COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Lecture 2 **(2.4.08)**

Digital Imaging Fundamentals:
Sampling & Quantization

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Announcement

- Assignments:
  - Due: 2/18/2007 (end of class)

Digital Image Acquisition:
From Physical Image to Digital Image

$$C(x, y, t, \lambda)$$

$$F(X_k, Y_k) = \int_{\Delta x}\int_{\Delta y}\int_{\Delta t} Y(X_k + x, Y_k + y, t)\,dx\,dy\,dt$$

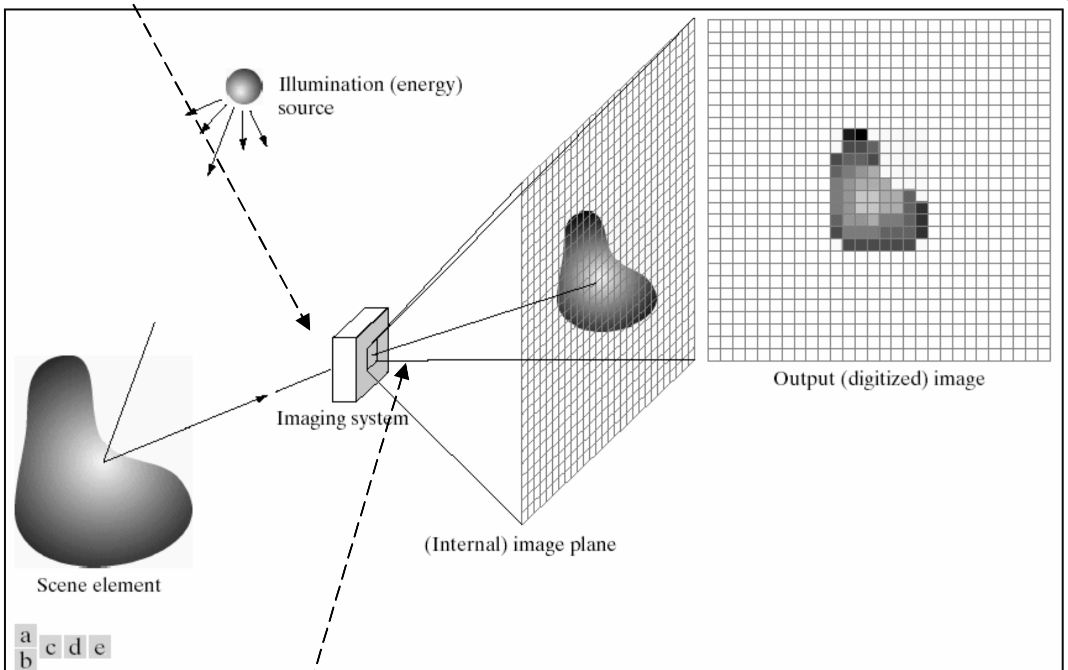
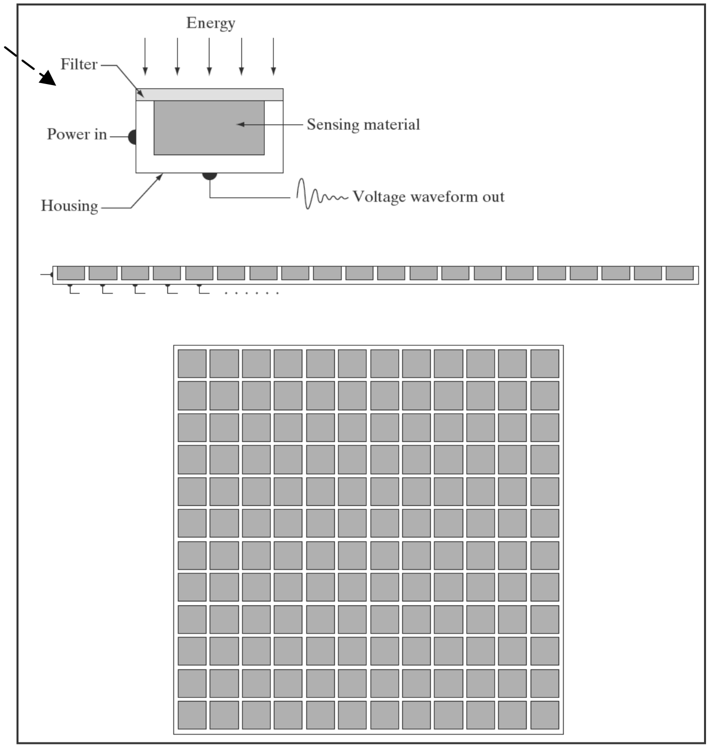


FIGURE 2.15 An example of the digital image acquisition process. (a) Energy ("illumination") source. (b) An element of a scene. (c) Imaging system. (d) Projection of the scene onto the image plane. (e) Digitized image.



CCD Array

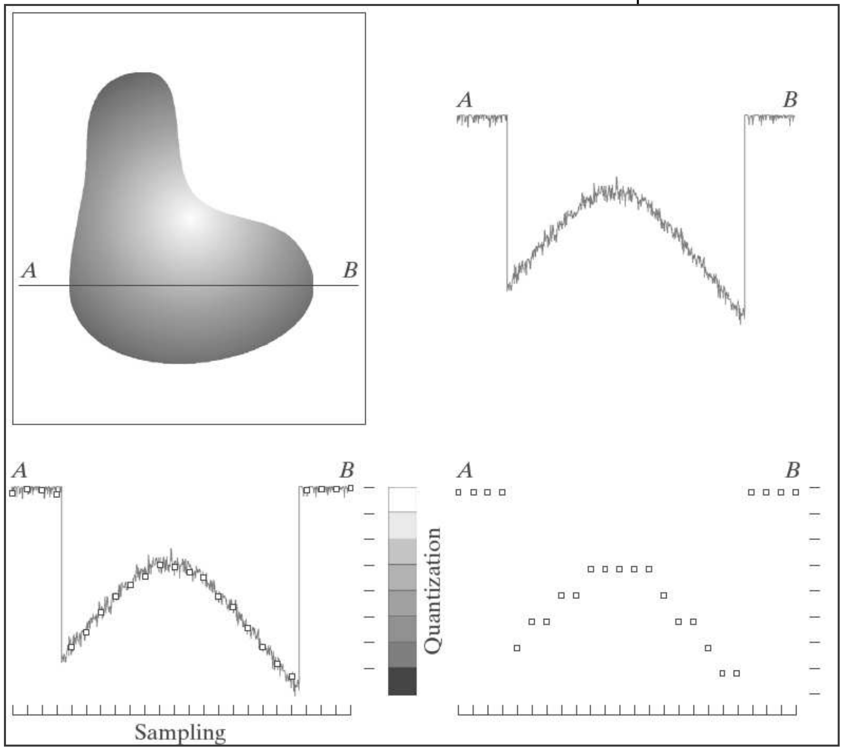
$$Y(x, y, t) = \int C(x, y, t, \lambda) U(\lambda)\, d\lambda$$

Digital Image Acquisition: From Physical Image to Digital Image

Generally an image cannot be represented by an analytic function

Instead represent it as a tabulated function

- Sampling
  - Process of mapping a continuous function to discrete
  - NOTE: pixels are samples from physical image
    - Photoreceptors in eye
    - CCD array
- Quantization
  - Process of mapping continuous variable to discrete



Spatial discretization

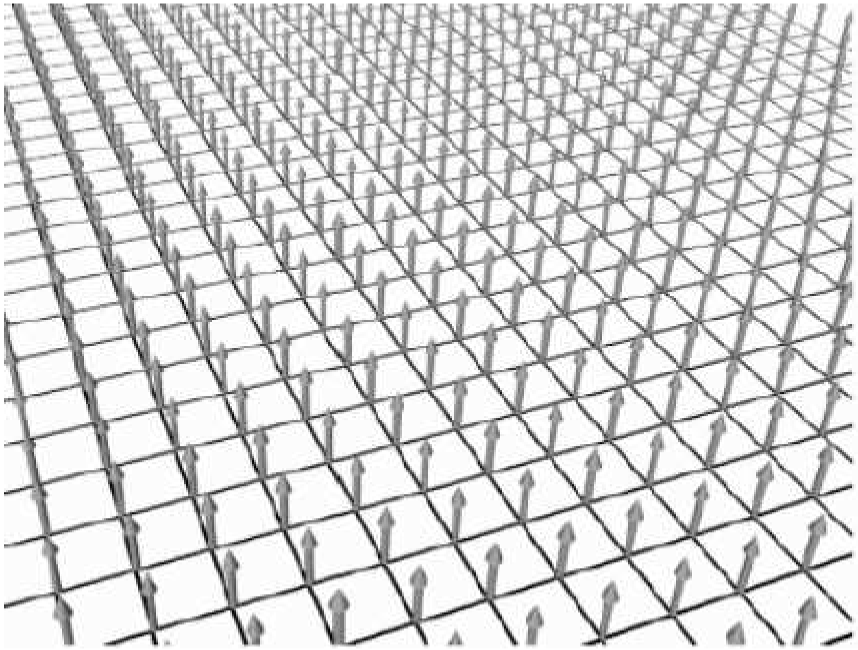
Intensity discretization

Preliminaries

Delta function: $\delta(x) = 0, x \neq 0$ $\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \delta(x) dx = 1$

Sampling function: $comb(x) = \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)$

$$s(x, y) = \sum_j \sum_k \delta(x - j\Delta x, y - k\Delta y)$$



Preliminaries – Fourier Transform

1-D FT: $f(t) \rightarrow F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-j2\omega t) dt$

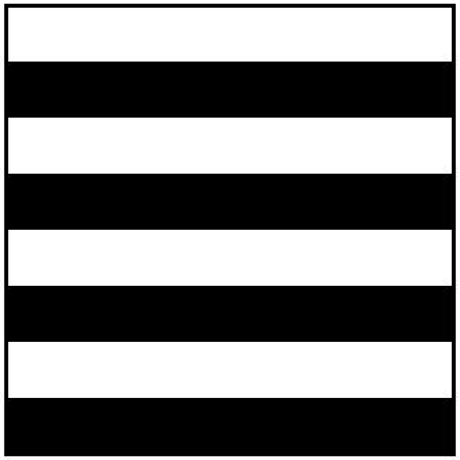
1-D Inverse FT: $F(\omega) \rightarrow f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(j2\omega t) d\omega$

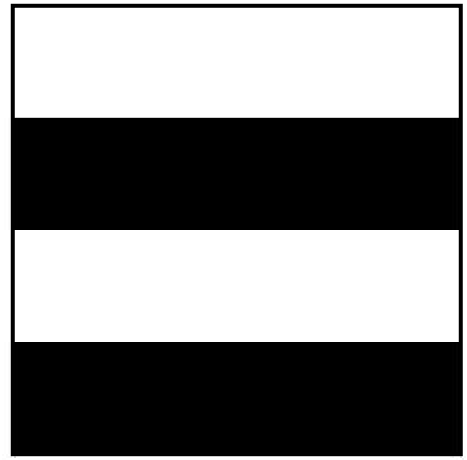
2-D FT: $f(x, y) \rightarrow F(\omega_x, \omega_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \exp[-j(\omega_x x + \omega_y y)] dx dy$

2-D Inverse FT: $F(\omega_x, \omega_y) \rightarrow f(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\omega_x, \omega_y) \exp[j(\omega_x x + \omega_y y)] d\omega_x d\omega_y$

