1. (P1.27) Let $X(t)$ be a stationary process with zero mean, autocorrelation function $R_X(\tau)$, and power spectral density $S_X(f)$. We are required to find a linear filter with impulse response $h(t)$, such that the filter output has the same statistical characteristics as $X(t)$ when the input is white noise of power spectral density $N_0/2$.

(a) Determine the condition which the impulse response $h(t)$ must satisfy to achieve this requirement.

\[
R_X(\tau) = \frac{N_0}{2} \delta(\tau).
\]

Thus, the condition which $h(t)$ must satisfy is:

\[
R_X(\tau) = \frac{N_0}{2} \int h(t-\tau)h(t)dt = \frac{N_0}{2}[h(t)*h(t)] \quad (1)
\]

(b) What is the corresponding condition on the frequency response $H(f)$ of the filter?

\[
|H(f)|^2 = \frac{2}{N_0} S_X(f) \quad (2)
\]
is the condition which $H(f)$ must provide in order to satisfy the given statistical characteristics.

**Notice:** We can check that (1) and (2) are two exchangeable forms having the same property, i.e. computing the Fourier transform of (1), we will get (2).

$$S_X(f) = \mathcal{F}[R_X(\tau)] = \mathcal{F} \left[ \frac{N_0}{2} \int_{t} h(t-\tau)h(t)dt \right]$$

$$= \frac{N_0}{2} \int_{\tau} \left[ \int_{t} h(t-\tau)h(t)dt \right] e^{-j2\pi f\tau}d\tau$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi f\tau}d\tau \cdot \int_{t} h(t)dt$$

Let $v = t - \tau$,

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} h(v)e^{-j2\pi f(t-v)}(-dv) \cdot \int_{t} h(t)dt$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} h(v)e^{j2\pi f\tau}dv \cdot \int_{t} h(t)dt$$

$$= \frac{N_0}{2} H^*(f) \cdot H(f)$$

$$= \frac{N_0}{2} |H(f)|^2$$

2. (P1.29) Assume that the narrowband noise $n(t)$ is Gaussian and its power spectral density $S_N(f)$ is symmetric about the midband frequency $f_c$. Show that the in-phase and quadrature components of $n(t)$ are statistically independent.

(Solution.)

Since $S_N(f)$ is symmetric about $f_c$ and is an even function, $S_N(f)$ is also symmetric about $-f_c$. Thus, $S_N(f + f_c) = S_N(f - f_c)$, if $-B \leq f \leq B$

where $2B$ is the bandwidth of the narrowband noise $n(t)$.

By applying this relation to eqn(1.102) in Haykin,

$$S_{N_i,N_Q}(f) = 0 \ , \ \text{for } \forall f$$

This implies that the crosscorrelation $R_{N_i,N_Q}(\tau) = E[N_i(t)N_Q(t+\tau)] = 0$ because $S_{N_i,N_Q}(f)$ and $R_{N_i,N_Q}(\tau)$ are a Fourier transform pair.

As mentioned on p.65 of Haykin, $n_i(t)$ and $n_Q(t)$ are jointly Gaussian if $n(t)$ is Gaussian.

In the case of jointly Gaussian random variables, $X$ and $Y$, if $R_{XY}(\tau) = 0$ ($X$ and $Y$ are uncorrelated), then $X$ and $Y$ are independent.

Thus, $N_i(t)$ and $N_Q(t)$ are statistically independent because $R_{N_i,N_Q}(\tau) = 0$. 

2
3. (P2.4) Consider the AM signal

\[ s(t) = A_c[1 + \mu \cos(2\pi f_m t)] \cos(2\pi f_c t) \]

produced by a sinusoidal modulating signal of frequency \( f_m \). Assume that the modulation factor is \( \mu = 2 \), and the carrier frequency \( f_c \) is much greater than \( f_m \). The AM signal \( s(t) \) is applied to an ideal envelope detector, producing the output \( v(t) \).

(a) Determine the Fourier series representation of \( v(t) \).

