Problem 1

1. Discrete-Time Fourier Transform:

\[ F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k} \]

2. Discrete-Time Fourier Series:

\[ f[k] = \sum_{r=0}^{N_0-1} D_r e^{jr\Omega_0 k} \]

where

\[ D_r = \frac{1}{N_0} \sum_{k=0}^{N_0-1} f[k]e^{-jr\Omega_0 k}, \quad \Omega_0 = \frac{2\pi}{N_0} \]

3. Discrete Fourier Transform:

\[ F_r = \sum_{k=0}^{N_0-1} f[k]e^{-jr\Omega_0 k}, \quad r = 0, 1, 2, \ldots, N_0 - 1 \]

Proof:

\[ \mathcal{F}\{f[k] * h[k]\} = \sum_{k=-\infty}^{\infty} (f[k] * h[k])e^{-j\Omega k} \]

\[ = \sum_{k=-\infty}^{\infty} (\sum_{m=-\infty}^{\infty} f[m]h[k-m])e^{-j\Omega k} \]

\[ = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f[m]h[k-m]e^{-j\Omega k} \]

\[ = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f[m]h[k-m]e^{-j\Omega m}e^{-j\Omega (k-m)} \]

\[ = \sum_{m=-\infty}^{\infty} f[m]e^{-j\Omega m} \sum_{k=-\infty}^{\infty} h[k-m]e^{-j\Omega (k-m)} \]

\[ = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k} \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k} \]

\[ = \mathcal{F}\{f[k]\}\mathcal{F}\{h[k]\} \]
Problem 2

\[ y[n + 2] - 0.1y[n + 1] + 0.24y[n] = 2.5f[n + 1] - f[n] \]

To determine the impulse response, we must first solve the characteristic equation,

\[ \gamma^2 - 0.1\gamma + 0.24 = 0 \]

Using the quadratic formula, we obtain,

\[ \gamma = \frac{0.1 \pm \sqrt{0.01 - 0.96}}{2} = 0.05 \pm j(0.5)(\sqrt{0.95}) \]

\[ |\gamma| = \sqrt{(0.05)^2 + (0.25)(0.95)} \approx 0.49 \]

\[ \angle \gamma = \arctan\left(\frac{0.5\sqrt{0.95}}{0.05}\right) \approx 0.47\pi \]

\[ \gamma = 0.49e^{\pm j0.47\pi} \]

and

\[ y[k] = C(0.49)^k \cos(0.47k + \Theta) \]

Then we have

\[ h[k] = \frac{b_0}{a_0} \delta[k] + y[k]u[k] \]

\[ = \frac{-1}{0.24} + C(0.49)^k \cos(0.47\pi k + \Theta)u[k] \]

Substituting into the original equation,

\[ h[n + 2] - 0.1h[n + 1] + 0.24h[n] = 2.5\delta[n + 1] - \delta[n] \]

we have:

\[ k = -2 \Rightarrow h[0] = 0 = -\frac{1}{0.24} + C \cos \Theta \Rightarrow C = \frac{1}{0.24 \cos \Theta} \]

and

\[ k = -1 \Rightarrow h[-1] = 2.5 = C(0.49) \cos(0.47\pi + \Theta) \]

Solving these equations, making use of the identity \( \cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B) \), we get

\[ \tan \Theta = \frac{\sin \Theta}{\cos \Theta} = \frac{1 - 0.82 \cos(0.47\pi)}{0.82 \sin(0.47\pi)} \]

giving \( \Theta \approx -0.27\pi \) and \( C \approx 6.28 \).

So, finally, we have:

\[ h[k] = -4.17\delta[k] + 6.28(0.49)^k \cos(0.47k - 0.27\pi)u[k] \]
And, given the input \( f[k] = 2\delta[k] + 3\delta[k-2] \), the result is simply
\[
y[k] = 2h[k] + 3h[k-2]
\]
Since the magnitude of the characteristic roots, \(|\gamma| = 0.49 < 1\), the system is stable.