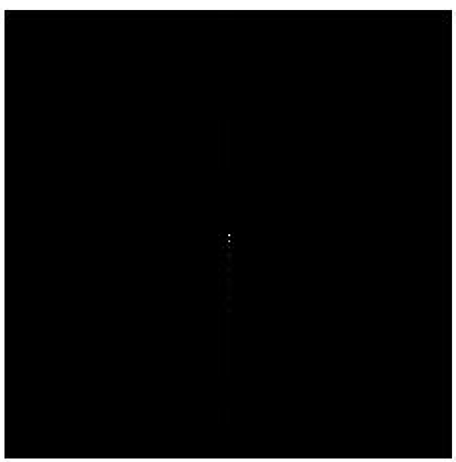
Fourier Transform Pair - example

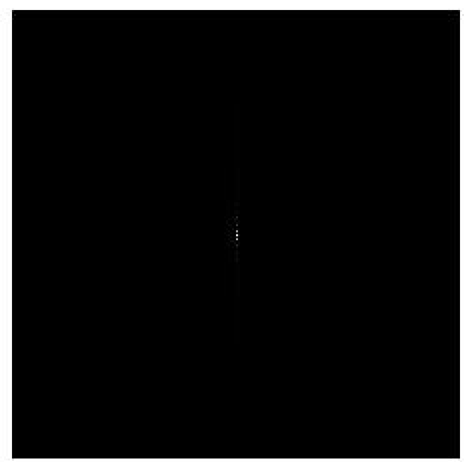
Image





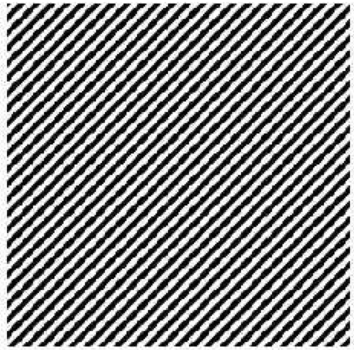
Fourier Transform



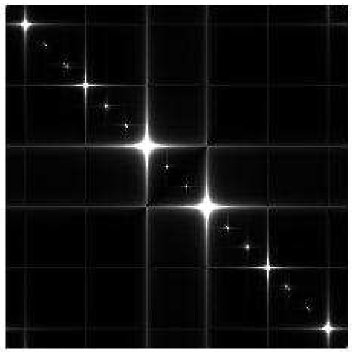


Fourier Transform Pair - example

Image



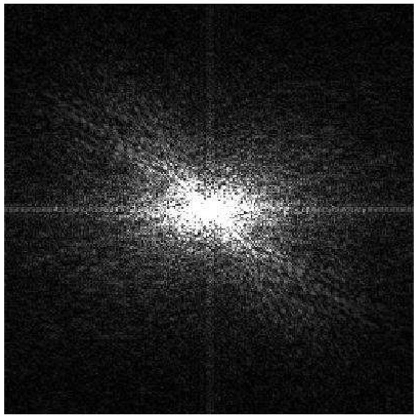
Fourier Transform



Fourier Transform Pair - example

Image

Fourier Transform



Preliminaries – Convolution

$$f(x)\otimes g(x)=\int_{-\infty}^{\infty} f(\lambda)g(x-\lambda)d\lambda$$

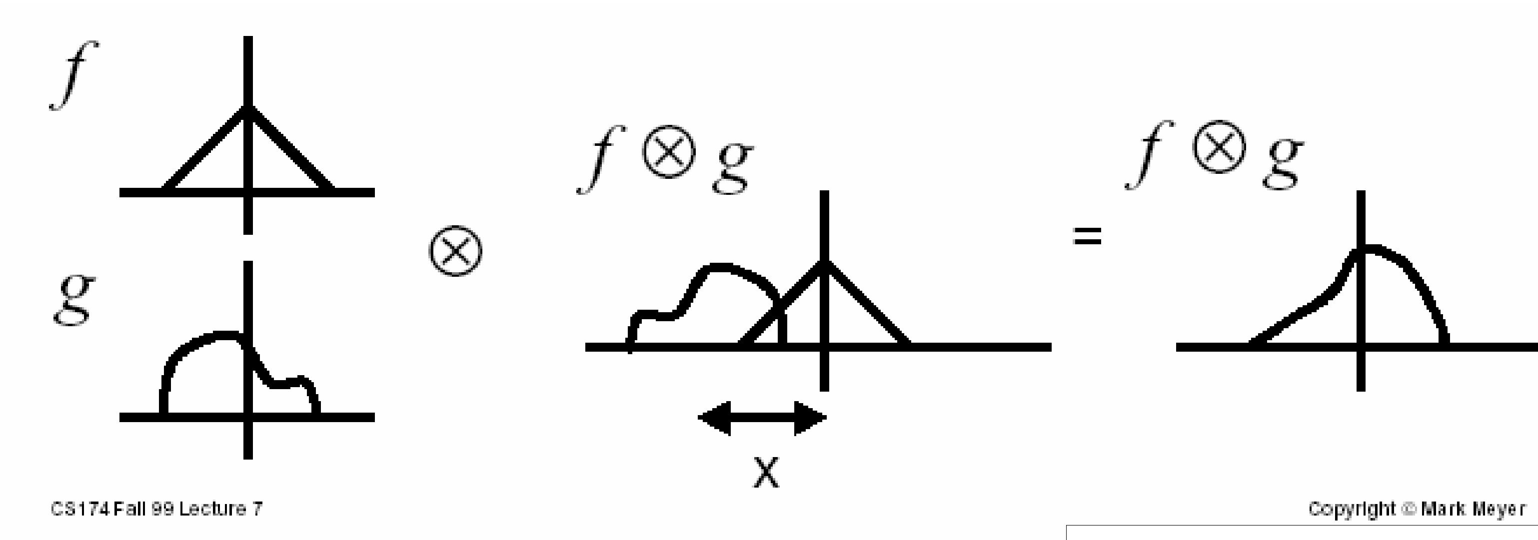


Image from Mark Meyer
http://www.gg.caltech.edu/~cs174ta/

Convolution Theorem:

$$f(x)\otimes h(x)\rightarrow F(\omega)H(\omega)$$

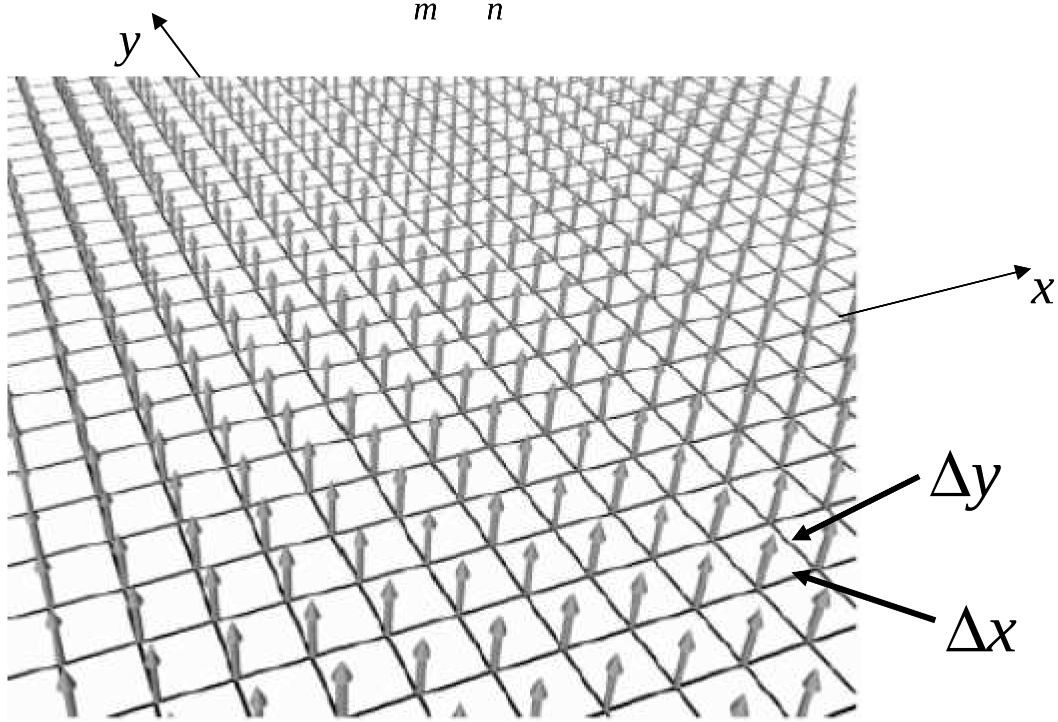
$$F(\omega)\otimes H(\omega)\rightarrow f(x)h(x)$$

Physical → Discrete Image

Physical Image: $f_I(x, y)$

Sampling Function: $s(x, y) = \sum_m \sum_n \delta(x - m\Delta x, y - n\Delta y)$

Sampled Image: $f_S(x, y) = f_I(x, y)s(x, y) = \sum_m \sum_n f_I(m\Delta x, n\Delta y)\delta(x - m\Delta x, y - n\Delta y)$



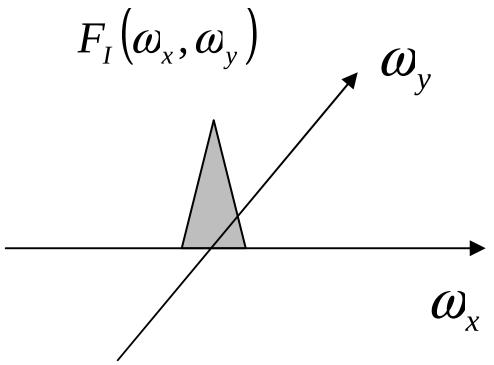
Spatial Sampling

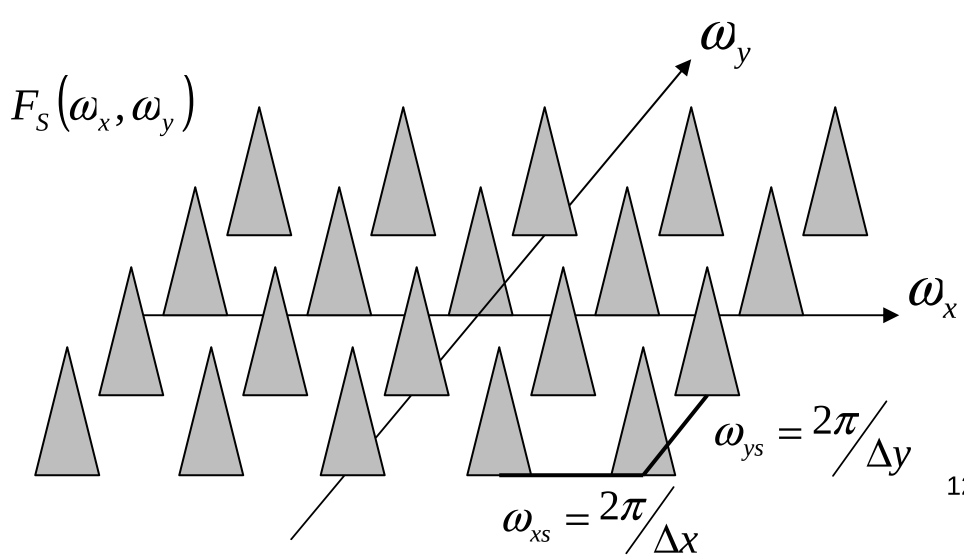
$$f(x,y) \rightarrow F_S(\omega_x,\omega_y) = \frac{1}{4\pi^2} F_I(\omega_x,\omega_y) \otimes S(\omega_x,\omega_y)$$

$$s(x,y) \rightarrow S(\omega_x,\omega_y) = \frac{4\pi^2}{\Delta x \Delta y} \sum_m \sum_n \delta(\omega_x - m\omega_{xs}, \omega_y - n\omega_{ys})$$

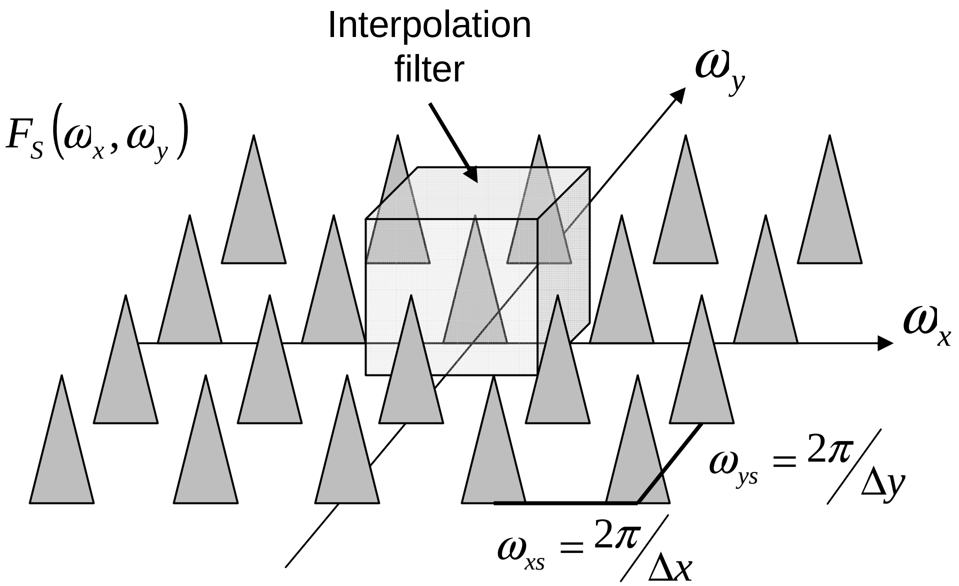
$$F_S(\omega_x,\omega_y) = \frac{1}{\Delta x \Delta y} \iint F_I(\omega_x - \alpha, \omega_y - \beta) \sum_m \sum_n \delta(\alpha - m\omega_{xs}, \beta - n\omega_{ys})$$