(Solution.)

\[ s(t) = A_c[1 + 2 \cos(2\pi f_m t)] \cos(2\pi f_c t) \]

The ideal envelope detector takes the absolute value of its input signal and then passes the result through a lowpass filter. Let \( v(t) \) be the signal resulting from the ideal envelope detector.

\[ v(t) = A_c|1 + 2 \cos(2\pi f_m t)| \]

Fig. 1 shows two signals, \( s(t) \) and \( v(t) \).

![Figure 1: s(t) and v(t)](image)

Here, \( v(t) \) is a periodic signal with a period of \( 1/f_m \). Thus, we can compute the Fourier series of \( v(t) \). The Fourier series representation of a periodic signal is:

\[ v(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kf_m t} \]

with \( a_k = \int_{-\infty}^{\infty} v(t)e^{-j2\pi kf_m t} dt \)
Since \( v(t) \) is an even function,

\[
a_k = f_m \int_{-1/2f_m}^{1/2f_m} A_c [1 + 2 \cos(2\pi f_m t)] \cos(2\pi k f_m t) dt
\]

\[
= 2A_c f_m \int_{0}^{1/2f_m} [1 + 2 \cos(2\pi f_m t)] \cos(2\pi k f_m t) dt
\]

\[
= 2A_c f_m \int_{0}^{1/2f_m} [1 + 2 \cos(2\pi f_m t)] \cos(2\pi k f_m t) dt
\]

\[
= 2A_c f_m \int_{0}^{1/3f_m} [1 + 2 \cos(2\pi f_m t)] \cos(2\pi k f_m t) dt - 2A_c \int_{1/3f_m}^{1/2f_m} [1 + 2 \cos(2\pi f_m t)] \cos(2\pi k f_m t) dt
\]

By using \( \int [1 + 2 \cos(2\pi f_m t)] \cos(2\pi k f_m t) dt = \frac{\sin(2\pi k f_m t)}{2\pi k f_m} + \frac{\sin(2\pi (k+1) f_m t)}{2\pi (k+1) f_m} + \frac{\sin(2\pi (k-1) f_m t)}{2\pi (k-1) f_m} \),

\[
a_k = A_c \left[ \frac{2 \sin\left(\frac{2}{3}\pi k\right) - \sin(\pi k)}{\pi k} + \frac{2 \sin\left(\frac{2}{3}\pi (k+1)\right) - \sin((k+1))}{\pi (k+1)} + \frac{2 \sin\left(\frac{2}{3}\pi (k-1)\right) - \sin((k-1))}{\pi (k-1)} \right]
\]

(b) What is the ratio of second-harmonic amplitude to fundamental amplitude in \( v(t) \)?

\[a_1 = A_c \left( \frac{1}{3} + \frac{\sqrt{3}}{2\pi} \right)\]

\[a_2 = A_c \left( \frac{\sqrt{3}}{2\pi} \right)\]

The ratio

\[
\frac{a_2}{a_1} = 0.4527
\]

4. (P2.6) Consider a square-law detector, using a nonlinear device whose transfer characteristic is defined by

\[v_2(t) = a_1 v_1(t) + a_2 v_1^2(t)\]

where \( a_1 \) and \( a_2 \) are constants, \( v_1(t) \) is the input, and \( v_2(t) \) is the output. The input consists of the AM wave

\[v_1(t) = A_c [1 + k_a m(t)] \cos(2\pi f_c t)\]

(a) Evaluate the output \( v_2(t) \)

\[v_2(t) = a_1 A_c [1 + k_a m(t)] \cos(2\pi f_c t) + a_2 A_c^2 [1 + k_a m(t)]^2 \cos^2(2\pi f_c t)\]

\[= \frac{a_2 A_c^2}{2} [1 + 2K_a m(t) + K_a^2 m^2(t)] + a_1 A_c [1 + k_a m(t)] \cos(2\pi f_c t)\]

\[+ \frac{a_2 A_c^2}{2} [1 + 2k_a m(t) + K_a^2 m^2(t)] \cos(4\pi f_c t)\]
(b) Find the conditions for which the message signal \( m(t) \) may be recovered from \( v_2(t) \).