**Problem 3**
There are several ways of describing the signal analytically. One is
\[
p[n] = u[n + m] - u[n - (m + 1)]
\]
So,
\[
p_2[n] = p[n] \ast p[n] = \sum_{k=-\infty}^{\infty} \infty(u[k + m] - u[k - (m + 1)])u[(n - k) + m] - u[(n - k) - (m + 1)]
\]
This simplifies to
\[
p_2[n] = \sum_{k=-n}^{n} nu[n - k + m] - u[n - k - (m + 1)]
\]
\[
= \sum_{k=-n}^{n} nu[n - k + m] - \sum_{k=-n}^{0} nu[n - k - (m + 1)]
\]
\[
= \begin{cases} 
0, & n < -2m \\
2m + 1 + n, & -2m \leq n < 0 \\
2m + 1 - n, & 0 \leq n < 2m + 1 \\
0, & 2m + 1 \leq n 
\end{cases}
\]
As we iterate the convolution of \( p[n] \), the signal will smooth out in the sense that higher and higher derivatives will come to exist. Also, the width of the non-zero part of the signal will increase. In the limit, the signal will look like a bell curve.

**Problem 4**
Given:
\[
X(\Omega) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\Omega k}
\]
For \( g[k] = x[2k] \), find \( G(\Omega) \).
\[
G(\Omega) = \sum_{k=-\infty}^{\infty} g[k]e^{-j\Omega k}
\]
\[
= \sum_{k=-\infty}^{\infty} x[2k]e^{-j\Omega k}
\]
\[
= \sum_{m=-\infty}^{\infty} x[m]e^{-j\Omega m/2}
\]
Many students made the mistake of stating that
\[
G(\Omega) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\Omega m/2}
\]
by setting \( m = 2k \). However, this is NOT true as we should only take the values of \( x[m] \) when \( m \) is even. As you can see, the summation in the above equation takes all values of \( x[m] \), whether \( m \) is even or odd. To fix this, we define a function \( y[k] \) as follows:

\[
y[k] = \begin{cases} 
  x[k] & \text{if } k \text{ is even,} \\
  0 & \text{otherwise.}
\end{cases}
\]

Then, we have \( g[k] = y[2k] \) and,

\[
G(\Omega) = \sum_{k=-\infty}^{\infty} g[k]e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} y[2k]e^{-j\Omega k} = \sum_{m=-\infty}^{\infty} y[m]e^{-j\Omega m/2}
\]

Since \( y[k] \) has value 0 when \( k \) is not even, we can safely state that

\[
G(\Omega) = \sum_{m=-\infty}^{\infty} y[m]e^{-j\Omega m/2}
\]

by setting \( m = 2k \). Then, we get

\[
G(\Omega) = Y(\Omega/2)
\]

Now, we need to find the relationship between \( Y(\Omega) \) and \( X(\Omega) \). From the definition of \( y[k] \), we have

\[
y[k] = \frac{1}{2} [\cos(\pi k) + 1] x[k] = \frac{1}{2} \left[ e^{-j\pi k} + 1 \right] x[k]
\]

Then, we have

\[
Y(\Omega) = \sum_{k=-\infty}^{\infty} y[k]e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} \frac{1}{2} \left[ e^{-j\pi k} + 1 \right] x[k]e^{-j\Omega k}
\]

\[
= \frac{1}{2} \left( \sum_{k=-\infty}^{\infty} x[k]e^{-j\pi k} e^{-j\Omega k} + \sum_{k=-\infty}^{\infty} x[k]e^{-j\Omega k} \right)
\]

\[
= \frac{1}{2} \left( X(\Omega + \pi) + X(\Omega) \right)
\]

Finally, we have

\[
G(\Omega) = Y(\Omega/2) = \frac{1}{2} X(\Omega/2 + \pi) + \frac{1}{2} X(\Omega/2)
\]

Adding a one sample delay, we have

\[
f[2k - 1] = g[k - \frac{1}{2}] = G(\Omega)e^{-j\Omega/2}
\]