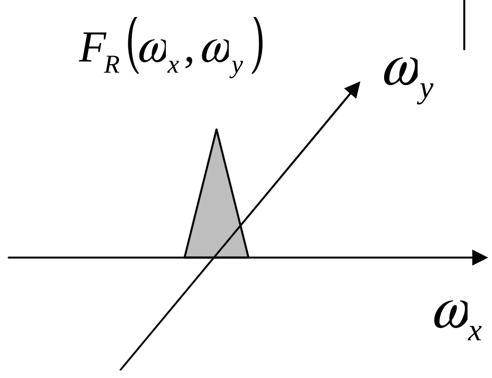
$$\boxed{F_S(\omega_x,\omega_y) = \frac{1}{\Delta x \Delta y} \sum_m \sum_n F_I(\omega_x - m\omega_{xs}, \omega_y - n\omega_{ys})}$$





Recovering Original Image from Samples



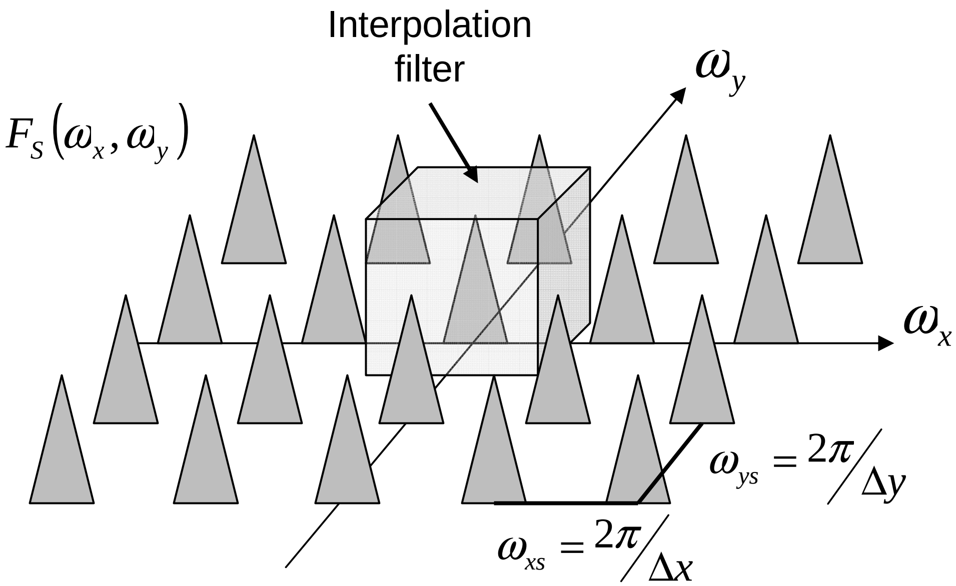


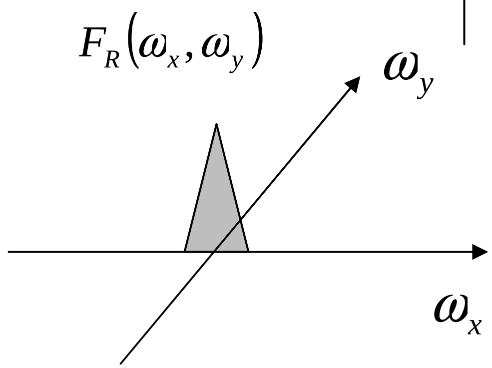
- Should have no spectrum overlap in $f_s(x, y)$
- $r(x, y)$ should be able to filter out $f_s(x, y)$ when $m, n \neq 0$

$$f_S(x, y) \longrightarrow \boxed{r(x, y)} \longrightarrow f_R(x, y)$$

$$f_R(x, y) = \sum_m \sum_n f_I(m\Delta x, n\Delta y) r(x - m\Delta x, y - n\Delta y)$$

Recovering Original Image from Samples

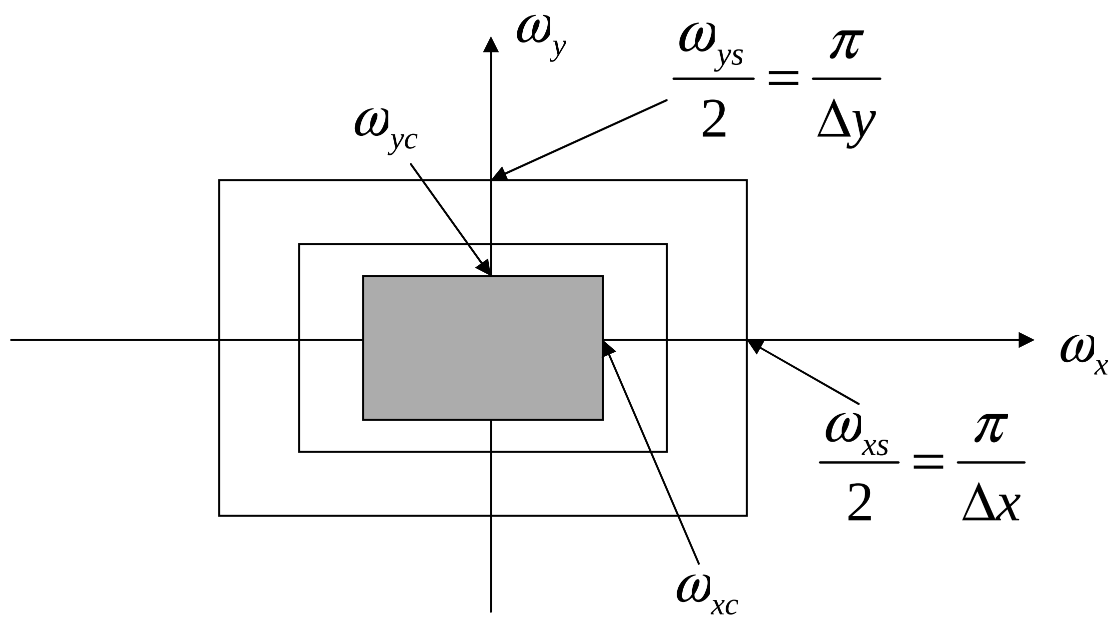




$$F_R(\omega_x, \omega_y) = F_S(\omega_x, \omega_y) R(\omega_x, \omega_y)$$

$$F_R(\omega_x, \omega_y) = \frac{1}{\Delta x \Delta y} R(\omega_x, \omega_y) \sum_m \sum_n F_I(\omega_x - m\omega_{xs}, \omega_y - n\omega_{ys})$$

Nyquist Frequency for Sampling



Condition for Perfect Reconstruction:

$$\omega_{xc} \le \frac{\omega_{xs}}{2} \qquad \omega_{yc} \le \frac{\omega_{ys}}{2}$$

- OR -

$$\Delta x \le \frac{\pi}{\omega_{xc}} \qquad \Delta y \le \frac{\pi}{\omega_{yc}}$$

Nyquist Rate:

$$\omega_{xs} = 2\omega_{xc}$$
$$\omega_{ys} = 2\omega_{yc}$$

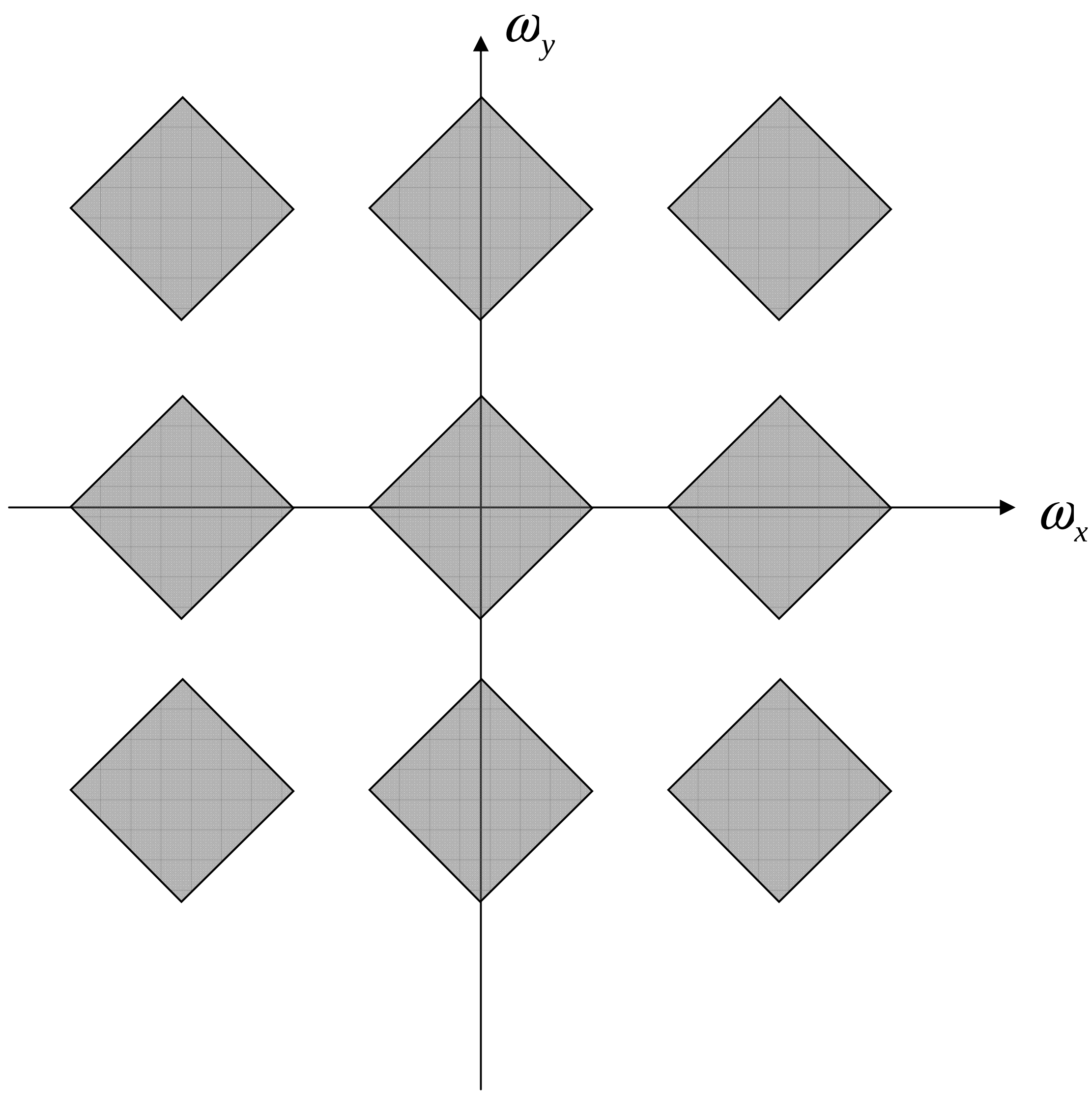
Oversampling:

$$\omega_{xs} > 2\omega_{xc}$$
$$\omega_{ys} > 2\omega_{yc}$$

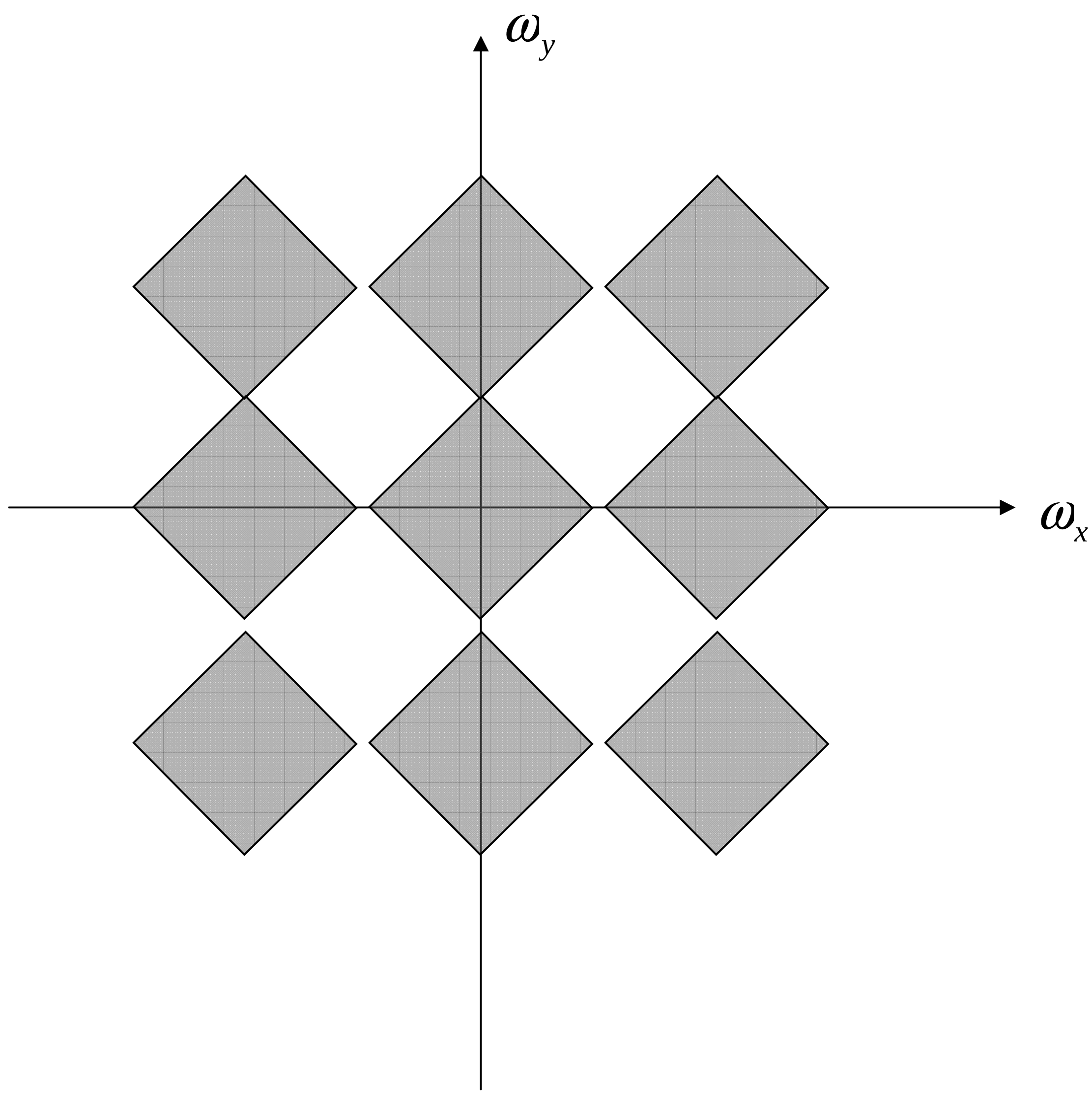
Undersampling:

$$\omega_{xs} < 2\omega_{xc}$$
$$\omega_{ys} < 2\omega_{yc}$$

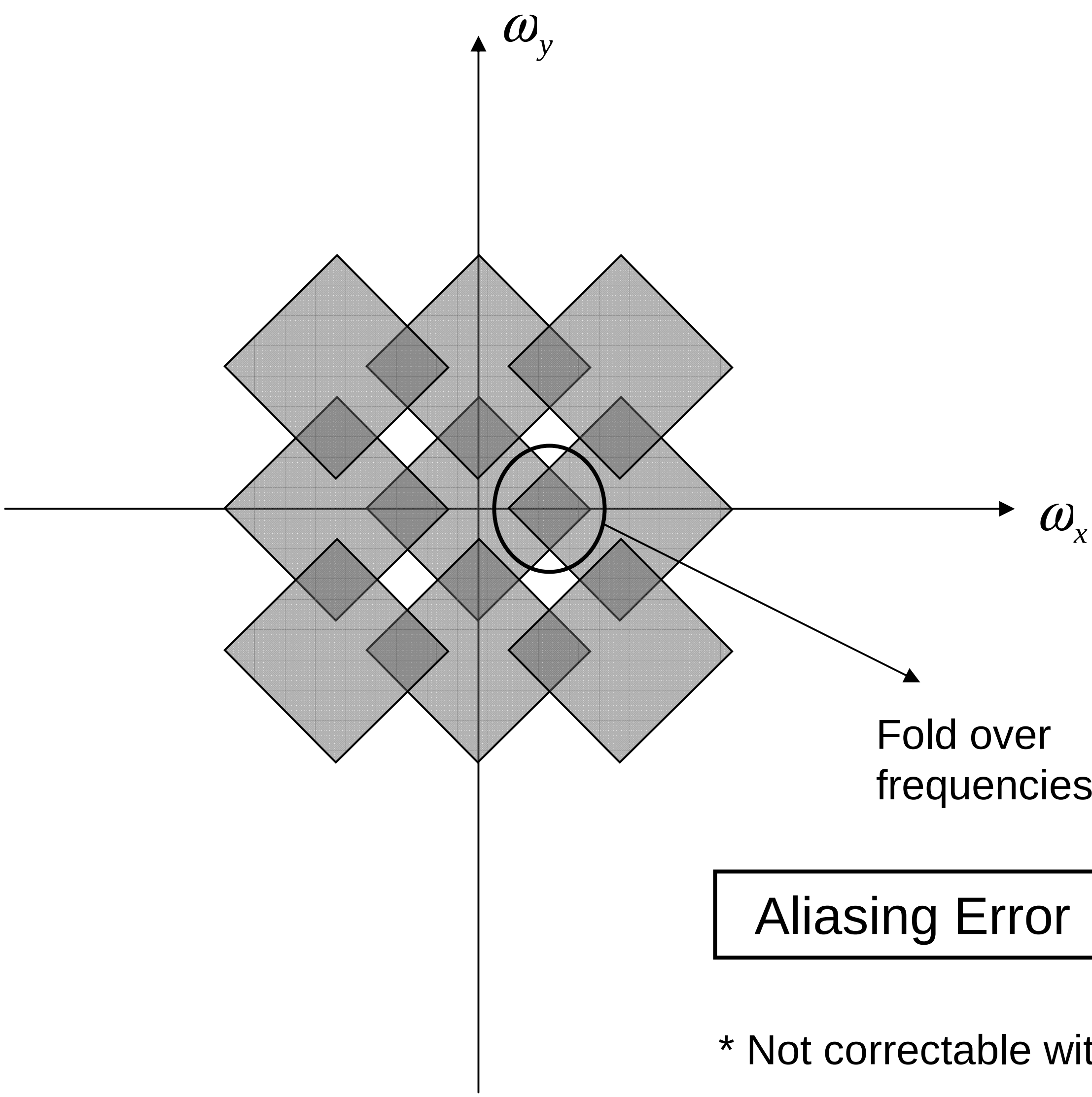
Problem with Undersampling



Problem with Undersampling



Problem with Undersampling



Aliasing Error

* Not correctable with post filtering

Aliasing

- At reconstruction signal will be mistaken for some simpler signal (there is no enough knowledge about the signal)

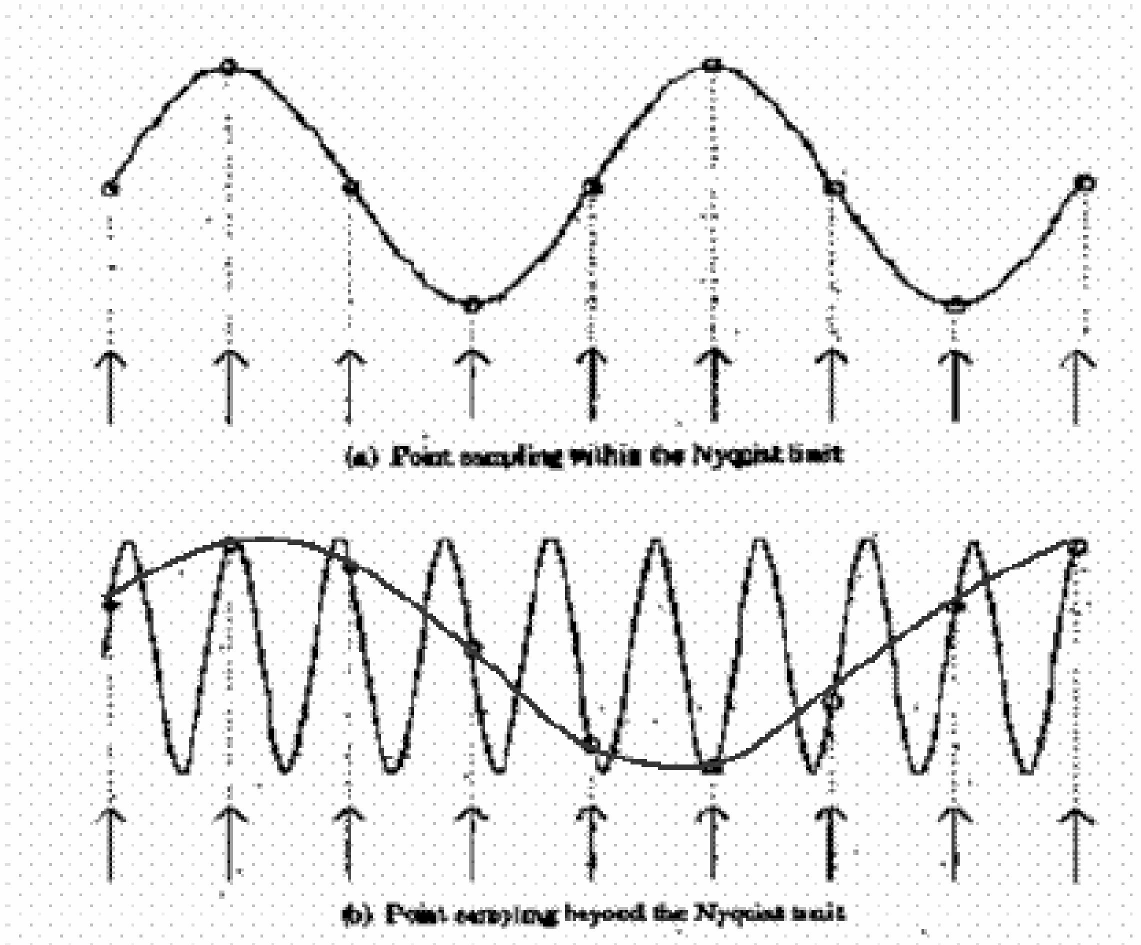
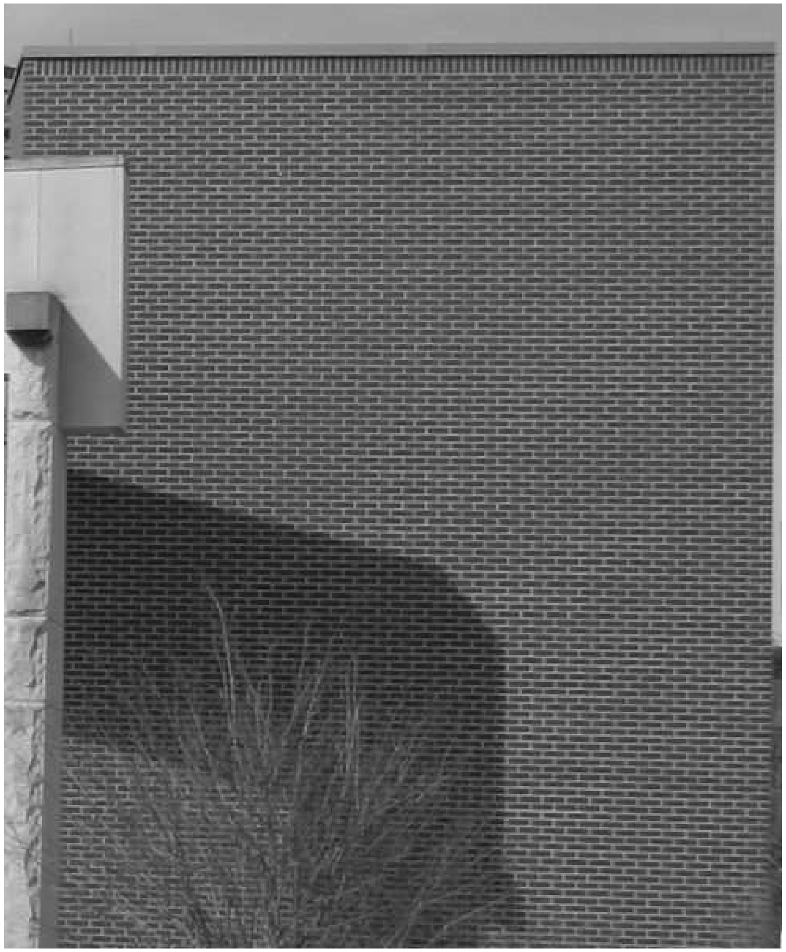
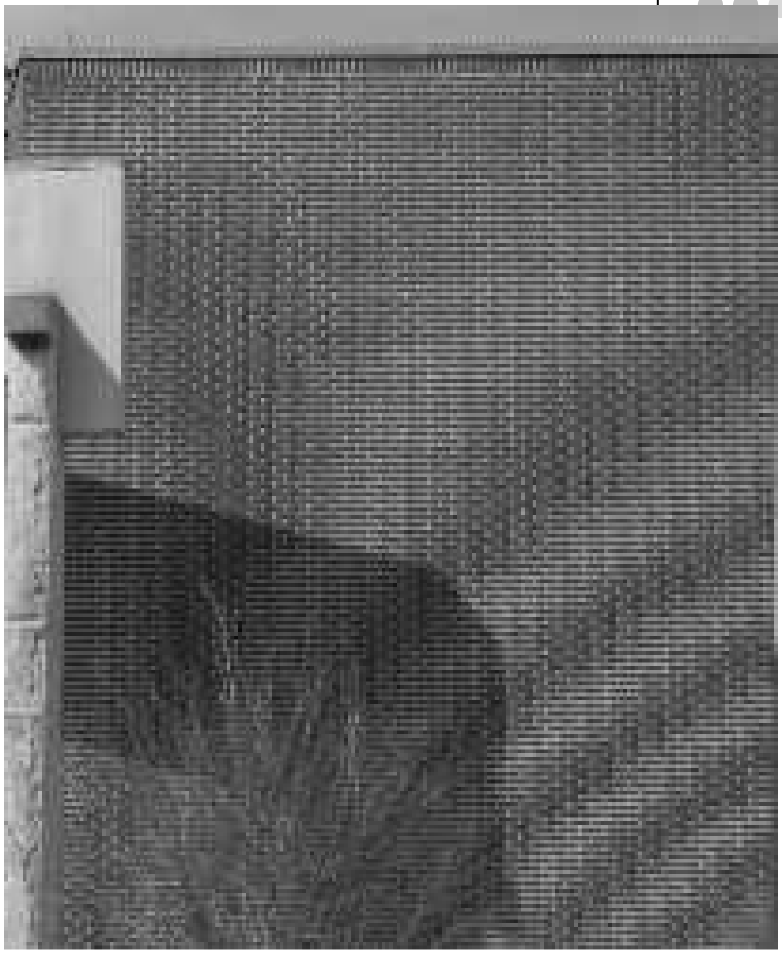