(Solution.)

If \( v_1(t) \) is bandlimited, \( f_c \gg f_{v_1,\text{max}} \), then we can filter out all components from \( v_2(t) \) except baseband component by using a low-pass filter. Then the resulting signal, \( v_3(t) \), is:

\[
v_3(t) = \frac{a_2 A_c^2}{2} \left[ 1 + 2K_a m(t) + K_a^2 m^2(t) \right]
\]

Because both \( m(t) \) and \( m^2(t) \) contribute for \( v_3(t) \), the power ratio between this two components is pertinent to the recovery of \( m(t) \) from \( v_2(t) \).

\[
\frac{K_a^2 m^2(t)}{2K_a m(t)} = \frac{K_a m(t)}{2}
\]

Thus, we have to set \( K_a \) small enough so that \( \frac{K_a m(t)}{2} \ll 1 \).

5. (P2.21) The single-tone modulating signal \( m(t) = A_m \cos(2\pi f_m t) \) is used to generate the VSB signal

\[
s(t) = \frac{1}{2} a A_m A_c \cos[2\pi(f_c + f_m)t] + \frac{1}{2} A_m A_c(1 - a) \cos[2\pi(f_c - f_m)t]
\]

where \( a \) is a constant, less than unity, representing the attenuation of the upper side frequency.

(a) Find the quadrature component of the VSB signal \( s(t) \).

(Solution.)

\[
s(t) = \frac{1}{2} a A_m A_c \cos[2\pi(f_c + f_m)t] + \frac{1}{2} A_m A_c(1 - a) \cos[2\pi(f_c - f_m)t]
\]

\[
= \frac{1}{2} A_m A_c \cos(2\pi f_m t) \cos(2\pi f_c t) + \frac{1}{2} A_m A_c(1 - 2a) \sin(2\pi f_m t) \sin(2\pi f_c t)
\]

Linear modulated signal \( s(t) \) can be represented as:

\[
s(t) = s_I(t) \cos(2\pi f_c t) - s_Q(t) \sin(2\pi f_c t)
\]

Then,

\[
s_Q(t) = -\frac{1}{2} A_m A_c(1 - 2a) \sin(2\pi f_m t)
\]

(b) The VSB signal, plus the carrier \( A_c \cos(2\pi f_c t) \), is passed through an envelope detector. Determine the distortion produced by the quadrature component.

(Solution.)

\[
s(t) + A_c \cos(2\pi f_c t) = A_c \left[ \frac{1}{2} A_m \cos(2\pi f_m t) + 1 \right] \cos(2\pi f_c t) + \frac{1}{2} A_m A_c(1 - 2a) \sin(2\pi f_m t) \sin(2\pi f_c t)
\]

Let \( v(t) \) denote the envelope signal.

\[
v(t) = A_c \sqrt{\left[ \frac{1}{2} A_m \cos(2\pi f_m t) + 1 \right]^2 + \left[ \frac{1}{2} A_m(1 - 2a) \sin(2\pi f_m t) \right]^2}
\]

\[
= A_c \left[ \frac{1}{2} A_m \cos(2\pi f_m t) + 1 \right] \cdot \sqrt{1 + \left[ \frac{1}{2} A_m(1 - 2a) \sin(2\pi f_m t) \right]^2}
\]
Here, $\left[ \frac{1}{2} A_m \cos(2\pi f_m t) + 1 \right]$ is the signal to be delivered and the remaining term is the distortion term. Let $d(t)$ denote the distortion part.

$$d(t) = \sqrt{1 + \left[ \frac{1}{2} A_m (1 - 2a) \sin(2\pi f_m t) \right]^2}$$

(c) What is the value of constant $a$ for which this distortion reaches its worst possible condition?

(Solution.)

Assuming $0 \leq a$, $d(t)$ has its maximum at $a = 0$. ( $a$ is less than 1)