Problem 4.4.3

Given:

\[ f_1(t) = 10^4 \text{rect}(10^4 t) \]
\[ f_2(t) = \delta(t) \]

From part (a) in Table 8.4 and time-shifting property, we get:

\[ F_1(\omega) = \text{sinc} \left( \frac{\omega}{20000} \right) \]
\[ F_2(\omega) = 1 \]

(a)

Impulse response is noncausal as the filter is causal. The exponential decay at \( x = 0 \) is from its maximum at \( t = 0 \), where \( T_0 \) is given by the formula:

\[ T_0 = \frac{\ln(100x)}{-10^5} \]

(b)

Choosing a decay rate \( \gamma = 1.8 \) and \( T_0 = 10^{-5} \), \( T_0 = 40 \mu \text{sec} = 40 \times 10^{-6} \text{sec} \).

(c)

\[ y_1(\omega) = H_1(\omega) f_1(\omega) \]
\[ y_2(\omega) = H_2(\omega) f_2(\omega) \]

(d)

Bandwidth of \( y_1(\omega) \rightarrow 10 \text{ kHz} \)

Bandwidth of \( y_2(\omega) \rightarrow 5 \text{ kHz} \)

Since \( y(t) = y_1(t) \cdot y_2(t) \Rightarrow Y(\omega) = Y_1(\omega) \ast Y_2(\omega) \).

From the width property of convolution, bandwidth of \( Y(\omega) \) is the sum of bandwidths of \( Y_1(\omega) \) and \( Y_2(\omega) \). Therefore, bandwidth of \( Y(\omega) = 15 \text{ kHz} \).
#2) (4.5-2)

Given: \( H(\omega) = \left[ \frac{2 \times 10^5}{\omega^2 + 10^5} \right] e^{-j\omega t_0} \)

From pair 3, Table 4.1 and time-shifting property, we get:

\[ h(t) = e^{-10^5|t-t_0|} \]

The impulse response is noncausal, and the filter is unrealizable.

The exponential decays to \( x/100 \) at \( t=0 \) from its maximum at \( t_0 \), where \( t_0 \) is given by the following formula:

\[ \frac{x}{100} = \text{max value} = \frac{1}{e^{-10^5(t-t_0)}} = e^{-10^5t_0} \Rightarrow t_0 = \frac{\ln(100x)}{-10^5} = -10^5 \ln\left(\frac{x}{100}\right) \]

Choosing a decay of 1.8%, \( x=1.8 \) and \( t_0 = -10^5 \ln\left(\frac{1.8}{100}\right) \approx 40 \mu\text{sec} = 4 \times 10^5 \)

At such a small value at \( t=0 \), the filter is approx. realizable.

Also, \( f^{\infty}(t) \leftrightarrow \frac{1}{2\pi} F_1(\omega) + F_2(\omega) \) and from the width property of convolution, the bandwidth of \( f^{\infty}(t) \) is twice the bandwidth of \( f_1(t) \), but of \( f^{\infty}(t) \) is three times the bandwidth of \( f_2(t) \). Similarly, the bandwidth of \( f_1(t) \) is the sum of the bandwidths of \( f_1(t) \) and \( f_2(t) \). Therefore, the Nyquist rate for \( f_1(t) \) is 400kHz, for \( f_2(t) \) is 900kHz, and for \( f(t) \) is 500kHz.
E3801 Signals + Systems HW7 Solutions

#3) (4.6-1)

\[ E_f = \int_{-\infty}^{\infty} f^2(t) dt = \frac{1}{2\pi \sigma^2} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} dt \]

Let \( \frac{t^2}{\sigma^2} = \frac{x^2}{2}, \quad \frac{t}{\sigma} = \frac{x}{\sqrt{2}} \) and \( dt = \frac{\sigma}{\sqrt{2}} dx \)

So, \( E_f = \frac{1}{2\pi \sigma^2} \frac{\sigma}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \frac{\sqrt{2\pi}}{2\sqrt{2} \pi \sigma} = \frac{1}{2\sigma \sqrt{\pi}} \)