Image from Robert L. Cook, "Stochastic Sampling and Distributed Ray Tracing", An Introduction to Ray Tracing, Andrew Glassner, ed., Academic Press Limited, 1989.

Aliasing Error – Visual Perception



Original image



Down-sampled version

Spatial Sampling Theorem

Band-limited image $f(x, y)$, i.e. $F(\omega_x, \omega_y) = 0, \quad |\omega_x| > \omega_{xc}, \quad |\omega_y| > \omega_{yc}$
Sampled uniformly on the rectangular grid at $\Delta x, \Delta y$ intervals is recoverable without error from the sample values $f(m\Delta x, n\Delta y)$
if sampling rate is greater than the Nyquist rate!

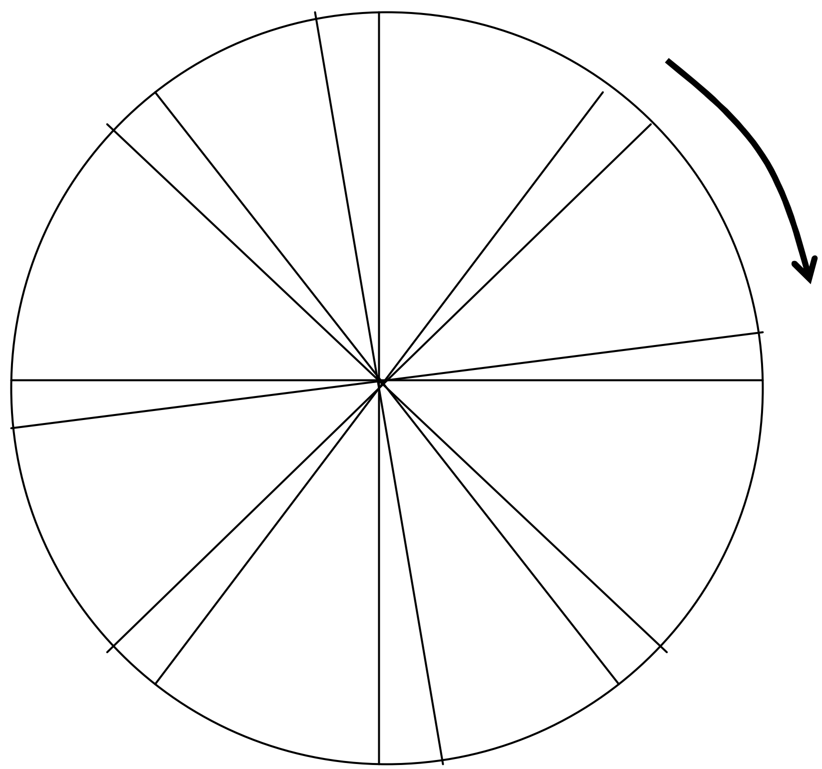
The recovered signal is given by:

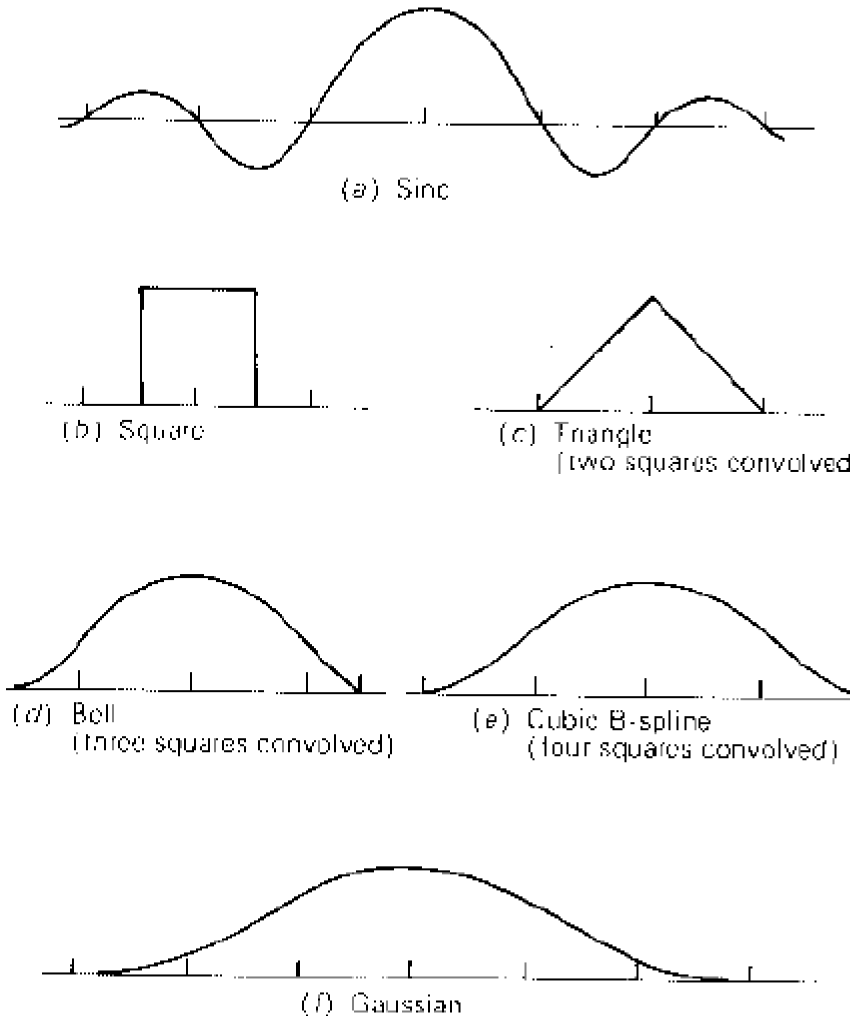
$$f_R(x, y) = \sum_m \sum_n f_I(m\Delta x, n\Delta y) r(x - m\Delta x, y - n\Delta y)$$

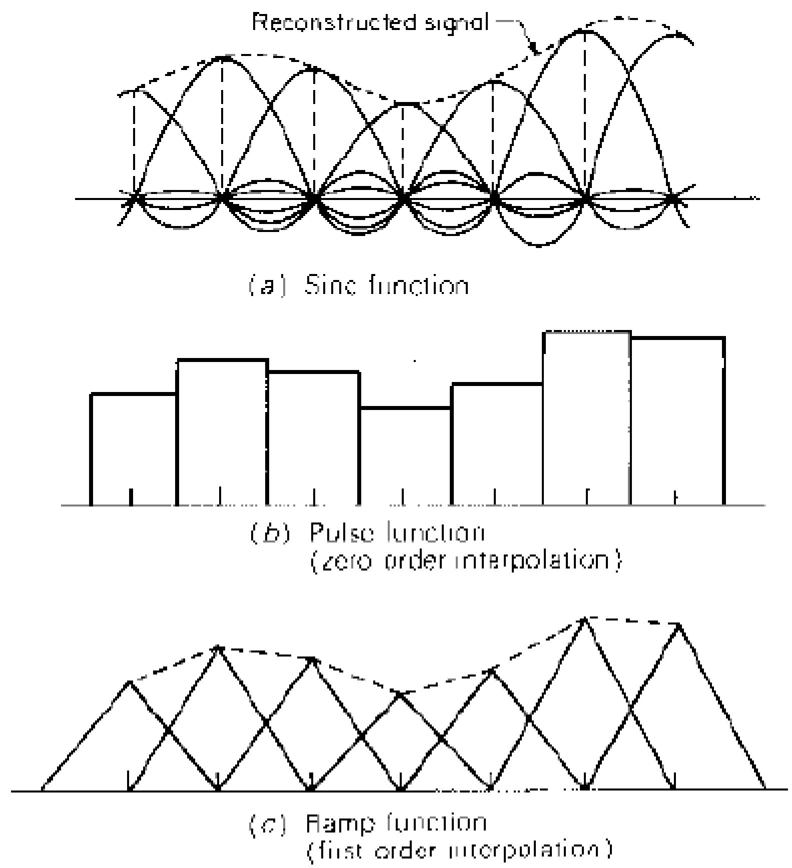
$$r(x, y) = \frac{K\omega_{xL}\omega_{yL}}{\pi^2} \frac{\sin(\omega_{xL} x)}{(\omega_{xL} x)} \frac{\sin(\omega_{yL} y)}{(\omega_{yL} y)}$$

Temporal Aliasing - Video

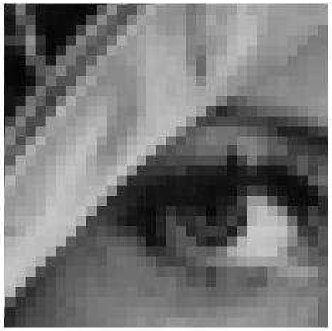
- Strobing effect
- Flickering effect



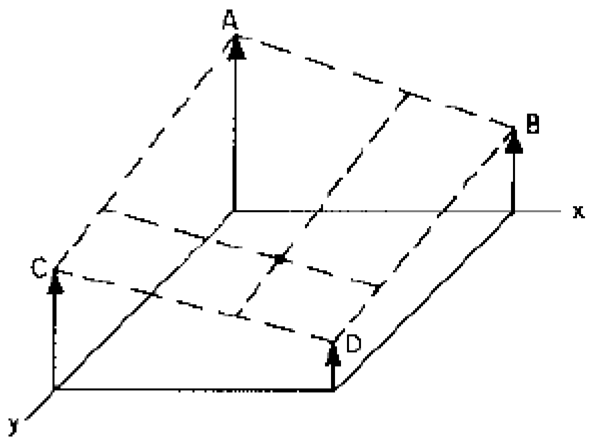

(a) Sinc
(b) Square
(c) Triangle
(two squares convolved
(d) Bell
(three squares convolved)
(e) Cubic B-spline
(four squares convolved)
(f) Gaussian


Reconstructed signal
(a) Sinc function
(b) Pulse function
(zero order interpolation)
(c) Ramp function
(first order interpolation)

