Homework Set no. 8

1. Convexity and Binary Channels.
   (a) Prove that $I(X;Y)$ is a convex function of $p(y|x)$ for fixed $p(x)$. You may use the fact that $D(p||q)$ is convex in the pair $(p,q)$.
   (b) For this part you may assume the result of the previous part. Consider the channel shown in Figure 1. Assume $P(X = 0) = \lambda$ where $0 < \lambda < 1$. Show that replacing both $p$ and $q$ with $p + q/2$ will either lower the mutual information or leave it alone. This new channel has a certain symmetry, but it is not generally the standard binary symmetric channel.
   (c) (1 pt) For fixed $\lambda$, compute the mutual information for the reduced-mutual-information channel you created in the previous part.
   (d) 1 pt Argue that for fixed $p$, one choice of $q$ that certainly minimizes $I(X;Y)$ is $q = p$.

2. Maximum number of configurations with assigned minimum distance
   Consider a binary code (i.e., the alphabet is $\{0,1\}$ of length $n$. An important question in coding is to determine the maximum number of configurations (codewords) having minimum pairwise distance $d$. Call this number $A(n,d)$. This is a difficult problem, but it is easy to come up with bounds and equalities. Among these equalities, two are of interest and are not too difficult to prove.
   - Show that $A(n,d) \leq 2A(n-1,d)$;
   - Show that $A(n+1,2s) = A(n,2s-1)$;
   
   Note: though it is not too difficult to come up with the answers, the problem requires a bit of thought. Writing up an example might give you an insight of what a proof might be.

3. Ternary Channel
   Consider the ternary channel (i.e., where both inputs and outputs are ternary) defined by the conditional probability matrix
   
   $$P_{i,j} = P(Y = i \mid X = j) = \begin{bmatrix} 2/3 & 0 & 1/3 \\ 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \end{bmatrix}.$$
Compute the capacity of this channel.

4. **Differential entropy (EIT 9.1).** Evaluate the differential entropy $h(X) = -\int f \ln f$ for the following:

(a) The exponential density, $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$.

(b) The Laplace density, $f(x) = \frac{1}{2} \lambda e^{-\lambda |x|}$.

(c) The sum of $X_1$ and $X_2$, where $X_1$ and $X_2$ are independent normal random variables with means $\mu_i$ and variances $\sigma_i^2$, $i = 1, 2$.

*Hint: this is essentially a mechanical problem.*