COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

General Image Transforms and Applications

Lecture 6, March 3rd, 2008

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EE4830 Digital Image Processing
http://www.ee.columbia.edu/~xlx/ee4830/

thanks to G&W website, Min Wu, Jelena Kovacevic and Martin Vetterli for slide materials

announcements

- HW#2 due today
- HW#3 will be out by Wednesday

- Midterm on March 10^{th}
  - "Open-book"
    - YES: text book(s), class notes, calculator
    - NO: computer/cellphone/matlab/internet
  - 5 analytical problems
  - Coverage: lecture 1-6
    - intro, representation, color, enhancement, transforms and filtering (until DFT and DCT)
  - Additional instructor office hours
    - 2-4 Monday March $10^{th,}$ Mudd 1312

- Grading breakdown
  - HW-Midterm-Final: 30%-30%-40%

outline

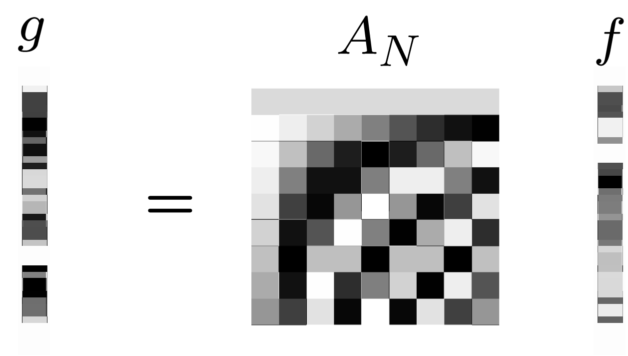
- Recap of DFT and DCT
- Unitary transforms
- KLT
- Other unitary transforms
- Multi-resolution and wavelets
- Applications

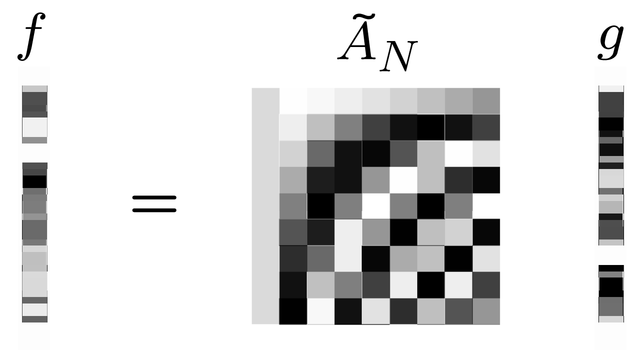
- Readings for today and last week: G&W Chap 4, 7, Jain 5.1-5.11

recap: transform as basis expansion

$$g(u) = \sum_{n=0}^{N-1} f(n)\ a_N^{un}$$

$$f(n) = \sum_{u=0}^{N-1} g(u)\ \tilde{a}_N^{un}$$





DFT: $a_N^{un} = e^{-j2\pi\frac{un}{N}}, \quad \tilde{a}_N^{un} = a_N^{*un}, \quad \tilde{A}_N = A_N^{*T}$

DCT:

$$a_N^{0n} = \sqrt{\frac{1}{N}} \qquad u = 0$$

$$a_N^{un} = \sqrt{\frac{2}{N}} cos\frac{\pi(2n+1)u}{2N} \qquad u = 1, \ldots, N-1$$

$$\tilde{a}_N^{un} = a_N^{un}, \quad \tilde{A}_N = A_N^T$$

recap: DFT and DCT basis

1D-DCT

$$a_N^{0n} = \sqrt{\frac{1}{N}} \qquad u = 0$$

$$a_N^{un} = \sqrt{\frac{2}{N}} cos \frac{\pi(2n+1)u}{2N}$$

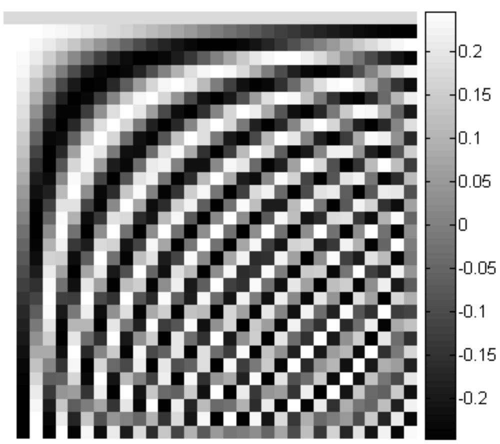
$$u = 1, 2, \ldots, N-1$$

1D-DFT

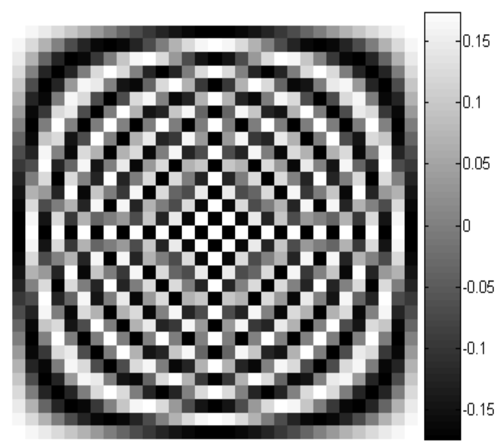
$$\begin{aligned} a_N^{un} &= e^{-j2\pi\frac{un}{N}} \\ &= cos(2\pi\frac{un}{N}) - jsin(2\pi\frac{un}{N}) \end{aligned}$$

N=32

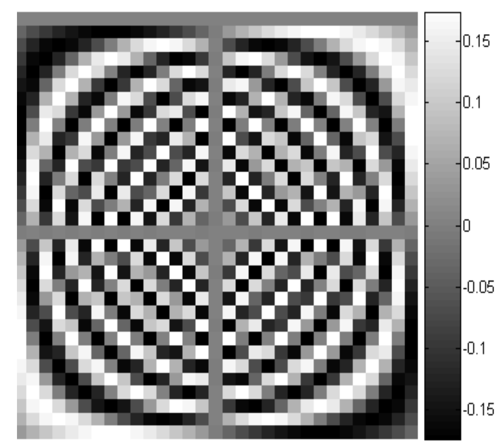
A



real(A)



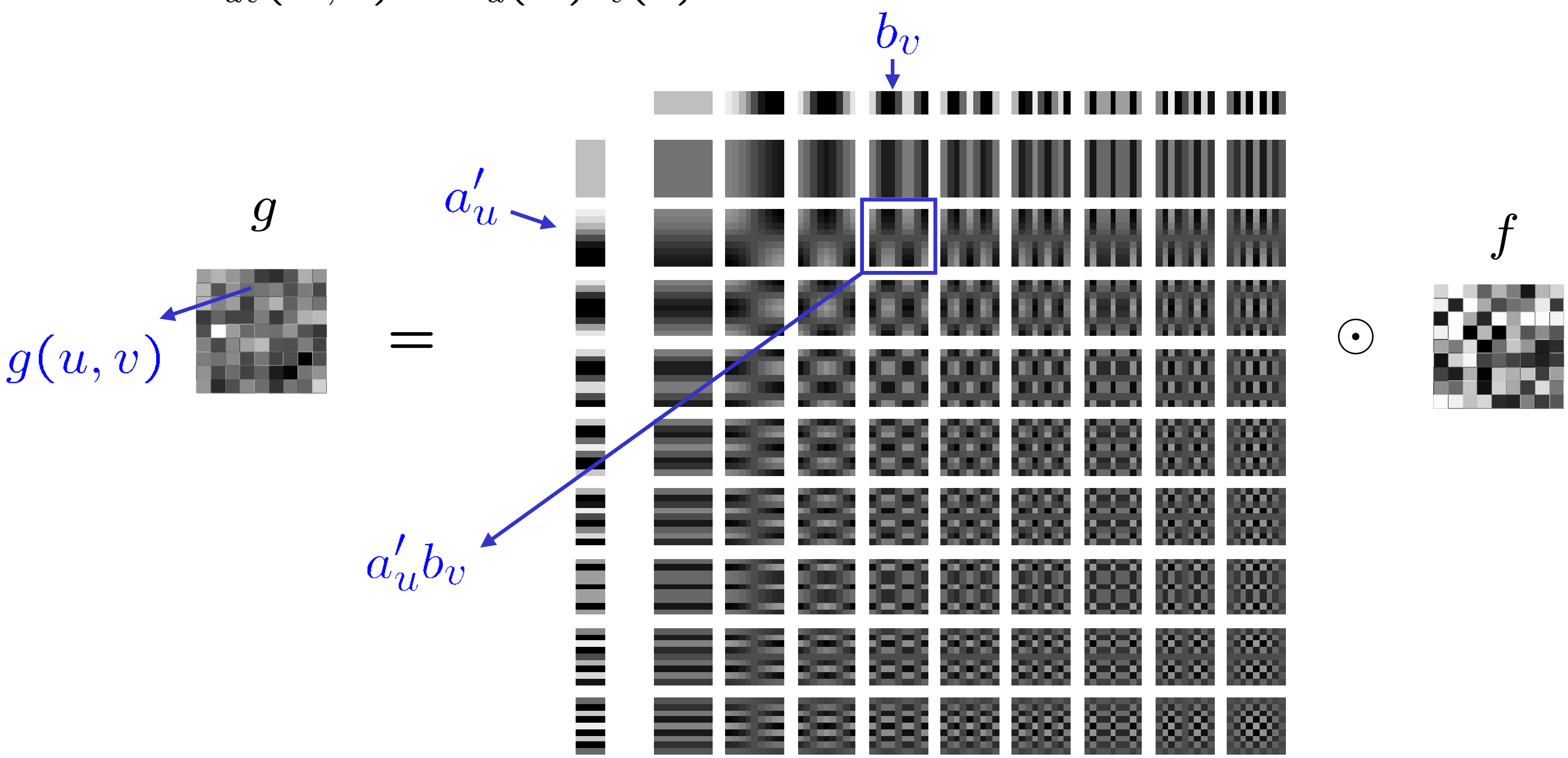
imag(A)



recap: 2-D transforms

$$g(u,v) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)a_{uv}(m,n), \quad f(m,n) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} g(u,v)\tilde{a}_{uv}(m,n)$$

the transform is *separable*,
when $a_{uv}(m,n) = a_u(m)b_v(n)$.



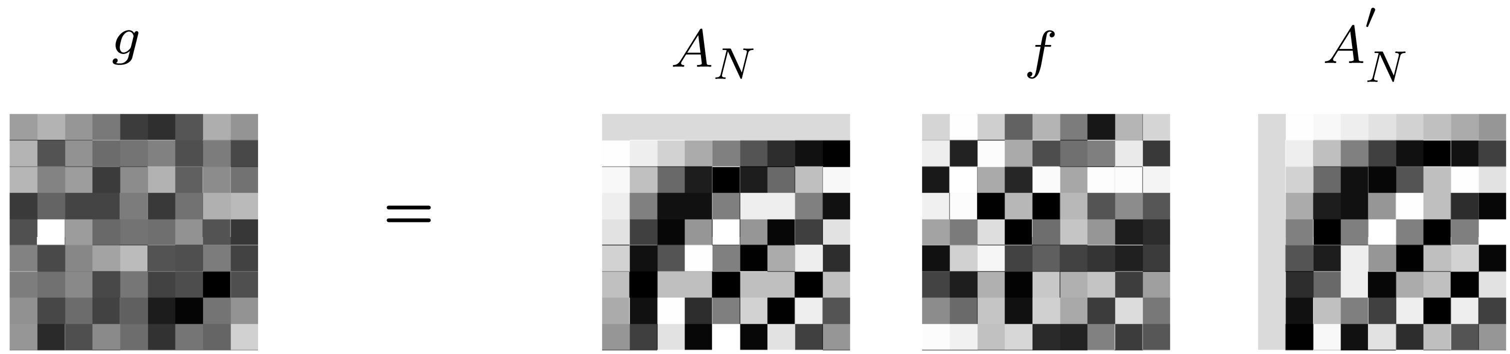
2D-DFT and 2D-DCT are separable transforms.

separable 2-D transforms

when $a = b,\ M = N$

$$g(u,v) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} a_N^{um}\ f(m,n)\ a_N^{vn}$$

$$f(m,n) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \tilde{a}_N^{um}\ g(u,v)\ \tilde{a}_N^{vn}$$



Symmetric 2D separable transforms can be expressed with the notations of its corresponding 1D transform.