Zoom-in Reconstruction

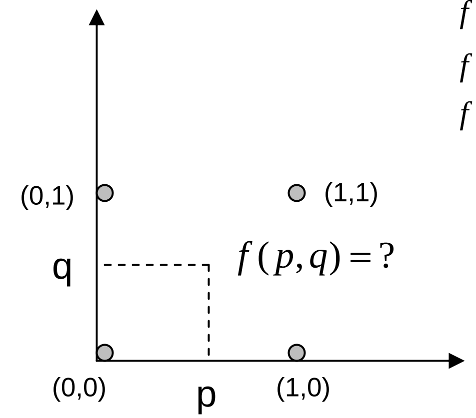


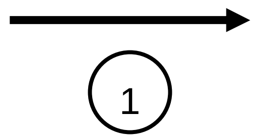
Bilinear Interpolation

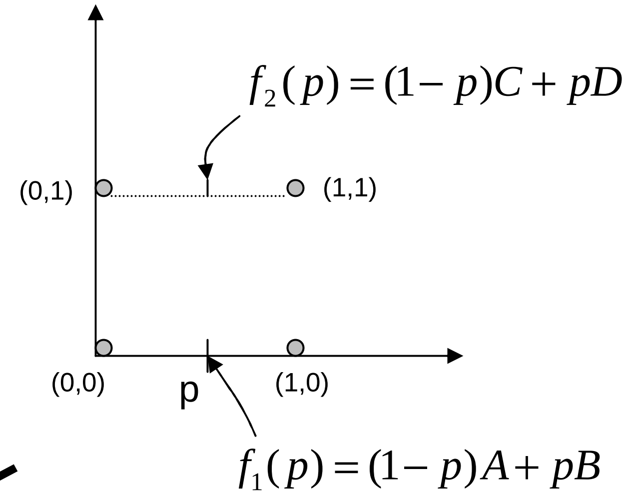


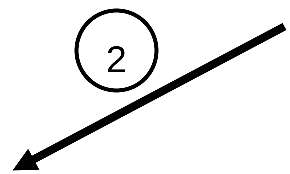
Bilinear Interpolation

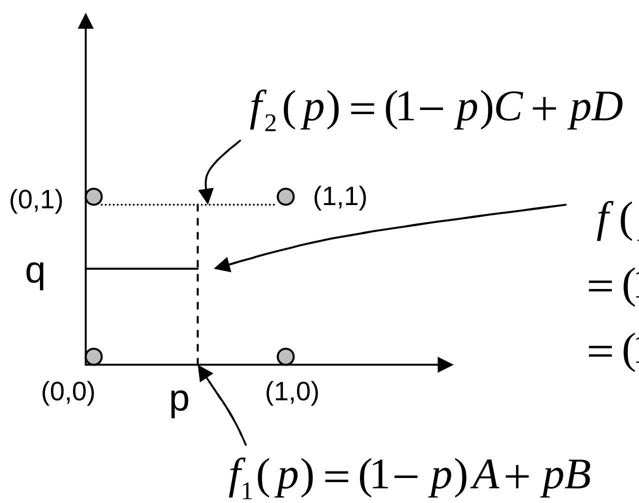
$f(0,0) = A$
$f(1,0) = B$
$f(0,1) = C$
$f(1,1) = D$









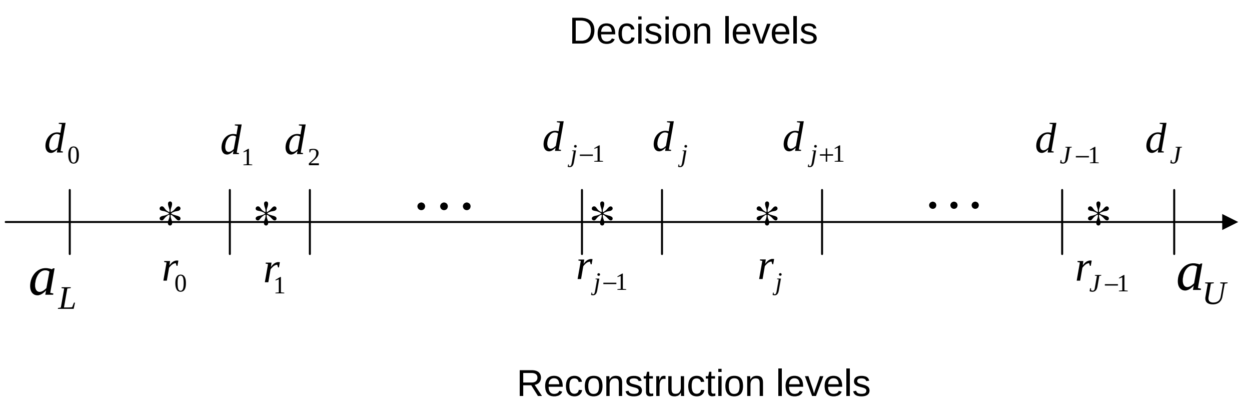


$$f(p,q) = (1-q)f_1(p) + qf_2(p)$$
$$= (1-q)[(1-p)A + pB] + q[(1-p)C + pD]$$
$$= (1-p)(1-q)A + p(1-q)B + (1-p)qC + pqD$$

Quantization

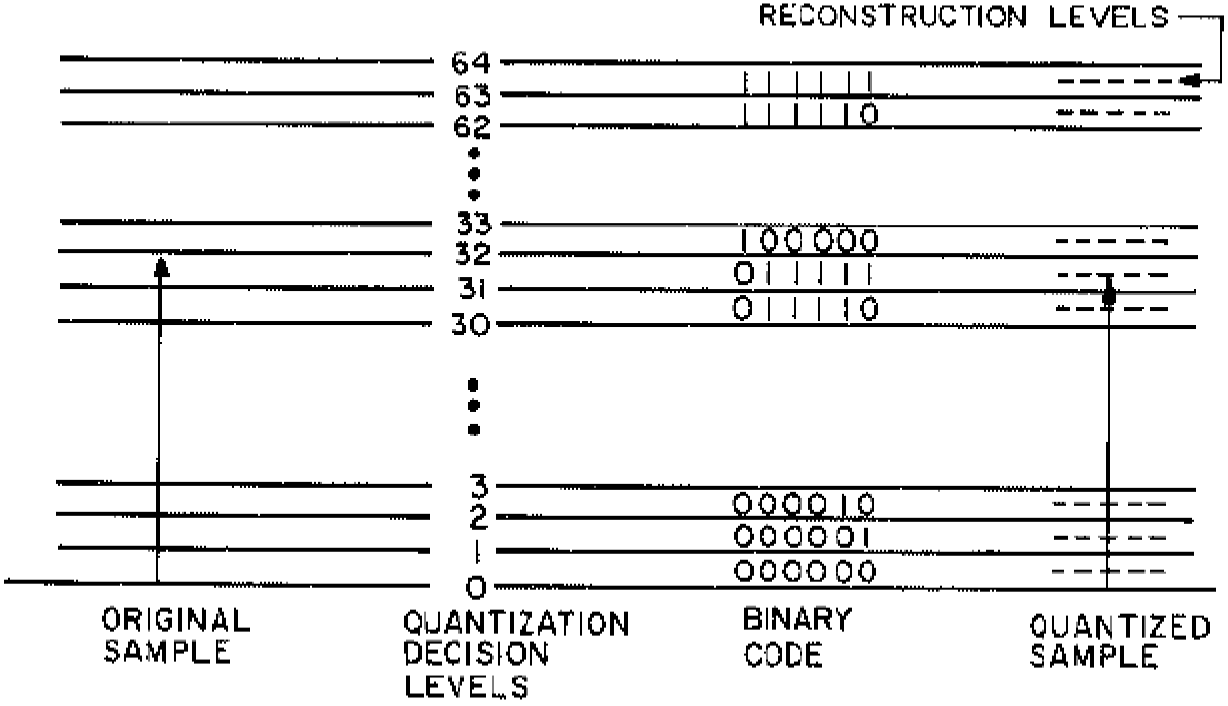
Quantization

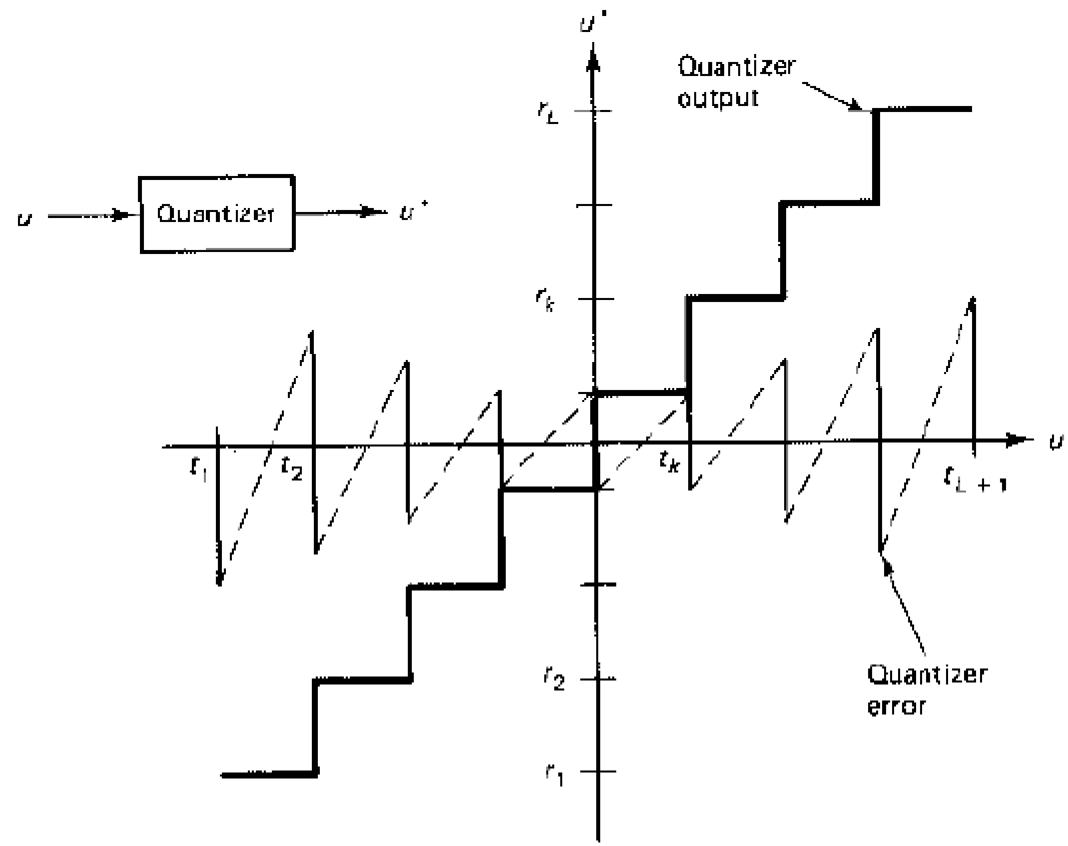
- Continuous Valued Samples → Discrete Valued Samples



- Issues:
  - J=?
  - Placement of reconstruction levels ?
  - Placement of decision levels ?
  - Error ?

Quantization





u
Quantizer
u*
Quantizer
output
Quantizer
error

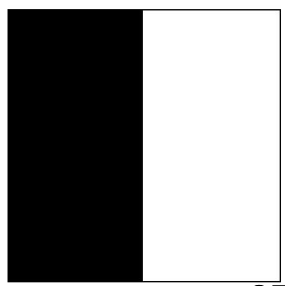
Image Histogram

$$f_{MxN} \quad , f(m,n) \in \{r_0, r_1, \cdots, r_k, \cdots, r_{L-1}\}$$

$$H = (h(0), h(1), \cdots, h(k), \cdots, h(L-1)) \quad , h(k) = \frac{n_k}{MxN} = \frac{\sum_m \sum_n \delta(f(m,n) - r_k)}{MxN}$$

