u-v) s_1(u)s_1(v) du dv
\]

\[
= \frac{EN_0}{2},
\]
therefore, by noting that $S_1 - S_0$ is a Gaussian random variable with variance $\frac{EN_0}{2} + \frac{EN_0}{2} = EN_0$, we can compute the conditional probability

$$P[\text{error}|X(t) = s_1(t)] = P[E + S_1 - S_0 > 0] = Q\left(\sqrt{\frac{E}{N_0}}\right).$$

Using a similar argument, we can calculate

$$P[\text{error}|X(t) = s_0(t)] = Q\left(\sqrt{\frac{E}{N_0}}\right),$$

which implies

$$P[\text{error}] = P[X(t) = s_0(t)]P[\text{error}|X(t) = s_0(t)] + P[X(t) = s_1(t)]P[\text{error}|X(t) = s_1(t)]$$

$$= Q\left(\sqrt{\frac{E}{N_0}}\right).$$

c) The two-branch matched filter for this modulation technique is shown below. Sample at $T$.

d) Using the same argument as in b), we know that the error probability is $Q\left(\sqrt{\frac{E}{N_0}}\right)$. 

![Diagram of the two-branch matched filter](image_url)