→ We only need to discuss 1D transforms

two properties of DFT and DCT

$$g(u) = \sum_{n=0}^{N-1} f(n)\ a_N^{un} \qquad \tilde{A}_N = A_N^{*T}$$

$$f(n) = \sum_{u=0}^{N-1} g(u)\ \tilde{a}_N^{un}$$

- Orthonormal (Eq 5.5 in Jain)
  : no two basis represent the same information in the image

$$\sum_n a_N^{un} a_N^{*vn} = \delta(u-v)$$

- Completeness (Eq 5.6 in Jain)
  : all information in the image are represented in the set of basis functions

$$\sum_u a_N^{um} a_N^{*un} = \delta(m-n)$$

⇨

for $Q < N$, let $f_Q(n) = \sum_{u=0}^{Q-1} \hat{g}(u) a_N^{*un}$

$\sigma_Q^2 = \sum_{n=1}^{N-1} [f(n) - f_Q(n)]^2$ minimized when $\hat{g}(u) = g(u)$

$f - f_Q = 0$, iff. $Q = N$

Exercise

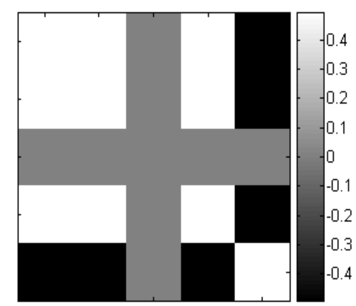
- How do we decompose this picture?

DCT2 →

?

$$A_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

DCT2 basis image



■ =-1

□ =1

$$\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

■ =0

□ =1

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{DCT2}} \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

What if black=0, does the transform coefficients look similar?

Unitary Transforms

A linear transform:

$$\mathcal{R}^N \rightarrow \mathcal{R}^N \qquad g = A_N\ f, \quad f = A_N^{*T}\ g$$

The Hermitian of matrix A is: $A^H = A^{*T}$

This transform is called "unitary" when A is a unitary matrix,
"orthogonal" when A is unitary and real.

$$A^{-1} = A^H, \ AA^H = A^* A^T = I$$

- Two properties implied by construction
  - Orthonormality

$$\sum_n a_N^{un} a_N^{*vn} = \delta(u-v)$$

  - Completeness

$$\sum_u a_N^{um} a_N^{*un} = \delta(m-n)$$

Exercise

- Are these transform matrixes unitary/orthogonal?

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2} & j \\ -j & \sqrt{2} \end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & j \\ j & 1 \end{bmatrix}$$

properties of 1-D unitary transform

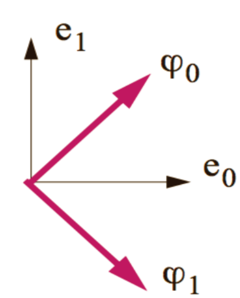
- energy conservation $\|g\|^2 = \|f\|^2$

$$\|g\|^2 = \|Af\|^2 = (Af)^{*T}(Af) = f^{*T}A^{*T}Af = f^{*T}f = \|f\|^2$$

- rotation invariance
  - the angles between vectors are preserved

$$cos\theta = \frac{f_1 \cdot f_2}{\|f_1\|\|f_2\|} \qquad g_1 \cdot g_2 = g_1^{*T}g_2 = (Af_1)^{*T}Af_2 = f_1 \cdot f_2$$

  - unitary transform: rotate a vector in R^n, i.e., rotate the basis coordinates



observations about unitary transform

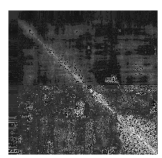
- Energy Compaction
  - Many common unitary transforms tend to pack a large fraction of signal energy into just a few transform coefficients
- De-correlation
  - Highly correlated input elements → quite uncorrelated output coefficients
  - Covariance matrix $R_g = cov(g) = E\{(g - E\{g\})(g - E\{g\})^{*T}\}$

  let $\hat{g} = g - E\{g\}$, then $R_{mn} = E\{\hat{g}_m \hat{g}_n\}$

f: columns of image pixels

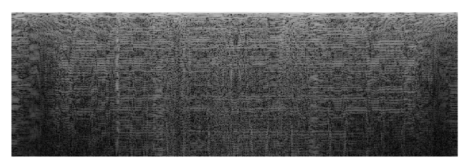
$g = DCT(f)$

$f_1, f_2, \ \ldots, \ f_{600}$ $cov(f)$ $g_1, g_2, \ \ldots, \ g_{600}$ $cov(g)$

linear display scale: g






display scale: log(1+abs(g))

one question and two more observations

- Is there a transform with
  - best energy compaction
  - maximum de-correlation
  - is also unitary… ?
- transforms so far are data-independent
  - transform basis/filters do not depend on the signal being processed
  - "optimal" should be defined in a statistical sense so that the transform would work well with many images
  - signal statistics should play an important role

review: correlation after a linear transform

- x is a zero-mean random vector in $\mathcal{R}^N$

  $E[x] = 0$

  - the covariance (autocorrelation) matrix of x

    $R_x = cov(x) = E[xx^H]$

    - $R_x(i,j)$ encodes the correlation between x_i and x_j
    - R_x is a diagonal matrix iff. all N random variables in x are uncorrelated

- apply a linear transform: $y = Ax$
  - What is the correlation matrix for y ?

$$R_y = cov(y) = E[yy^H] = E[Ax(Ax)^H]$$

$$= E[Axx^H A^H] = AE[xx^H]A^H = AR_x A^H$$

transform with maximum energy compaction

$$y = A'x$$

$$y(u) = a'_u x$$

$$A' = \begin{bmatrix} a'_0 \\ a'_1 \\ \cdots \\ a'_{N-1} \end{bmatrix}$$

$$a'_u a^*_u = 1$$
$$a'_u a^*_v = 0 \quad \forall u \neq v$$

$$\|x\|^2 = E[x^H x] = \sum_u R_x(u,u)$$
$$\|y\|^2 = E[y^H y] = \|x\|^2$$
$$\|y_Q\|^2 = \sum_{u=0}^{Q-1} y^2(u)$$

max. $E[y_Q^H y_Q]$

s.t. $y(u) = a'_u x, \quad a'_u a^*_u = 1, \; a'_u a^*_v = 0 \quad \forall u \neq v$

proof. maximum energy compaction

max. $E[y_Q^H y_Q] = E[(A_Q x)^H A_Q x]$

$$A_Q = \begin{pmatrix} a_0^{'} \\ \cdots \\ a_{Q-1}^{'} \\ \cdots \\ 0 \end{pmatrix}$$

$$= E\left[x^H \begin{pmatrix} a_0^* & \ldots & a_{Q-1}^* & \ldots 0 \end{pmatrix} \begin{pmatrix} a_0^{'} \\ \cdots \\ a_{Q-1}^{'} \\ \cdots \\ 0 \end{pmatrix} x\right]$$

$$= E[x^H \sum_{u=0}^{Q-1} a_u^* a_u^{'} \; x]$$

$a_u^{'} a_v^* = 0$

$$= \sum_{u=0}^{Q-1} a_u^{'} R_x a_u^*$$

$a_u^{'} a_u^* = 1$

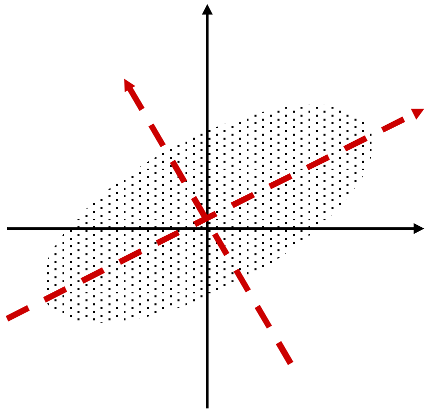
let $L = \sum_{u=0}^{Q-1} a_u^{'} R_x a_u^* - 2 \sum_{u=0}^{Q-1} \lambda_u (1 - a_u^{'} a_u^*)$

$$\frac{\partial L}{\partial a_u^*} = 2 R_x a_u^* - 2 \lambda_u a_u^* = 0$$

⇨ a^*_u are the eigen vectors of R_x

$$R_x a_u^* = \lambda_u a_u^*$$

Karhunen-Loève Transform (KLT)



- a unitary transform with the basis vectors in A being the "orthonormalized" eigenvectors of R_x

  $$y = A^T x,\ x = Ay,$$

  $$\text{with } A \in \mathcal{R}^{N\times N},\ A = [a_0, \ldots, a_{N-1}]$$

  $$R_x a_u = \lambda_u a_u,\ u = 0, \ldots, N-1$$

  - assume real input, write A^T instead of A^H
  - denote the inverse transform matrix as A, $AA^T=I$
  - R_x is symmetric for real input, Hermitian for complex input i.e. $R_x^T=R_x$, $R_x^H = R_x$
  - R_x nonnegative definite, i.e. has real non-negative eigen values

- Attributions
  - Kari Karhunen 1947, Michel Loève 1948
  - a.k.a Hotelling transform (Harold Hotelling, discrete formulation 1933)
  - a.k.a. Principle Component Analysis (PCA, estimate R_x from samples)

Properties of K-L Transform

- Decorrelation by construction

$$R_y = E[yy^T] = AR_xA^T = \begin{pmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \cdots & \\ & & & \lambda_{N-1} \end{pmatrix}$$

  - note: other matrices (unitary or nonunitary) may also de-correlate the transformed sequence [Jain's example 5.5 and 5.7]

- Minimizing MSE under basis restriction
  - Basis restriction: Keep only a subset of m transform coefficients and then perform inverse transform (1≤ m ≤ N)

  → Keep the coefficients w.r.t. the eigenvectors of the first m largest eigenvalues

$$v = A^T u \quad w = I_m v \quad z = Bw$$

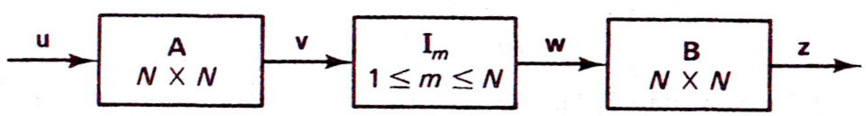


Figure 5 16 KL transform basis restriction

$$I_m = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \cdots & & \\ & & & 0 & \\ & & & & \cdots \end{pmatrix}$$

discussions about KLT

- The good
  - Minimum MSE for a “shortened” version
  - De-correlating the transform coefficients
- The ugly
  - Data dependent
    - Need a good estimate of the second-order statistics
    - Increased computation complexity

data: $x_1, \ldots, x_M \in \mathcal{R}^N$

estimate R_x: $O(MN)$

linear transform: $O(MN)$

compute eig R_x: $\sim O(N^3)$

fast transform: $O(M \log N)$

Is there a data-independent transform with similar performance?

energy compaction properties of DCT

- DCT is close to KLT when …
  - x is first-order stationary Markov $x_n = \rho x_{n-1} + z_n, \quad z_n \sim \mathcal{N}(0, \sigma_z^2), \ |\rho| < 1$

$$\rightarrow \quad E[x_n x_{n-1}] = \rho\sigma_x^2, \ E[x_n x_{n-2}] = \rho^2\sigma_x^2, \ \ldots \quad r(n) = \rho^{|n|}$$

$$\rightarrow \quad R_x = \begin{pmatrix} 1 & \rho & \rho^2 & \ldots & \\ \rho & 1 & \rho & & \\ \ldots & & \ldots & & \\ \rho^{n-1} & & & & 1 \end{pmatrix}$$

$$\beta^2 \triangleq \frac{\rho^2}{1+\rho^2}$$

$$\alpha \triangleq \frac{\rho}{1+\rho^2}$$

$$\rightarrow \quad \beta^2 R_x^{-1} = \begin{pmatrix} 1-\rho\alpha & -\alpha & & & \\ -\alpha & 1 & -\alpha & 0 & \\ \ldots & & \ldots & & \\ & 0 & & -\alpha & 1-\rho\alpha \end{pmatrix}$$