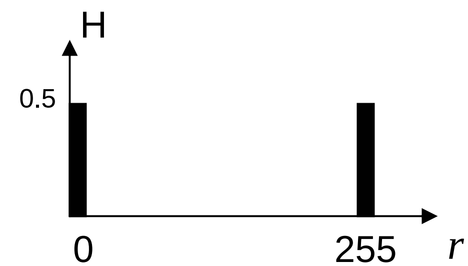
normalized

bi-level image

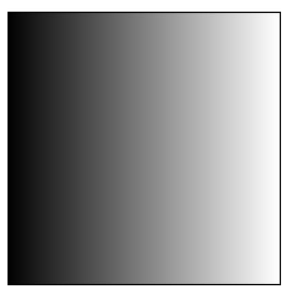


256x256

histogram



Pixel values linearly increasing from 0 to 255 with increasing column index



256x256

histogram

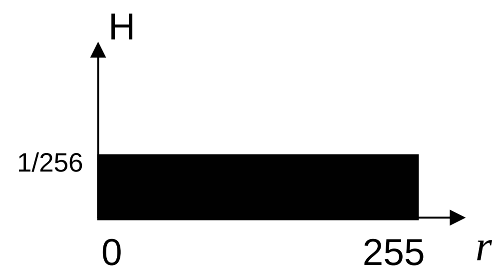


Image Histogram: example

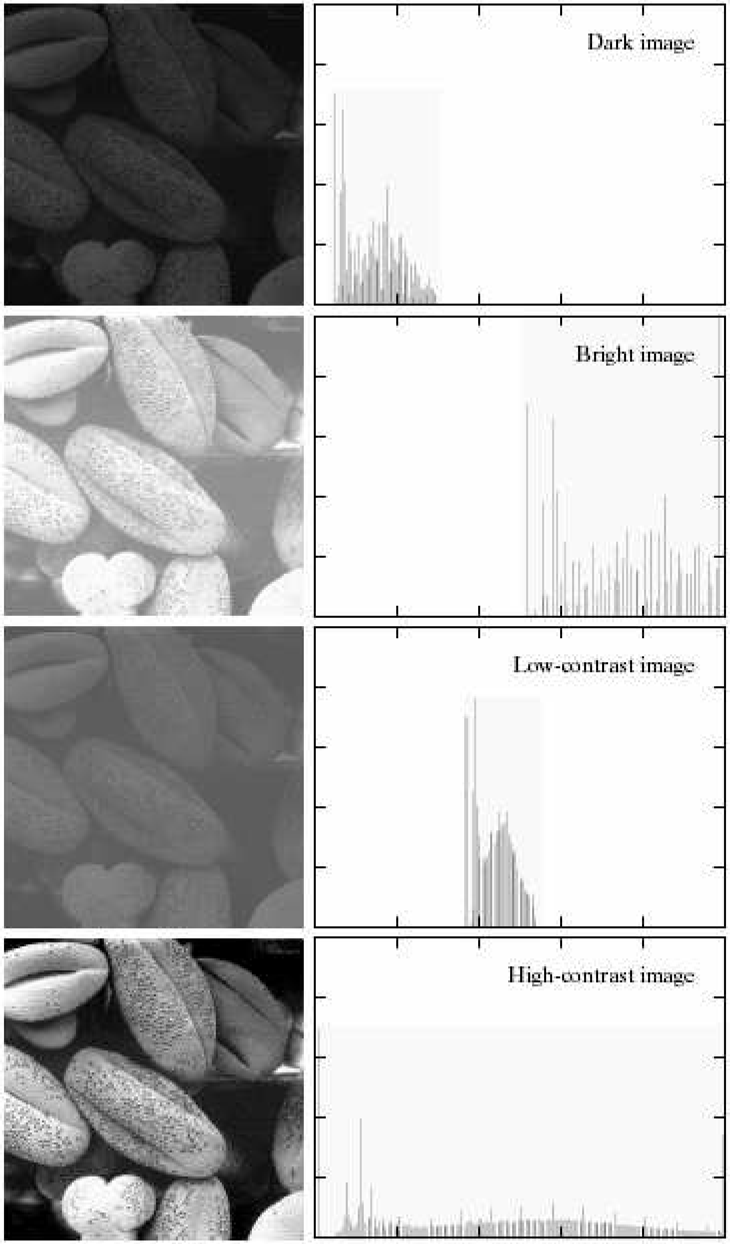


TABLE 6.1-1. Placement of decision and reconstruction levels for Max quantizer

	Uniform		Gaussian		Laplacian		Rayleigh	
Bits	d_i	r_i	d_i	r_i	d_i	r_i	d_i	r_i
1	−1.0000	−0.5000	$-\infty$	−0.7979	$-\infty$	−0.7071	0.0000	1.2657
	0.0000	0.5000	0.0000	0.7979	0.0000	0.7071	2.0985	2.9313
	1.0000		∞		∞		∞	
2	−1.0000	−0.7500	$-\infty$	−1.5104	$-\infty$	−1.8340	0.0000	0.8079
	−0.5000	−0.2500	−0.9816	−0.4528	−1.1269	−0.4198	1.2545	1.7010
	−0.0000	0.2500	0.0000	0.4528	0.0000	0.4198	2.1667	2.6325
	0.5000	0.7500	0.9816	1.5104	1.1269	1.8340	3.2465	3.8604
	1.0000		∞		∞		∞	
3	−1.0000	−0.8750	$-\infty$	−2.1519	$-\infty$	−3.0867	0.0000	0.5016
	−0.7500	−0.6250	−1.7479	−1.3439	−2.3796	−1.6725	0.7619	1.0222
	−0.5000	−0.3750	−1.0500	−0.7560	−1.2527	−0.8330	1.2594	1.4966
	−0.2500	−0.1250	−0.5005	−0.2451	−0.5332	−0.2334	1.7327	1.9688
	0.0000	0.1250	0.0000	0.2451	0.0000	0.2334	2.2182	2.4675
	0.2500	0.3750	0.5005	0.7560	0.5332	0.8330	2.7476	3.0277
	0.5000	0.6250	1.0500	1.3439	1.2527	1.6725	3.3707	3.7137
	0.7500	0.8750	1.7479	2.1519	2.3796	3.0867	4.2124	4.7111
	1.0000		∞		∞		∞	
4	−1.0000	−0.9375	$-\infty$	−2.7326	$-\infty$	−4.4311	0.0000	0.3057
	−0.8750	−0.8125	−2.4008	−2.0690	−3.7240	−3.0169	0.4606	0.6156
	−0.7500	−0.6875	−1.8435	−1.6180	−2.5971	−2.1773	0.7509	0.8863
	−0.6250	−0.5625	−1.4371	−1.2562	−1.8776	−1.5778	1.0130	1.1397
	−0.5000	−0.4375	−1.0993	−0.9423	−1.3444	−1.1110	1.2624	1.3850
	−0.3750	−0.3125	−0.7995	−0.6568	−0.9198	−0.7287	1.5064	1.6277
	−0.2500	−0.1875	−0.5224	−0.3880	−0.5667	−0.4048	1.7499	1.8721
	−0.1250	−0.0625	−0.2582	−0.1284	−0.2664	−0.1240	1.9970	2.1220
	0.0000	0.0625	0.0000	0.1284	0.0000	0.1240	2.2517	2.3814
	0.1250	0.1875	0.2582	0.3880	0.2644	0.4048	2.5182	2.6550
	0.2500	0.3125	0.5224	0.6568	0.5667	0.7287	2.8021	2.9492
	0.3750	0.4375	0.7995	0.9423	0.9198	1.1110	3.1110	3.2729
	0.5000	0.5625	1.0993	1.2562	1.3444	1.5778	3.4566	3.6403
	0.6250	0.6875	1.4371	1.6180	1.8776	2.1773	3.8588	4.0772
	0.7500	0.8125	1.8435	2.0690	2.5971	3.0169	4.3579	4.6385
	0.8750	0.9375	2.4008	2.7326	3.7240	4.4311	5.0649	5.4913
	1.0000		∞		∞		∞	

Compandor

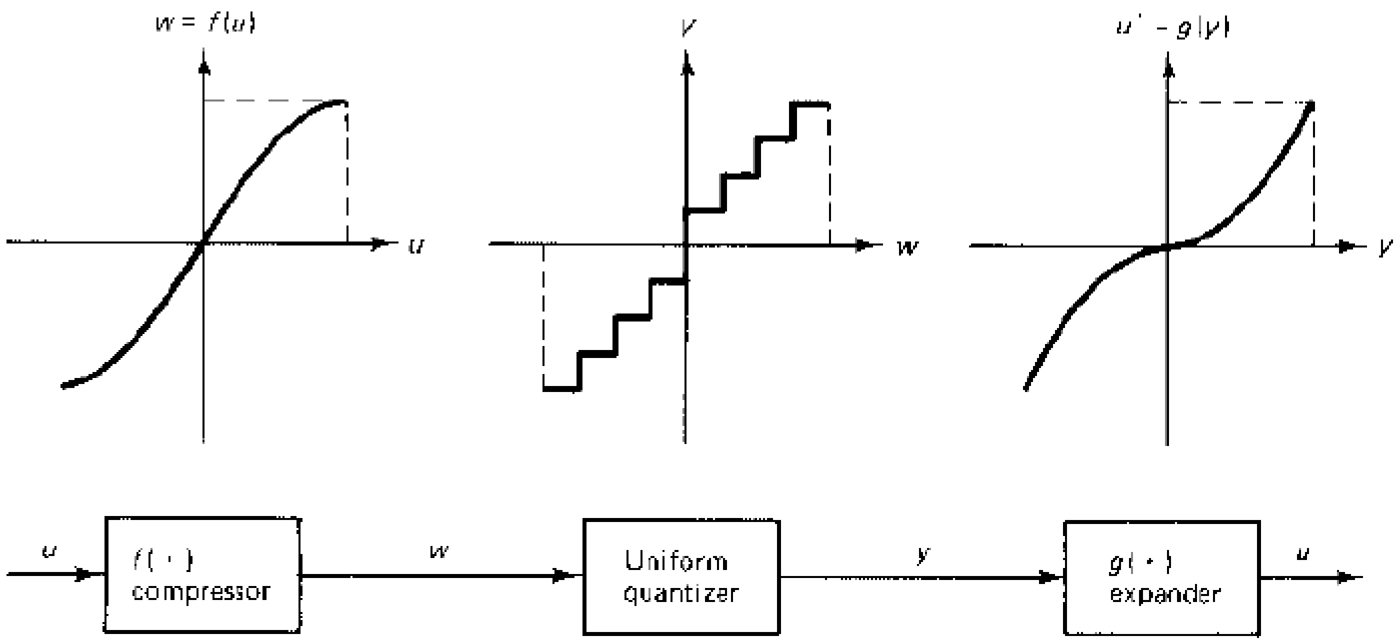
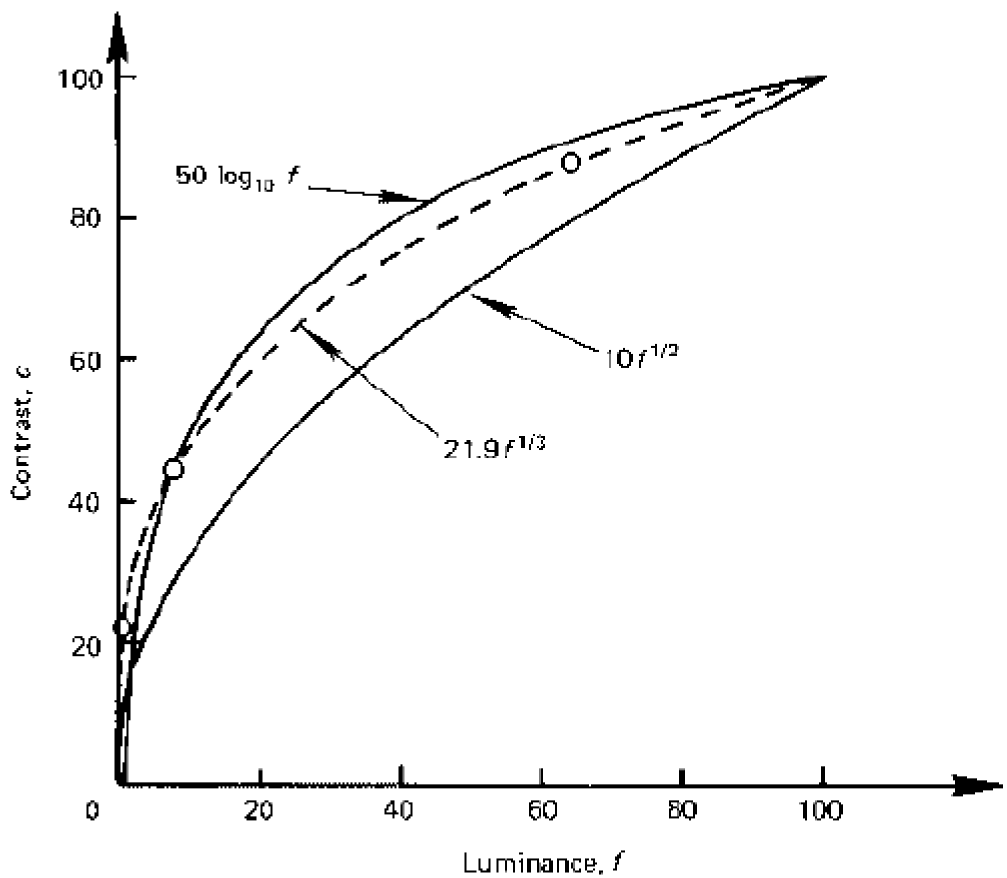


