1. (P1.15) A running integrator is defined by

\[ y(t) = \int_{t-T}^{t} x(\tau) d\tau \]  

(1)

where \( x(t) \) is the input, \( y(t) \) is the output, and \( T \) is the integration period. Both \( x(t) \) and \( y(t) \) are sample functions of stationary processes \( X(t) \) and \( Y(t) \), respectively. Show that the power spectral density of the integrator output is related to that of the integrator input by

\[ S_Y(f) = T^2 \text{sinc}^2(fT)S_X(f) \]  

(2)

(Solution)

The relation between \( S_Y(f) \) and \( S_X(f) \) can be described by (1.58) in Haykins.

\[ S_Y(f) = |H(f)|^2 S_X(f) \]

Thus, we need to show that \( |H(f)|^2 = T^2 \text{sinc}^2(fT) \).

The impulse response \( h(t) \) can be directly obtained from (1) by giving the impulse function, \( \delta(t) \), as an input.

\[
h(t) = \begin{cases} 
1, & 0 \leq t \leq T \\
0, & \text{otherwise}
\end{cases}
\]

By the Fourier transform,

\[
H(f) = \int_{0}^{T} 1 \cdot e^{-j2\pi f t} dt
\]

\[
= \frac{1}{-j2\pi f} \left( e^{-j2\pi f T} - 1 \right)
\]

\[
= \frac{1}{2\pi f} \cdot \left[ \sin 2\pi f T + j(\cos 2\pi f T - 1) \right]
\]

Then,

\[
|H(f)|^2 = \frac{1}{(2\pi f)^2} \cdot \left[ (\sin 2\pi f T)^2 + (\cos 2\pi f T - 1)^2 \right]
\]

\[
= \frac{1}{(2\pi f)^2} \cdot \left[ 4 \sin^2 \pi f T \right]
\]

\[
= \frac{2 \cdot 2 \cos 2\pi f T}{(2\pi f)^2}
\]

\[
= T^2 \text{sinc}^2(fT)
\]

Thus,

\[ S_Y(f) = T^2 \text{sinc}^2(fT)S_X(f) \]
2. (P1.16) A zero-mean stationary process \( X(t) \) is applied to a linear filter whose impulse response is defined by a truncated exponential:

\[
h(t) = \begin{cases} ae^{-at}, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}
\]

Show that the power spectral density of the filter output \( Y(t) \) is

\[
S_Y(f) = \frac{a^2}{a^2 + 4\pi^2 f^2} \left(1 - 2 \exp(-aT) \cos(2\pi f T) + \exp(-2aT)\right) S_X(f)
\]

where \( S_X(f) \) is the power spectral density of the filter input.

(Solution)

Since \( X(t) \) is a stationary process, then as in (P1.15), \( S_Y(f) \) can be described by the following equation.

\[
S_Y(f) = |H(f)|^2 S_X(f)
\]

The Fourier transform of the given \( h(t) \) is

\[
H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt = \int_0^T ae^{-(a+j2\pi f)t} dt = -\frac{a}{a+j2\pi f} \left[ e^{-(a+j2\pi f)T} - 1 \right]
\]

Then,

\[
|H(f)|^2 = H(f)H(f)^* = \left\{ -\frac{a}{a+j2\pi f} \left[ e^{-(a+j2\pi f)T} - 1 \right] \right\} \cdot \left\{ -\frac{a}{a-j2\pi f} \left[ e^{-(a-j2\pi f)T} - 1 \right] \right\} = \frac{a^2}{a^2 + 4\pi^2 f^2} \left[ 1 - 2e^{-aT} \cos(2\pi f T) + e^{2aT} \right]
\]

Thus,

\[
S_Y(f) = \frac{a^2}{a^2 + 4\pi^2 f^2} \left(1 - 2 \exp(-aT) \cos(2\pi f T) + \exp(-2aT)\right) S_X(f)
\]

3. (P1.17) The output of an oscillator is described by

\[
X(t) = A \cos(2\pi F t - \Theta)
\]

where \( A \) is a constant, and \( F \) and \( \Theta \) are independent random variables. The probability density function of \( \Theta \) is defined by

\[
f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \leq \theta \leq 2\pi \\ 0, & \text{otherwise} \end{cases}
\]

Find the power spectral density of \( X(t) \) in terms of the probability density function of the frequency \( F \). What happens to this power spectral density when the frequency assumes a constant value?

(Solution)

Because \( \Theta \) and \( F \) are independent, the joint probability density function becomes:

\[
f_{\Theta,F}(\theta, x) = f_\Theta(\theta) \cdot f_F(x)
\]
Then, autocorrelation function of $X(t)$ becomes:

$$R_X(t, u) = E[X(t)X(u)]$$

$$= \int_{\theta} \int_{x} f_{\theta, F}(\theta, x)X(t)X(u)d\theta dx$$

$$= \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{1}{2\pi} f_F(x)X(t)X(u)d\theta dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_F(x) \int_{0}^{2\pi} A^2 \cos(2\pi xt - \theta) \cdot \cos(2\pi xu - \theta)d\theta dx$$

$$= \frac{A^2}{2\pi} \int_{-\infty}^{\infty} f_F(x) \frac{1}{2} \int_{0}^{2\pi} \cos(2\pi x(t + u)) - 2\theta + \cos(2\pi x(t - u)) d\theta dx$$

$$= \frac{A^2}{2} \int_{-\infty}^{\infty} f_F(x) \cos(2\pi x(t - u)) dx$$

Let $\tau = t - u$,

$$R_X(\tau) = \frac{A^2}{2} \int_{-\infty}^{\infty} f_F(x) \cos(2\pi x\tau) dx$$  \hspace{1cm} (3)$$

And $R_X(\tau)$ can be represented as a Fourier transform pair of power spectral density, $S_X(x)$:

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(x) \exp(j2\pi x\tau) dx$$ \hspace{1cm} (4)$$

Since $R_X(\tau)$ is a real and even function,

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(x) \cos(2\pi x\tau) dx$$ \hspace{1cm} (5)$$

From the similarity between (3) and (5), we can get $S_X(x)$ without calculating the Fourier transform of $R_X(\tau)$ directly.

$$S_X(x) = \frac{A^2}{2} f_F(x)$$

If the frequency $x$ is a constant value, $x_0$, $f_F(x) = \frac{1}{2}\delta(x - x_0) + \frac{1}{2}\delta(x + x_0)$. Thus, $S_X(x)$ becomes:

$$S_X(x) = \frac{A^2}{4}\delta(x - x_0) + \frac{A^2}{4}\delta(x + x_0)$$