- R_x and $\beta^2 R_x^{-1}$ have the same eigen vectors
- $\beta^2 R_x^{-1} \sim Q_c$ when ρ is close to 1

  - DCT basis vectors are eigenvectors of a symmetric tri-diagonal matrix Q_c

$$Q_c = \begin{pmatrix} 1-\alpha & -\alpha & 0 & \ldots & \\ -\alpha & 1 & -\alpha & & \\ \ldots & & \ldots & & \\ 0 & & & -\alpha & 1-\alpha \end{pmatrix}$$

$$a_0 = const.$$

$$a_u \propto \left[1, \ cos\frac{\pi 3u}{2N}, \ \ldots, \ cos\frac{\pi u(2N-1)}{2N}\right]$$

$$\rightarrow \quad Q_c a_u = \lambda_u a_u$$

[trigonometric identity cos(a+b)+cos(a-b)=2cos(a)cos(b)]

DCT energy compaction

- DCT is close to KLT for
  - highly-correlated first-order stationary Markov source

- DCT is a good replacement for KLT
  - Close to optimal for highly correlated data
  - Not depend on specific data
  - Fast algorithm available

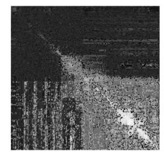
DCT/KLT example for vectors

x: columns of image pixels $\rho^* = 0.8786$

fraction of coefficient values in the diagonal

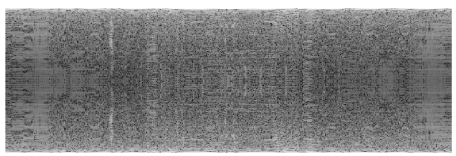
$x_1, x_2, \ldots, x_{600}$

R_x



0.0136

$abs(DFT_{1D}(x))$

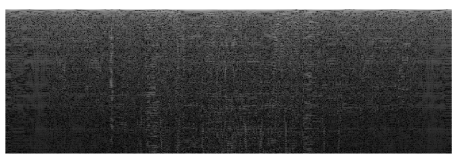


$R_{DFT(x)}$

0.1055

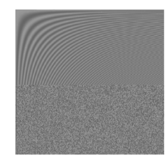
transform basis

$DCT_{1D}(x)$

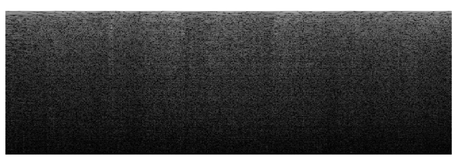


$R_{DCT(x)}$

0.1185

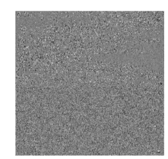


$KLT_{1D}(x)$



$R_{KLT(x)}$

1.0000



display scale: log(1+abs(g)), zero-mean

KL transform for images

- autocorrelation function 1D → 2D

  $x(1:n)$ $\quad\quad$ $R_x(n_1, n_2)$

  $x(1:m,\ 1:n)$ $\quad\quad$ $R_x(m_1, m_2, n_1, n_2)$

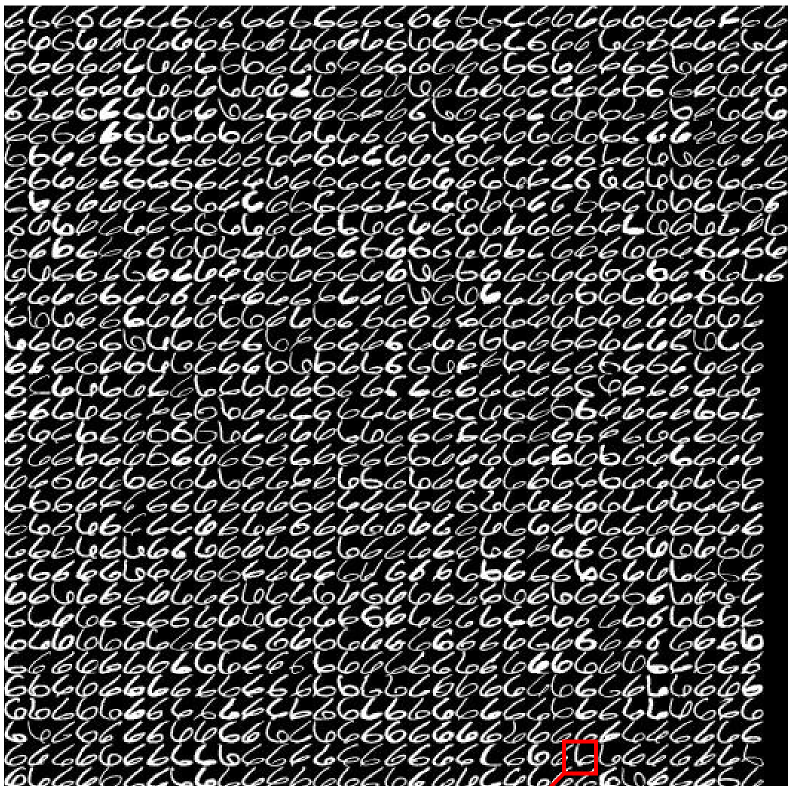
- KL basis images are the orthonormalized eigen-functions of *R*
- rewrite images into vector forms (N^2x1)
  - solve the eigen problem for N^2xN^2 matrix ~ $O(N^6)$

- if R_x is "separable"

  $$R_x(m_1, m_2, n_1, n_2) \rightarrow r(m_1, m_2) \cdot r(n_1, n_2)$$

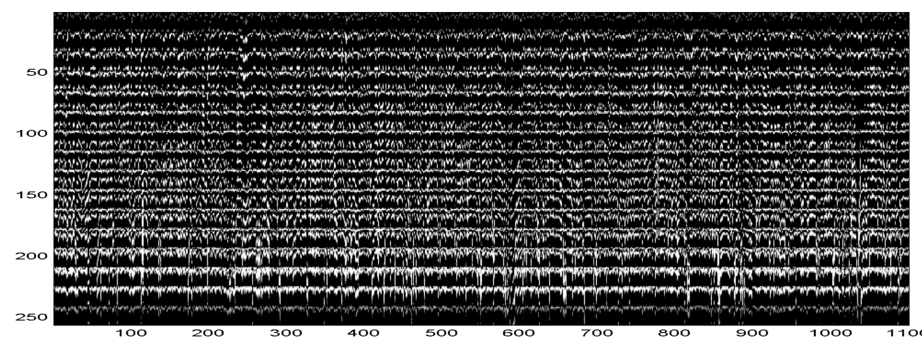
  - perform separate KLT on the rows and columns
  - transform complexity $O(N^3)$

KLT on hand-written digits …



1100 digits “6”
16x16 pixels

1100 vectors of size 256x1



The Desirables for Image Transforms

	DFT	DCT	KLT
Theory			
Inverse transform available	✓	✓	✓
Energy conservation (Parsevell)	✓	✓	✓
Good for compacting energy	?	?	✓
Orthonormal, complete basis	✓	✓	✓
(sort of) shift- and rotation invariant	✓	✓	?
Transform basis signal-independent			
Implementation			
Real-valued	x	✓	✓
Separable	✓	✓	x
Fast to compute w. butterfly-like structure	✓	✓	x
Same implementation for forward and inverse transform	✓	✓	x

Walsh-Hadamard Transform

$$H_0 = +1$$

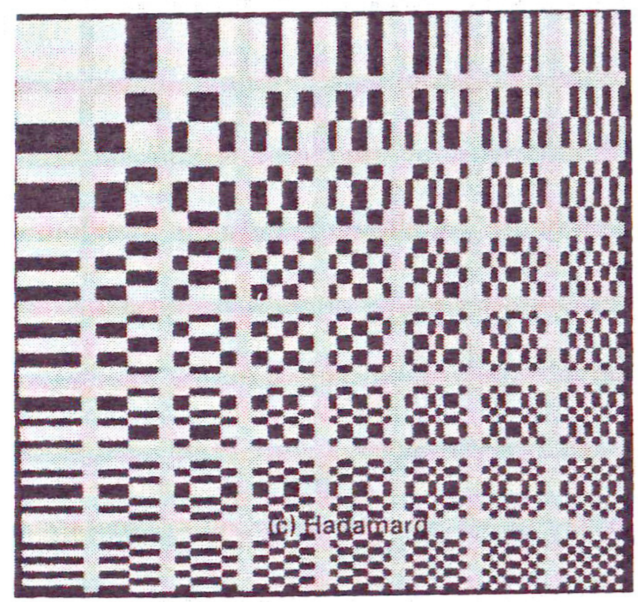
$$H_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$H_3 = \frac{1}{2^{3/2}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}.$$

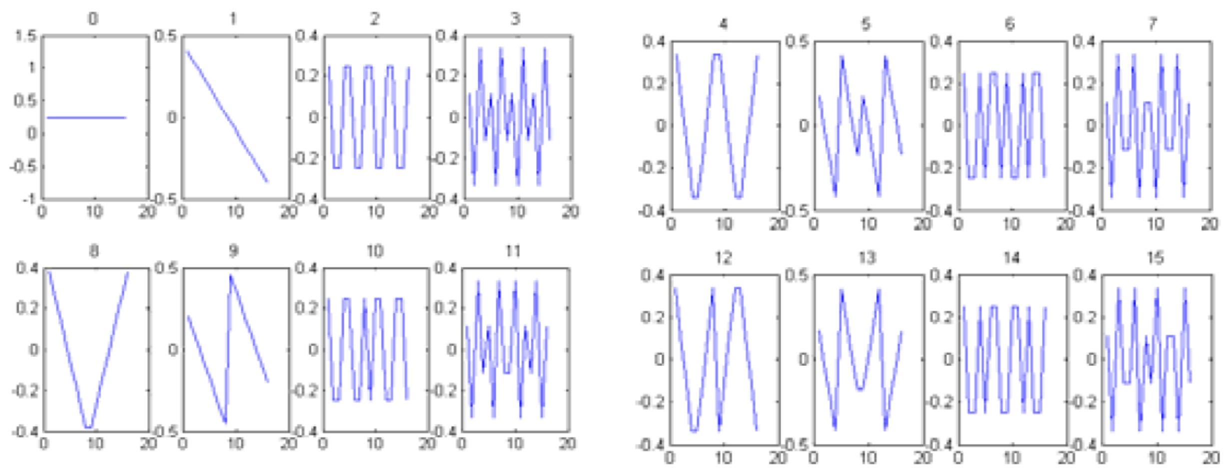
$$H_m = \frac{1}{\sqrt{2}}\begin{pmatrix} H_{m-1} & H_{m-1} \\ H_{m-1} & -H_{m-1} \end{pmatrix},$$

$$(H_m)_{k,n} = \frac{1}{2^{m/2}}(-1)^{\sum_j k_j n_j}$$



slant transform

$$
\begin{bmatrix}
0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 \\
0.5401 & 0.3858 & 0.2315 & 0.0772 & -0.0772 & -0.2315 & -0.3858 & -0.5401 \\
0.3536 & -0.3536 & -0.3536 & 0.3536 & 0.3536 & -0.3536 & -0.3536 & 0.3536 \\
0.1581 & -0.4743 & 0.4743 & -0.1581 & 0.1581 & -0.4743 & 0.4743 & -0.1581 \\
0.4743 & 0.1581 & -0.1581 & -0.4743 & -0.4743 & -0.1581 & 0.1581 & 0.4743 \\
0.2415 & -0.0345 & -0.3105 & -0.5866 & 0.5866 & 0.3105 & 0.0345 & -0.2415 \\
0.3536 & -0.3536 & -0.3536 & 0.3536 & -0.3536 & 0.3536 & 0.3536 & -0.3536 \\
0.1581 & -0.4743 & 0.4743 & -0.1581 & -0.1581 & 0.4743 & -0.4743 & 0.1581
\end{bmatrix}
$$