Also from pair 22 (Table 4.1)

\[ F(\omega) = e^{-\sigma^2 \omega^2} \Rightarrow E_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sigma^2 \omega^2} d\omega \]

Letting \( \sigma \omega = \sqrt{2} \) and \( d\omega = \left(\frac{1}{\sigma \sqrt{2}}\right) dx \)

\[ E_f = \left(\frac{1}{2\pi}\right) \frac{1}{\sigma \sqrt{2}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = \frac{\sqrt{2\pi}}{2\pi \sigma \sqrt{2}} = \frac{1}{2\sigma \sqrt{\pi}} \]

#4) (5.1-1)

The bandwidths of \( f_1(t) \) and \( f_2(t) \) are 100kHz and 150kHz, respectively. Therefore the Nyquist sampling rate for \( f_1(t) \) is 200kHz and for \( f_2(t) \) is 300kHz.

Also, \( f_1^2(t) \leftrightarrow \frac{1}{2\pi} F_1(\omega) * F_1(\omega) \) and from the width property of convolution, the bandwidth of \( f_1^2(t) \) is twice the bandwidth of \( f_1(t) \), that of \( f_2^3(t) \) is three times the bandwidth of \( f_2(t) \). Similarly, the bandwidth of \( f_1(t)f_2(t) \) is the sum of the bandwidths of \( f_1(t) \) and \( f_2(t) \). Therefore, the Nyquist rate for \( f_1^2(t) \) is 400kHz, for \( f_2^3(t) \) is 900kHz, and for \( f_1(t)f_2(t) \) is 500kHz.
Problem 5 5.1-3

\[ f(t) = \sin(200\pi t) \Rightarrow F(\omega) = 0.005 \text{rect} \left( \frac{\omega}{400\pi} \right) \]

Bandwidth of \( F(\omega) = 100 \text{ Hz} \) (200\pi \text{ rad/s}).

Therefore, Nyquist rate is 200 Hz.

The sampled spectrum consists of \( \frac{1}{T} F(\omega) = 0.005 \text{rect} \left( \frac{\omega}{400\pi} \right) \)

repeating periodically with period equal to sampling frequency.

(i) \( f_s = 150 \text{ Hz} \).

Undersampling \( \Rightarrow \)

\[\begin{array}{ccccccc}
-50\pi & -40\pi & -20\pi & -10\pi & 10\pi & 20\pi & 40\pi & 50\pi
\end{array}\]

\( \omega \) (rad/s)

(ii) \( f_s = 200 \text{ Hz} \).

Nyquist rate \( \Rightarrow \)

\[\begin{array}{cccccc}
-600\pi & -20\pi & 20\pi & 600\pi
\end{array}\]

\( \omega \) (rad/s)

(iii) \( f_s = 300 \text{ Hz} \).

Over-sampling \( \Rightarrow \)

\[\begin{array}{cccccc}
-800\pi & -400\pi & -200\pi & 200\pi & 400\pi & 800\pi
\end{array}\]

\( \omega \) (rad/s)

(b) \( f(t) \) can be recovered from the sampled signal in (ii) and (iii) by passing it through a low-pass filter, with sharp cut-off at 100 Hz in (ii) and with a cut-off freq. between 100 Hz and 200 Hz in (iii).
(i) \[ F(\omega) H(\omega) \]
\[ f_s = 150 \text{ Hz} \]

(ii) \[ f_s = 200 \text{ Hz} \]

(iii) \[ f_s = 300 \text{ Hz} \]

(c) Bandwidth of \( y_1(\omega) \rightarrow 10 \text{ kHz} \)
Bandwidth of \( y_2(\omega) \rightarrow 5 \text{ kHz} \)

Since \( y(\omega) = y_1(\omega) - y_2(\omega) \)
\[ \gamma(\omega) = \gamma_1(\omega) + \gamma_2(\omega) \]

From the cascade property of convolution, bandwidth of \( y(\omega) \) is sum of bandwidth of \( y_1(\omega) \) and \( y_2(\omega) \). Therefore
Bandwidth of \( y(\omega) = 15 \text{ kHz} \)