Nassiri et. al, “Texture Feature Extraction using Slant-Hadamard Transform”

energy compaction comparison

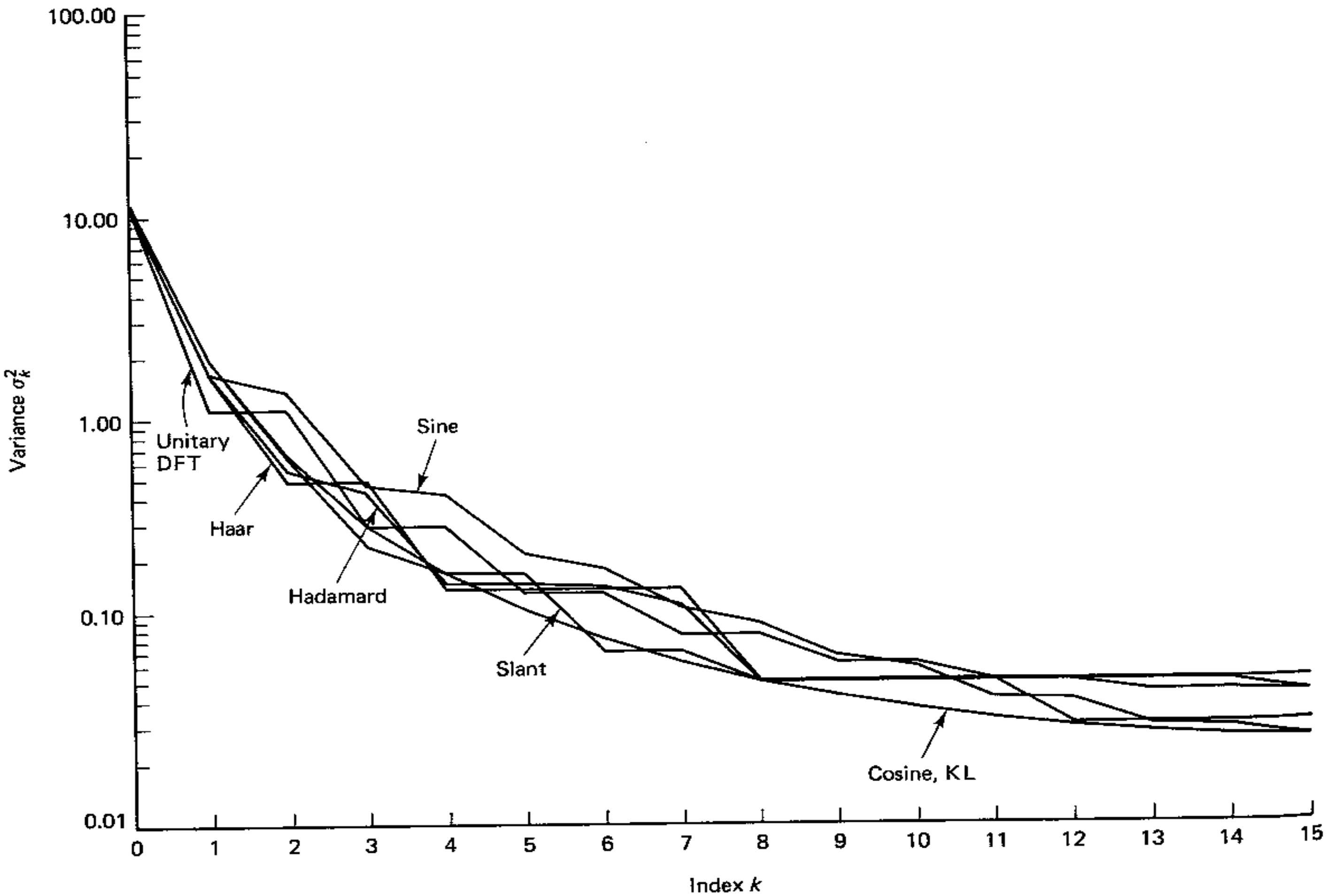
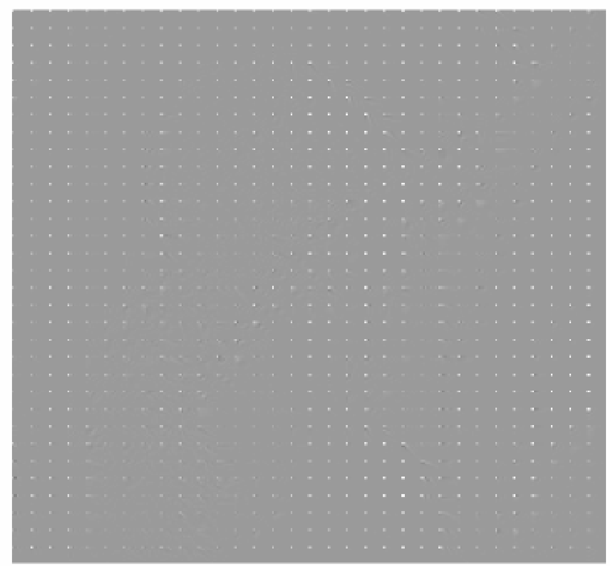


Figure 5.18 Distribution of variances of the transform coefficients (in decreasing order) of a stationary Markov sequence with $N = 16$, $\rho = 0.95$ (see Example 5.9).

implementation note: block transform

- similar to STFT (short-time Fourier transform)
  - partition a NxN image into mxn sub-images
  - save computation: O(N) instead of O(NlogN)
  - loose long-range correlation



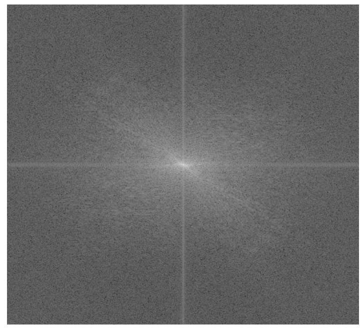
8x8 DCT coefficients

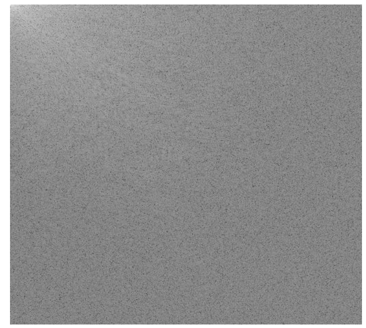
applications of transforms

- enhancement

- (non-universal) compression
- feature extraction and representation
- pattern recognition, e.g., eigen faces
- dimensionality reduction
  - analyze the principal ("dominating") components

Image Compression



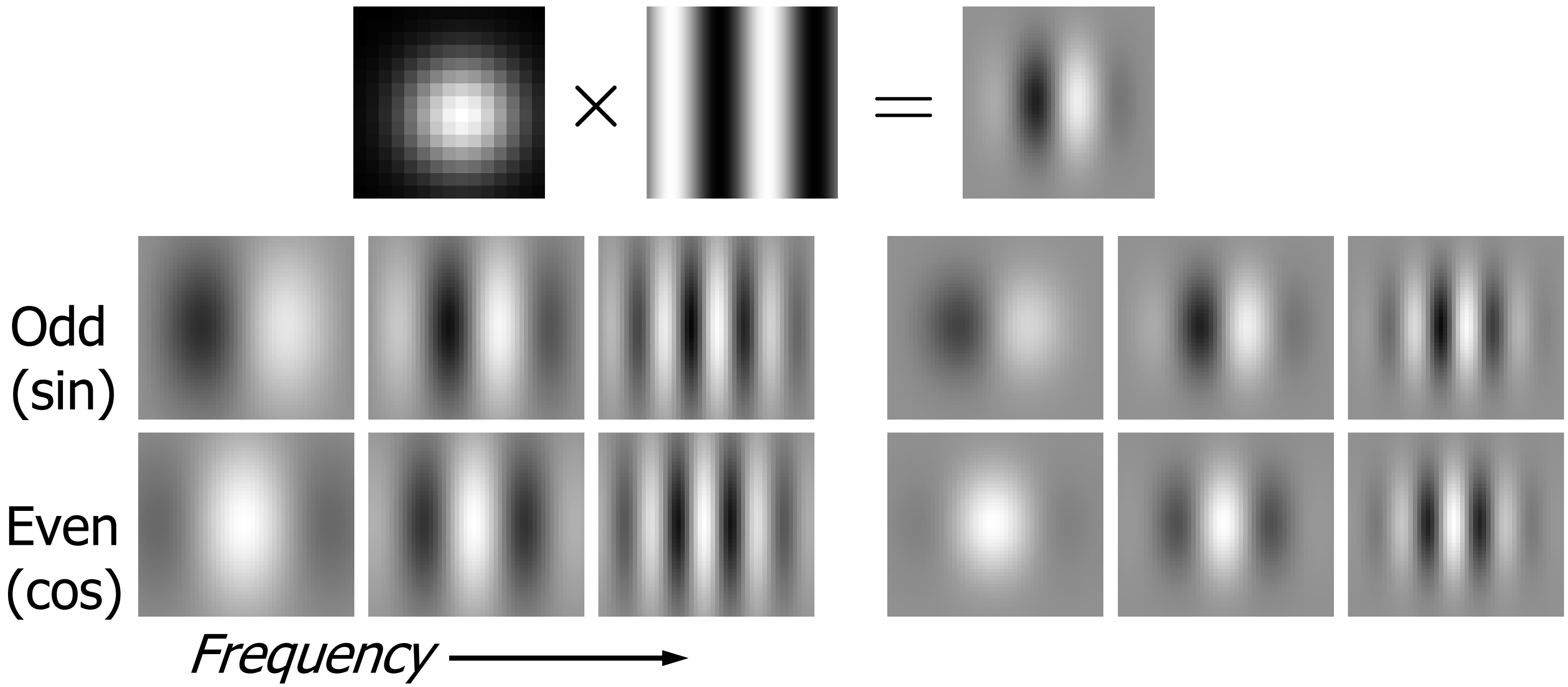


$$\mathrm{SNR(dB)} = 10\log_{10}\left(\frac{P_{\mathrm{signal}}}{P_{\mathrm{noise}}}\right) = 20\log_{10}\left(\frac{A_{\mathrm{signal}}}{A_{\mathrm{noise}}}\right)$$

where P is average power and A is RMS amplitude.

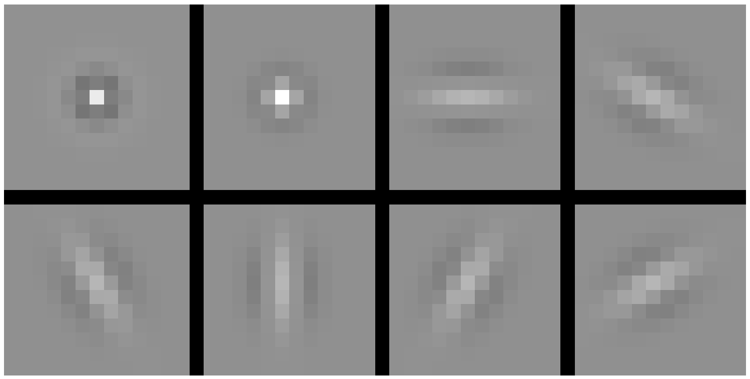
Gabor filters

- Gaussian windowed Fourier Transform
  - Make convolution kernels from product of Fourier basis images and Gaussians

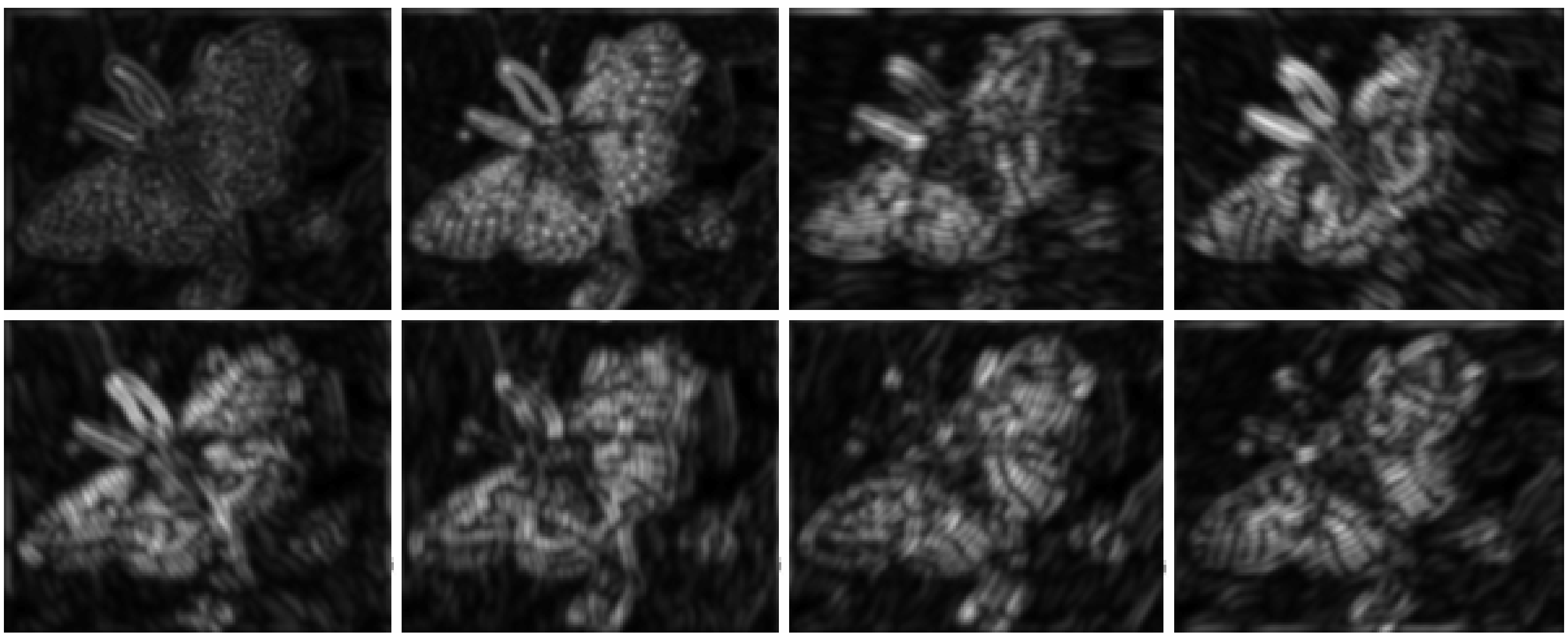


Example: Filter Responses

Filter bank



Input image

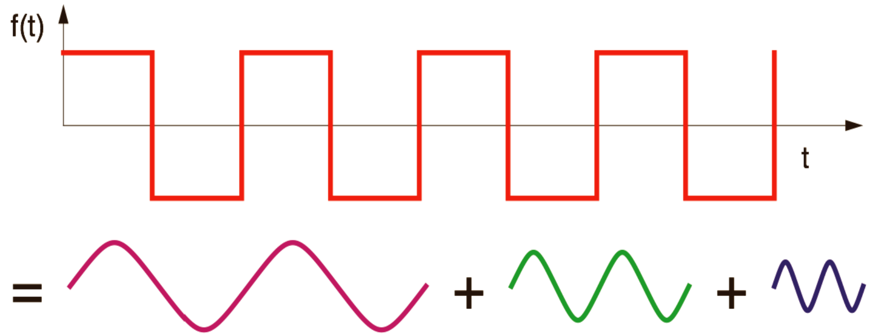


from Forsyth & Ponce

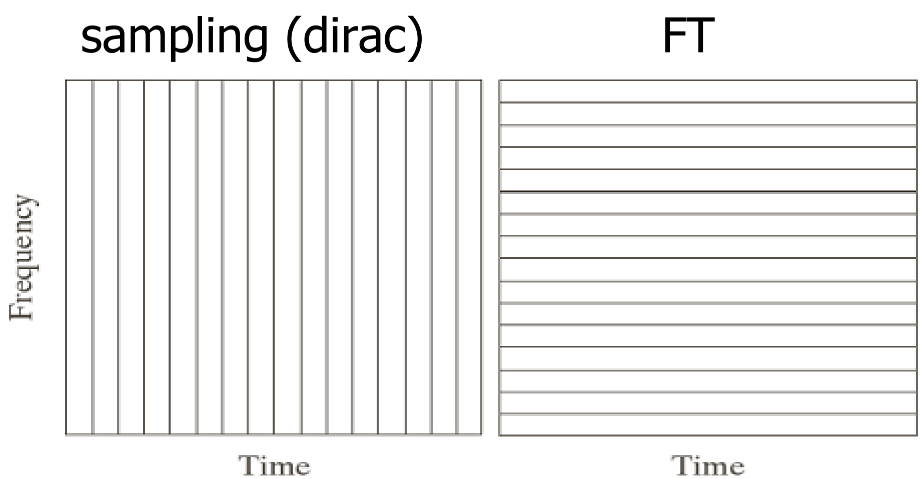
outline

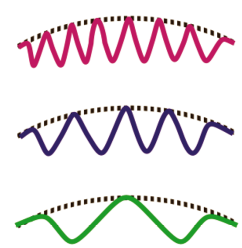
- Recap of DFT and DCT
- Unitary transforms
- KLT
- Other unitary transforms
- Multi-resolution and wavelets
- Applications

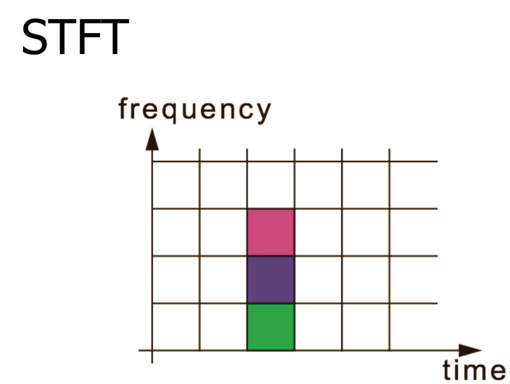
1807: Fourier upsets the French Academy....



Fourier Series: Harmonic series, frequency changes, f_0, $2f_0$, $3f_0$, ...

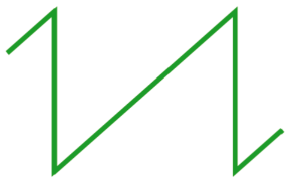






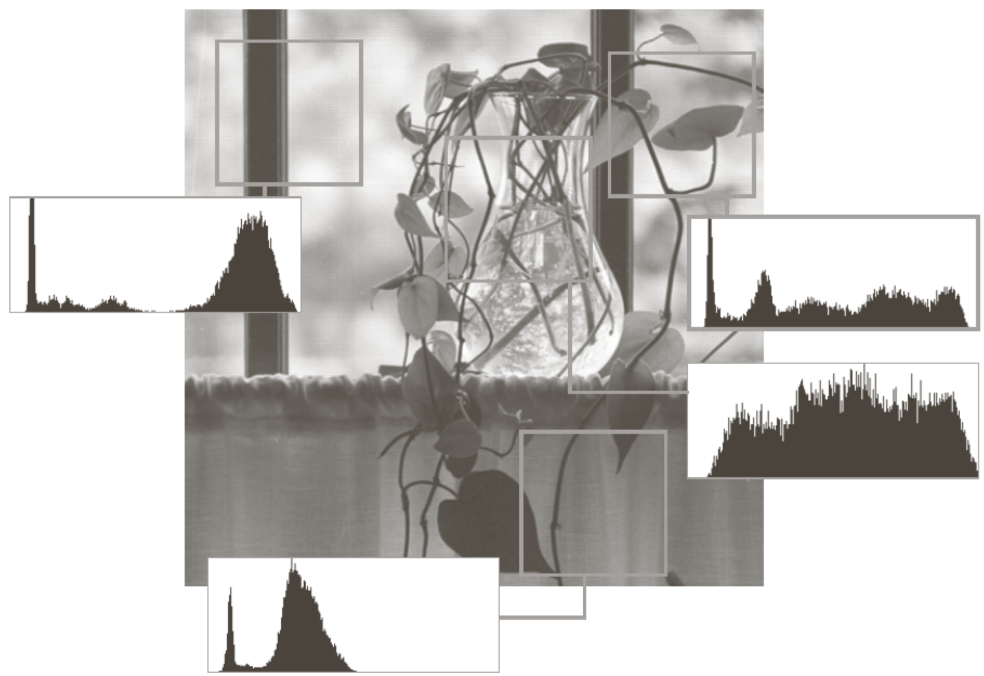
FT does not capture discontinuities well

But... 1898: Gibbs' paper

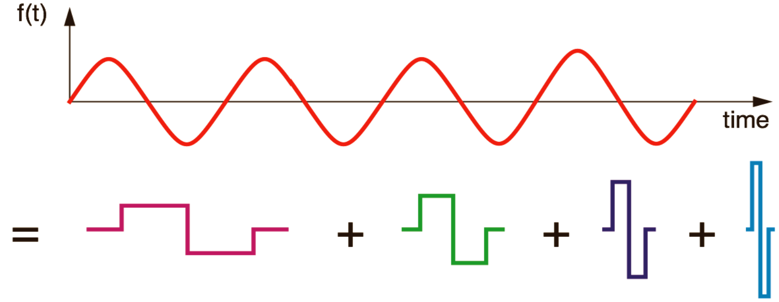


1899: Gibbs' correction

Orthogonality, convergence, complexity

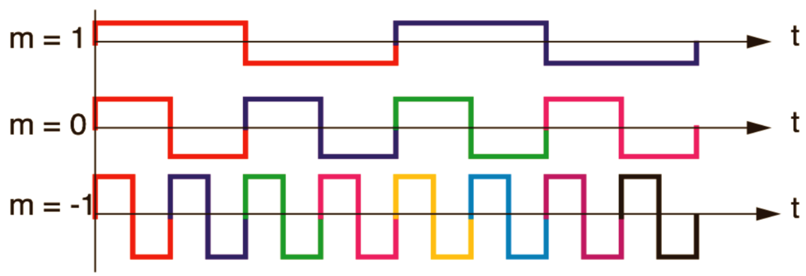


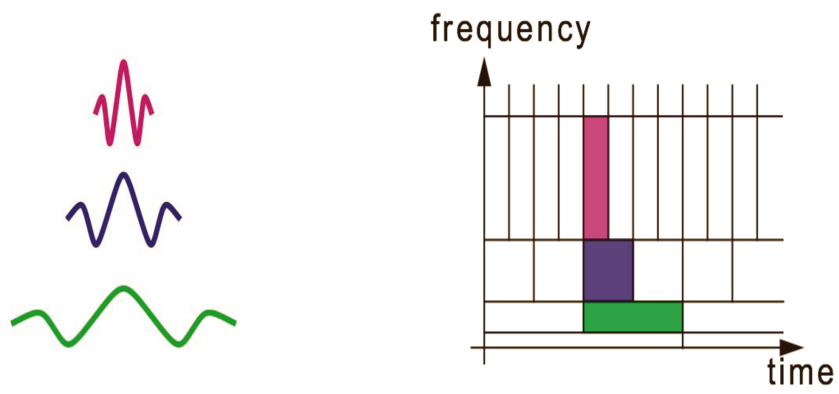
1910: Alfred Haar discovers the Haar wavelet
"dual" to the Fourier construction



Haar series:

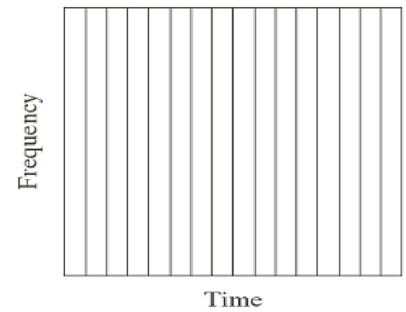
- Scale changes S_0, $2S_0$, $4S_0$, $8S_0$ ...
- orthogonality

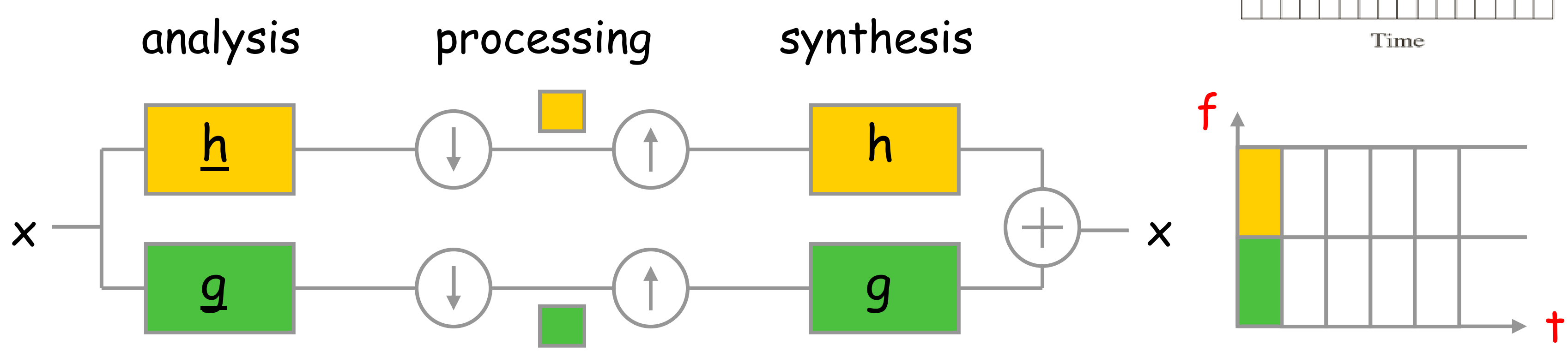




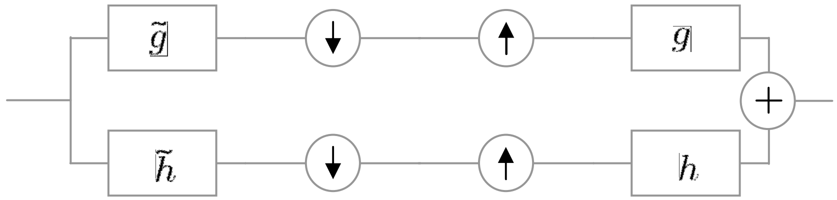
one step forward from dirac …

- Split the frequency in half means we can downsample by 2 to reconstruct upsample by 2.
- Filter to remove unwanted parts of the images and add
- Basic building block: Two-channel filter bank





orthogonal filter banks



1. Start from the reconstructed signal

$$
\begin{aligned}
x_{rec} &= x_V + x_W = \sum_{k\in\mathbb{Z}} \alpha_k g_{n-2k} + \sum_{k\in\mathbb{Z}} \beta_k h_{n-2k} \\
&= \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\dots & g_0 & h_0 & 0 & 0 & 0 & 0 & \dots \\
\dots & g_1 & h_1 & 0 & 0 & 0 & 0 & \dots \\
\dots & g_2 & h_2 & g_0 & h_0 & 0 & 0 & \dots \\
\dots & g_3 & h_3 & g_1 & h_1 & 0 & 0 & \dots \\
\dots & g_4 & h_4 & g_2 & h_2 & g_0 & h_0 & \dots \\
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
\vdots \\ \alpha_0 \\ \beta_0 \\ \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \vdots
\end{bmatrix} = \Phi X
\end{aligned}
$$

- Read off the basis functions

$$\Phi = \{\varphi_k\}_{k\in\mathbb{Z}} = \{\varphi_{2k}, \varphi_{2k+1}\}_{k\in\mathbb{Z}} = \{g_{\cdot-2k}, h_{\cdot-2k}\}_{k\in\mathbb{Z}}$$

orthogonal filter banks

2. We want the expansion to be orthonormal $\Phi\Phi^T = I$
   - The output of the analysis bank is

$$X = \tilde{\Phi}^T x = \Phi^T$$

3. Then
   - The rows of Φ^T are the basis functions $\{g_{\cdot-2k}, h_{\cdot-2k}\}_{k\in\mathbb{Z}}$
   - The rows of Φ^T are the reversed versions of the filters

$$\begin{aligned} \alpha_k &= \langle g_{\cdot-2k}, x\rangle = (g_{-n} * x_n)_{2k} &\Leftrightarrow& \quad \alpha = \Phi_g^T x, \\ \beta_k &= \langle h_{\cdot-2k}, x\rangle = (h_{-n} * x_n)_{2k} &\Leftrightarrow& \quad \beta = \Phi_h^T x. \end{aligned}$$

   - The analysis filters are

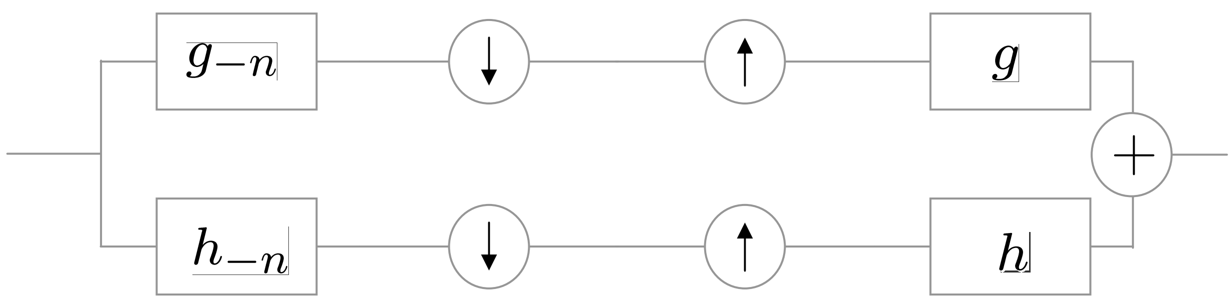
$$\tilde{g}_n = g_{-n}, \qquad \tilde{h}_n = h_{-n}$$

orthogonal filter banks

4. Since Φ is unitary, basis functions are orthonormal

$$\begin{aligned} \langle g_{\cdot-2k}, g\rangle &= \delta_k, \\ \langle h_{\cdot-2k}, h\rangle &= \delta_k, \\ \langle h_{\cdot-2k}, g\rangle &= 0. \end{aligned}$$

5. Final filter bank



orthogonal filter banks: Haar basis

Solve for the filter h explicitly.

$$g_n \; = \; \frac{1}{\sqrt{2}}\left(\delta_n + \delta_{n-1}\right).$$

Given that h_n must be of norm 1 and of same the length as g_n,

$$h_n \; = \; (\cos\alpha)\delta_n + (\sin\alpha)\delta_{n-1}.$$

Computing the inner product $\langle h_{\cdot-2k}, g\rangle = 0$:

$$\frac{1}{\sqrt{2}}(\cos\alpha + \sin\alpha) \; = \; 0.$$

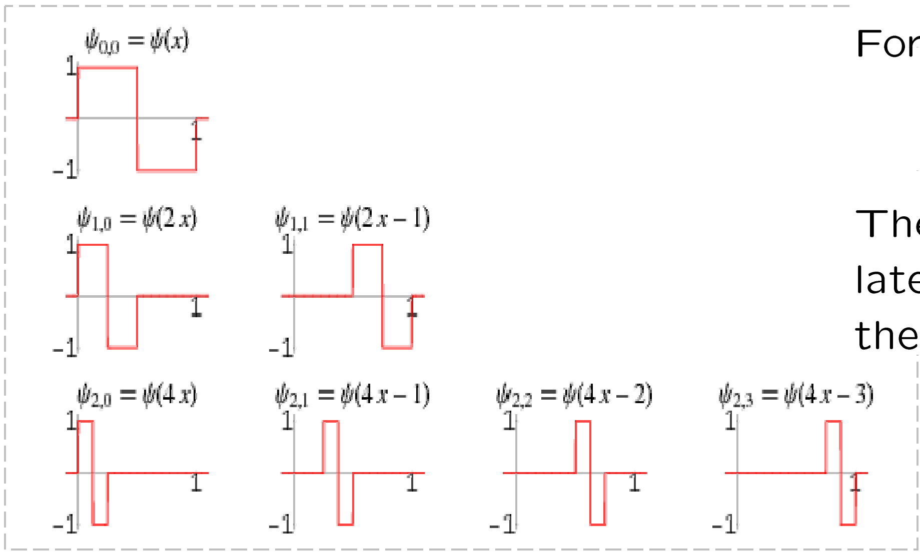
The solution to the above is:

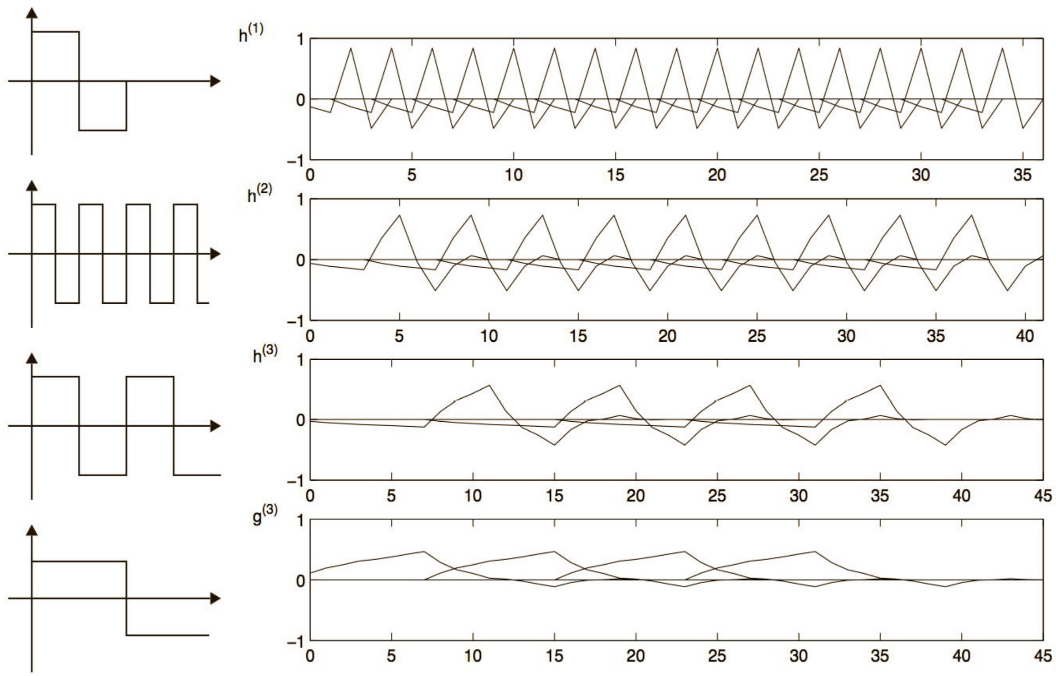
$$\sin\alpha \; = \; -\cos\alpha \qquad \Rightarrow \qquad \alpha \; = \; k\pi - \frac{\pi}{4}.$$

For $k = 0$, a solution to h_n is:

$$h_n \; = \; \frac{1}{\sqrt{2}}\left(\delta_n - \delta_{n-1}\right).$$

The above pair and their even translates translates constitute an ONB for $\ell^2(\mathbb{Z})$ and are called the *Haar filter pair*.

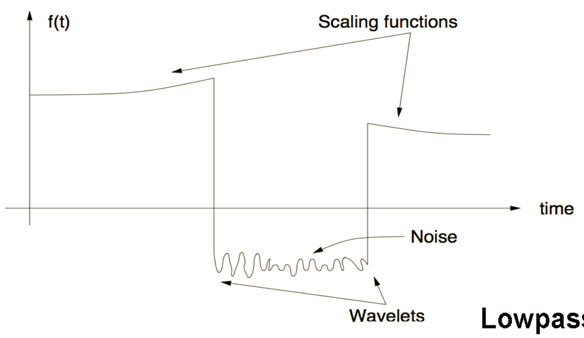




Haar

Daubechies, D_2

Goal: efficient representation of signals like



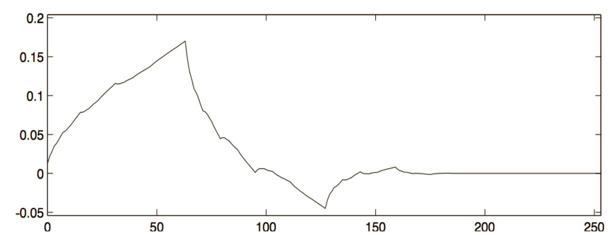
where:

- Wavelet act as singularity detectors
- Scaling functions catch smooth parts
- "Noise" is circularly symmetric

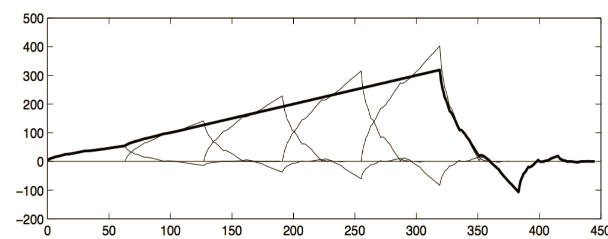
Note: Fourier gets all Gibbs-ed up!

Lowpass filters and scaling functions reproduce polynomials

- Iterate of Daubechies L=4 lowpass filter reproduces linear ramp



scaling function

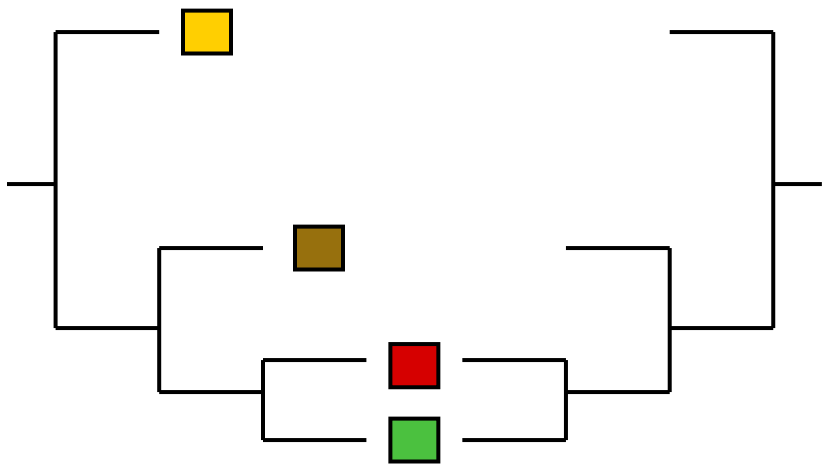


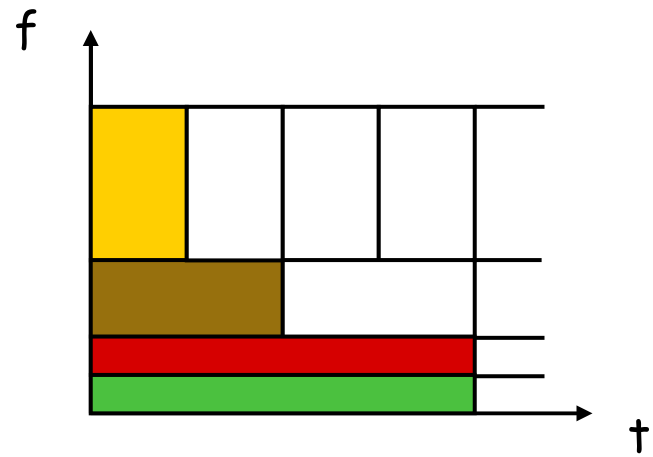
linear ramp

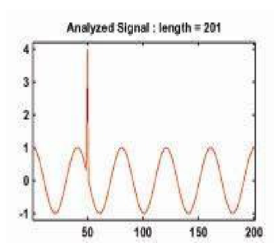
Scaling functions catch "trends" in signals

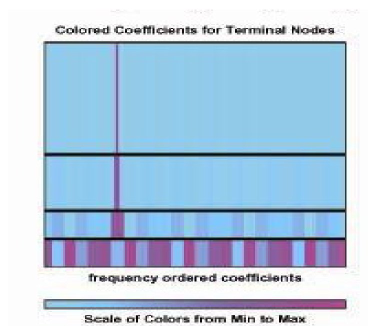
DWT

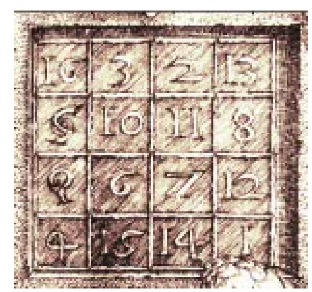
- Iterate only on the lowpass channel

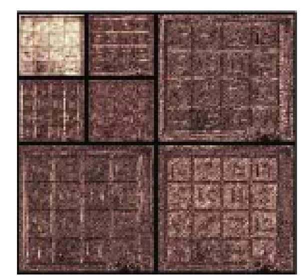




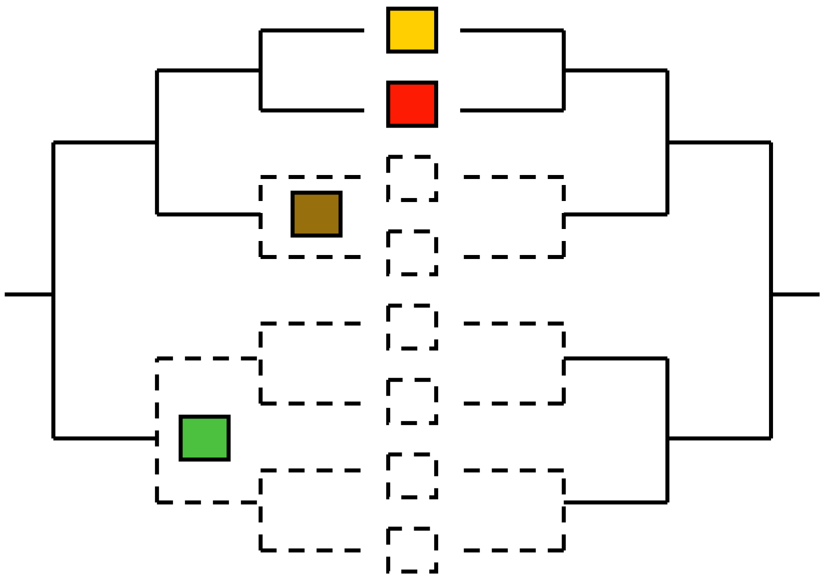


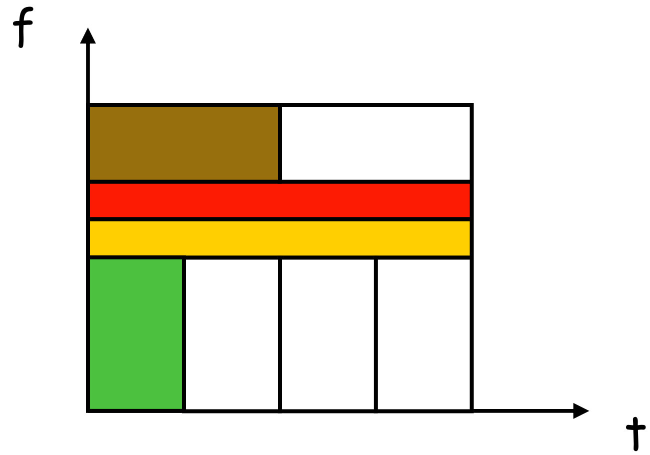






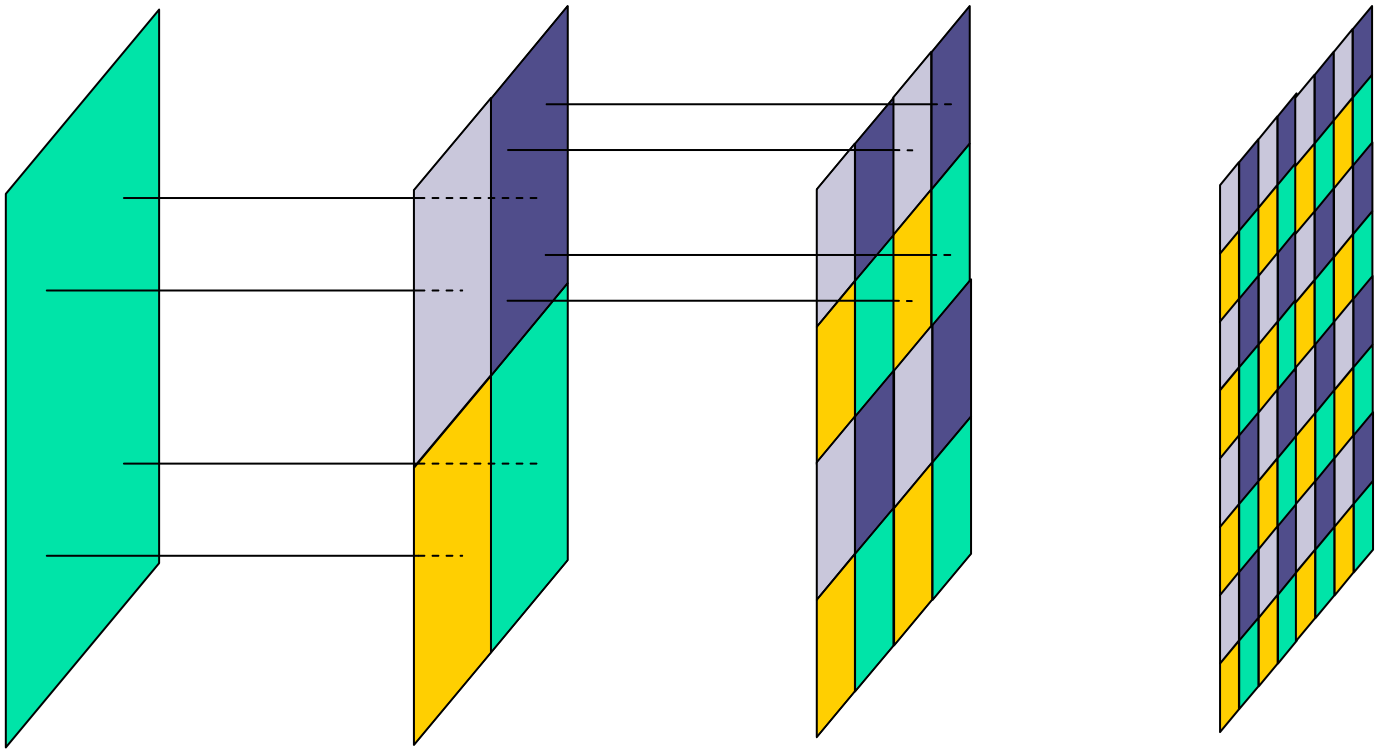
wavelet packet





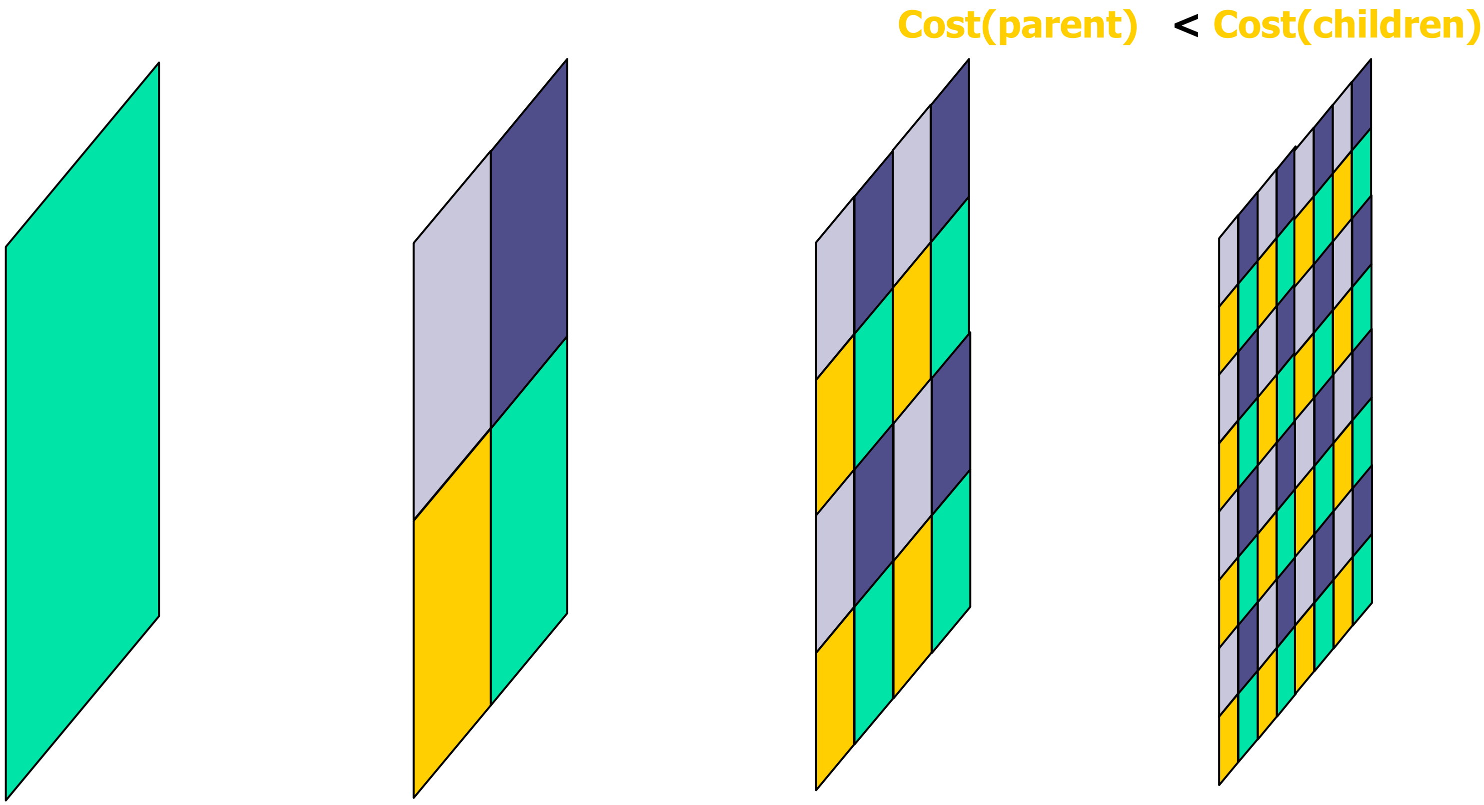
wavelet packet

- First stage: full decomposition

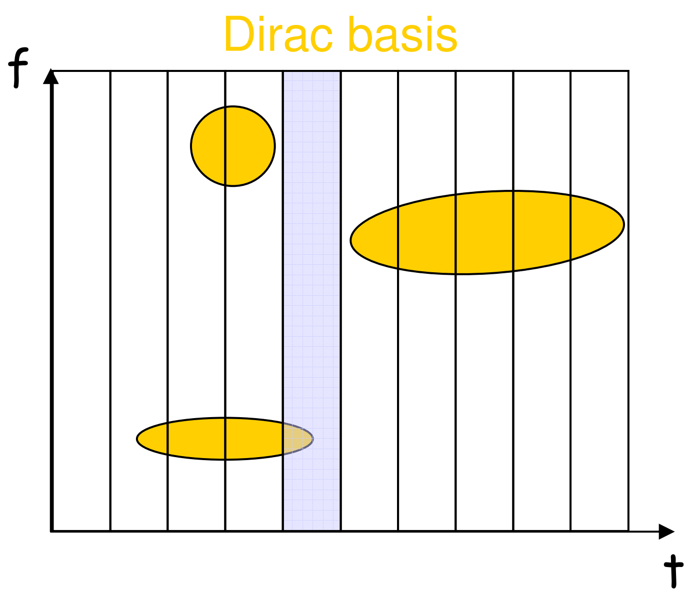


wavelet packet

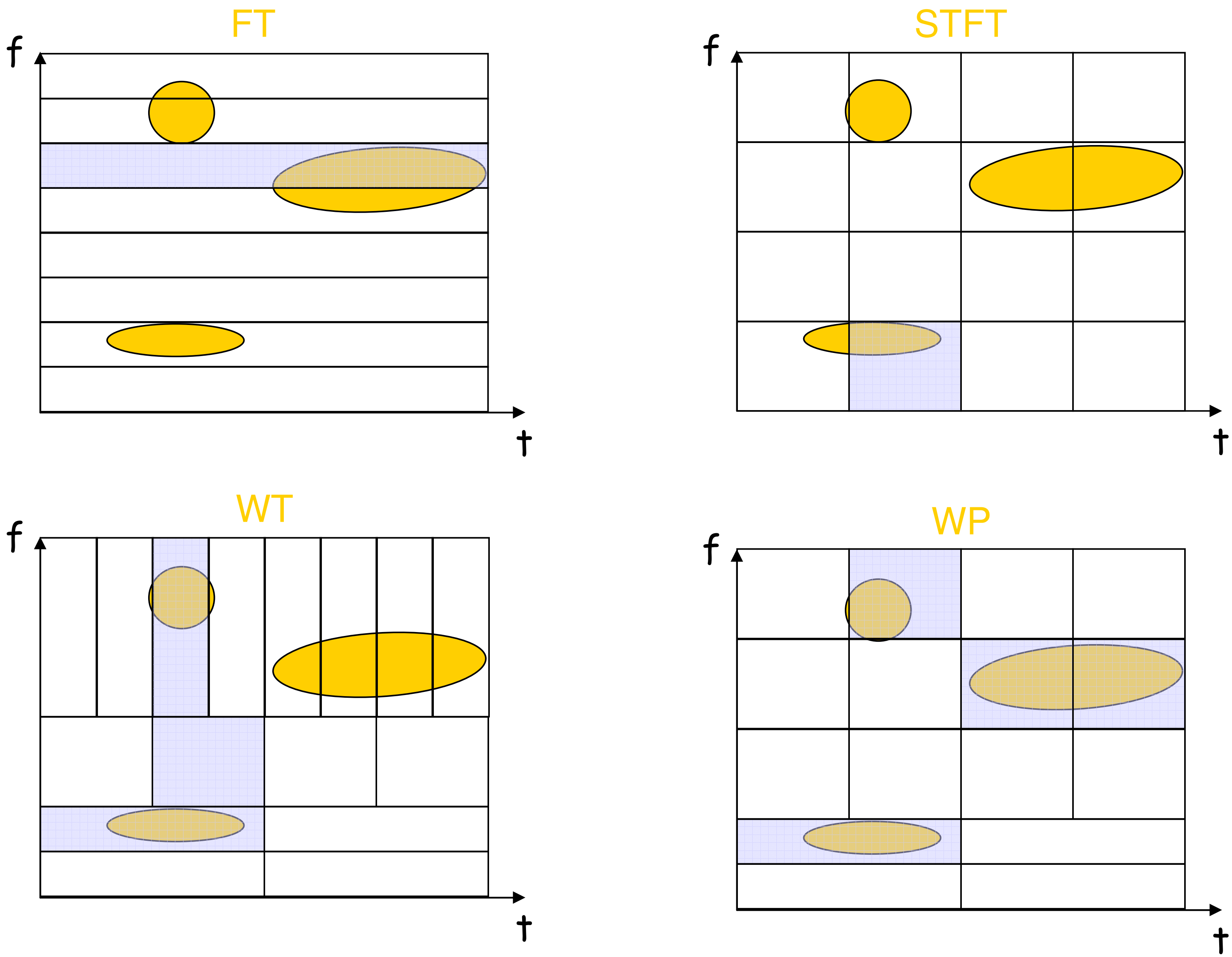
- Second stage: pruning



wavelet packet: why it works



- "Holy Grail" of Signal Analysis/Processing
  - Understand the "blob"-like structure of the energy distribution in the time-frequency space
  - Design a representation reflecting that


FT
STFT
WT
WP
f
t

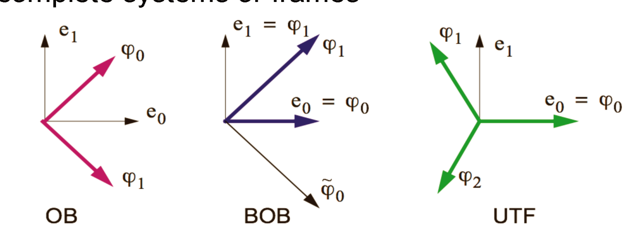
- are we solving x=x?
  - sort of: find matrices such that $x = Ix = \Phi\tilde{\Phi}^* x$
  - after finding those
    - Decomposition $X = \tilde{\Phi}^* x$
    - Reconstruction $x = \Phi X = \Phi\tilde{\Phi}^* x$

- in a nutshell
  - if Φ is square and nonsingular, Φ is a basis and $\tilde{\Phi}$ is its dual basis
  - if Φ is unitary, that is, $\Phi\,\Phi^* = I$, Φ is an orthonormal basis and $\tilde{\Phi} = \Phi$
  - if Φ is rectangular and full rank, Φ is a frame and $\tilde{\Phi}$ is its dual frame
  - if Φ is rectangular and $\Phi\,\Phi^* = I$, Φ is a tight frame and $\tilde{\Phi} = \Phi$

overview of multi-resolution techniques

Property	Orthonormal Basis	Biorthogonal Basis	Tight Frame	General Frame
Expansion Set	$\Phi = \{\varphi_i\}_{i=1}^n$ $\varphi_i \in \mathbb{C}^n$	$\Phi = \{\varphi_i\}_{i=1}^n$ $\tilde{\Phi} = \{\tilde{\varphi}_i\}_{i=1}^n$ $\varphi_i \in \mathbb{C}^n, \tilde{\varphi}_i \in \mathbb{C}^n$	$\Phi = \{\varphi_i\}_{i=1}^m$ $\varphi_i \in \mathbb{C}^n$, $m \geq n$	$\Phi = \{\varphi_i\}_{i=1}^m$ $\tilde{\Phi} = \{\tilde{\varphi}_i\}_{i=1}^m$ $\varphi_i \in \mathbb{C}^n, \tilde{\varphi}_i \in \mathbb{C}^n$, $m \geq n$
Self-Dual	Yes	No	Yes	No
Linearly Independent	Yes	Yes	No	No
Orthogonality Relations	$\langle \varphi_i, \varphi_j \rangle = \delta_{i-j}$	$\langle \varphi_i, \tilde{\varphi}_j \rangle = \delta_{i-j}$	None	None
Expansion	$x = \sum_{i=1}^n \langle \varphi_i, x \rangle \varphi_i$	$x = \sum_{i=1}^n \langle \tilde{\varphi}_i, x \rangle \varphi_i$	$x = \sum_{i=1}^m \langle \varphi_i, x \rangle \varphi_i$	$x = \sum_{i=1}^m \langle \tilde{\varphi}_i, x \rangle \varphi_i$
Matrix Representation	Φ of size $n \times n$ Φ unitary $\Phi\Phi^T = \Phi^T\Phi = I$	Φ of size $n \times n$ Φ full rank $\Phi\tilde{\Phi}^T = I$, $\tilde{\Phi} = (\Phi^T)^{-1}$	Φ of size $n \times m$ rows of Φ orthogonal $\Phi\Phi^T = I$	Φ of size $n \times m$ Φ full rank $\Phi\tilde{\Phi}^T = I$
Norm Preservation	Yes $\|x\|^2 = \sum_{i=1}^n \lvert\langle x, \varphi_i \rangle\rvert^2$	No	Yes	No
Successive Approximation	Yes $\hat{x}^{(k)} = \hat{x}^{(k-1)} + \langle x, \varphi_k \rangle \varphi_k$	No		
Redundant	No	No		

- orthonormal bases (e.g. Fourier series, wavelet series)
- biorthogonal bases
- overcomplete systems or frames



applications of wavelets

- enhancement and denoising
- compression and MR approximation
- fingerprint representation with wavelet packets
- bio-medical image classification
- subdivision surfaces “Geri’s Game”, “A Bug’s Life”, “Toy Story 2”

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