TABLE 6.1-2. Companding quantization transformations

	Probability Density	Forward Transformation	Inverse Transformation
Gaussian	$p(f) = (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{f^2}{2\sigma^2}\right\}$	$g = \frac{1}{2}\operatorname{erf}\left\{\frac{f}{\sqrt{2}\sigma}\right\}$	$\hat{f} = \sqrt{2}\sigma \operatorname{erf}^{-1}\{2\hat{g}\}$
Rayleigh	$p(f) = \frac{f}{\sigma^2}\exp\left\{-\frac{f^2}{2\sigma^2}\right\}$	$g = \frac{1}{2} - \exp\left\{-\frac{f^2}{2\sigma^2}\right\}$	$\hat{f} = [2\sigma^2 \ln\{1/(\frac{1}{2} - \hat{g})\}]^{1/2}$
	$p(f) = \frac{\alpha}{2}\exp\{-\alpha\lvert f\rvert\}$	$g = \frac{1}{2}[1 - \exp\{-\alpha f\}]$ $\quad f \geq 0$	$\hat{f} = -\frac{1}{\alpha}\ln\{1 - 2\hat{g}\}$ $\quad \hat{g} \geq 0$
Laplacian	$\alpha = \frac{\sqrt{2}}{\sigma}$	$g = -\frac{1}{2}[1 - \exp\{\alpha f\}]$ $\quad f < 0$	$\hat{f} = \frac{1}{\alpha}\ln\{1 + 2\hat{g}\}$ $\quad \hat{g} < 0$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\{-y^2\}\,dy$

Luminance - Contrast



Quantization: Contouring Problem

1-bit

2-bit

3-bit

4-bit

5-bit

6-bit

7-bit

8-bit

Dithering

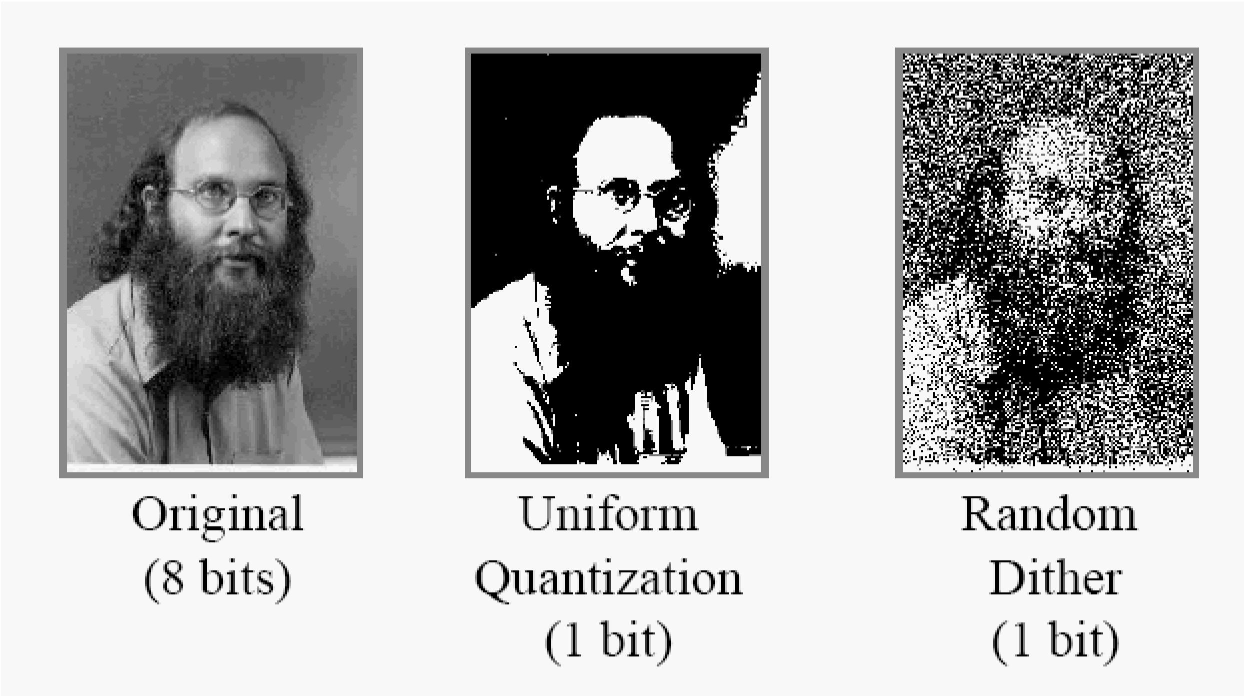


Image from Thomas Funkhouser's lecture in Princeton University.

Dithering: Ordered vs. Random

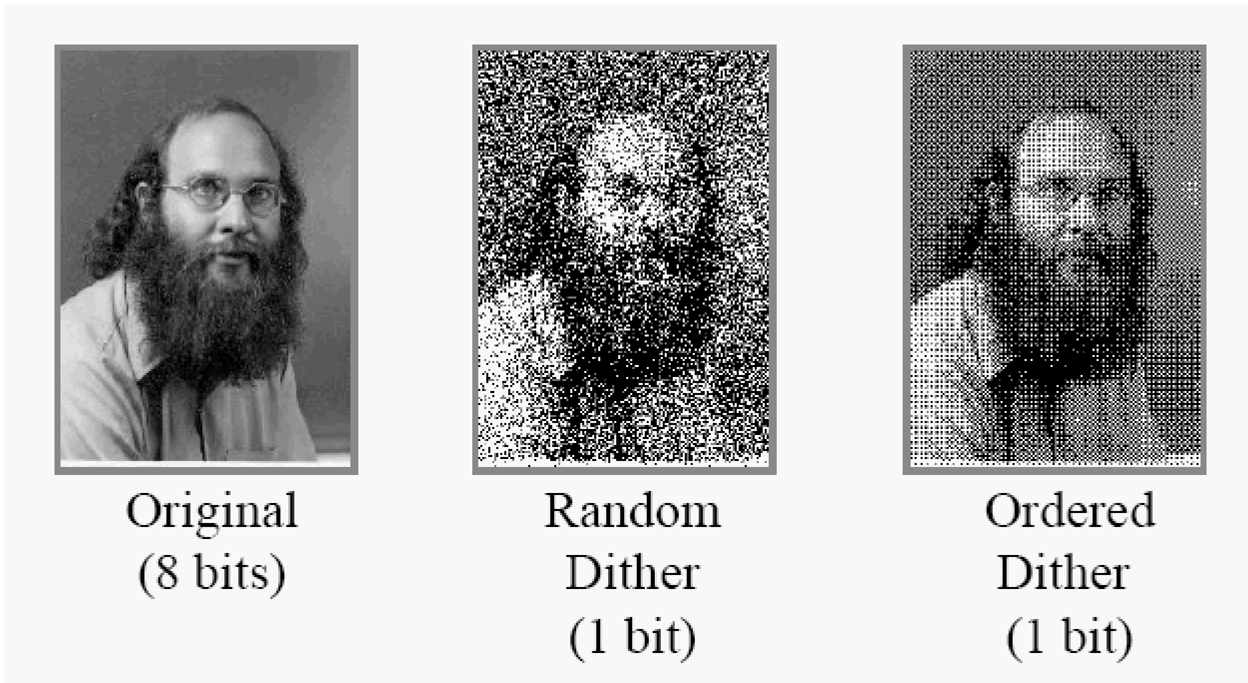


Image from Thomas Funkhouser's lecture in Princeton University.

Half-tone

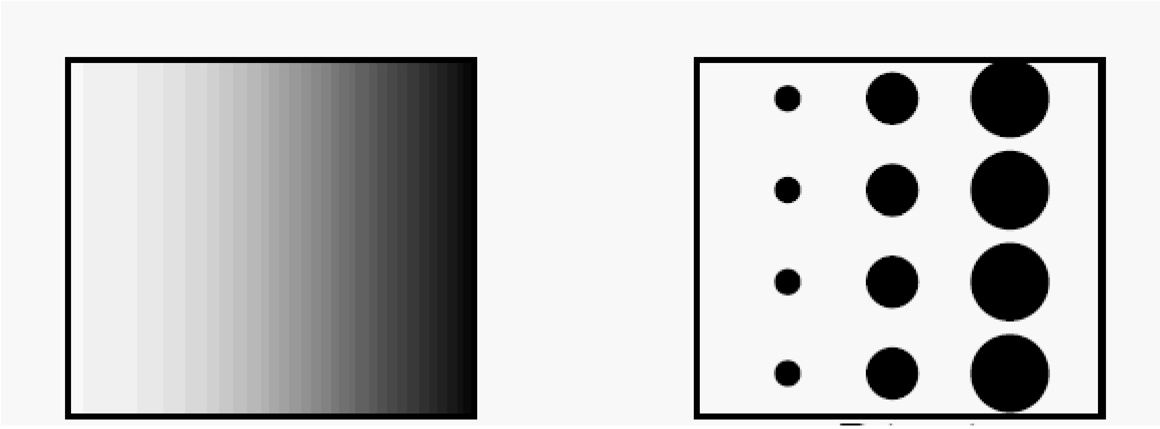
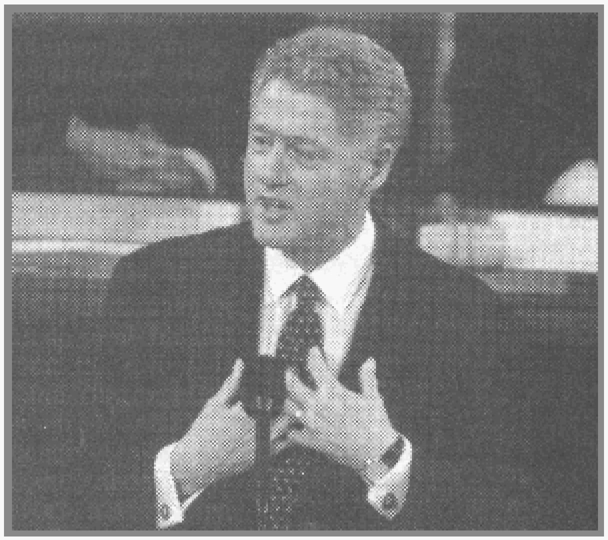
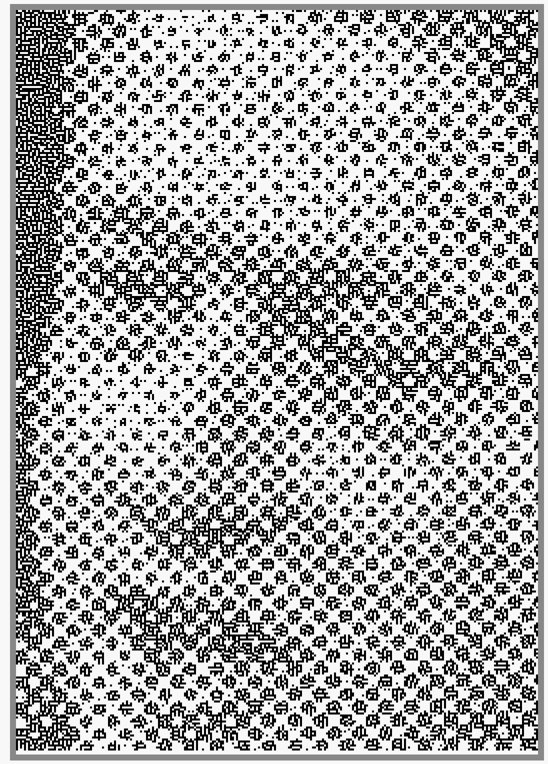


Image from Thomas Funkhouser's lecture in Princeton University.

Half-tone



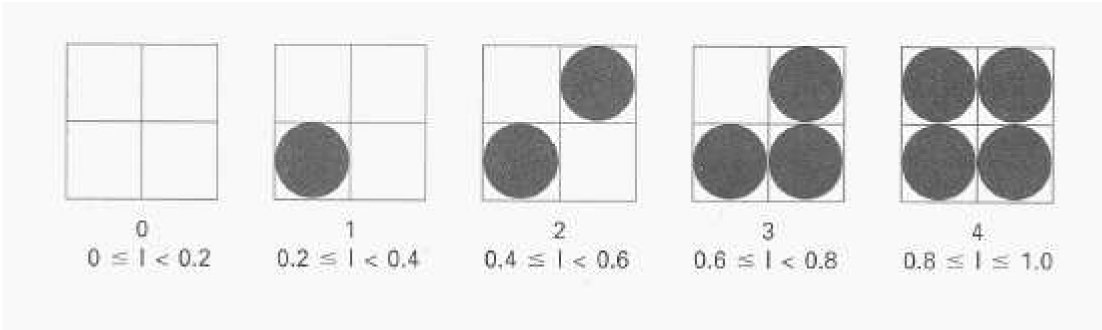
Newspaper Image



From New York Times, 9/21/99

Image from Thomas Funkhouser’s lecture in Princeton University.

Half-tone: how many gray-levels per block?



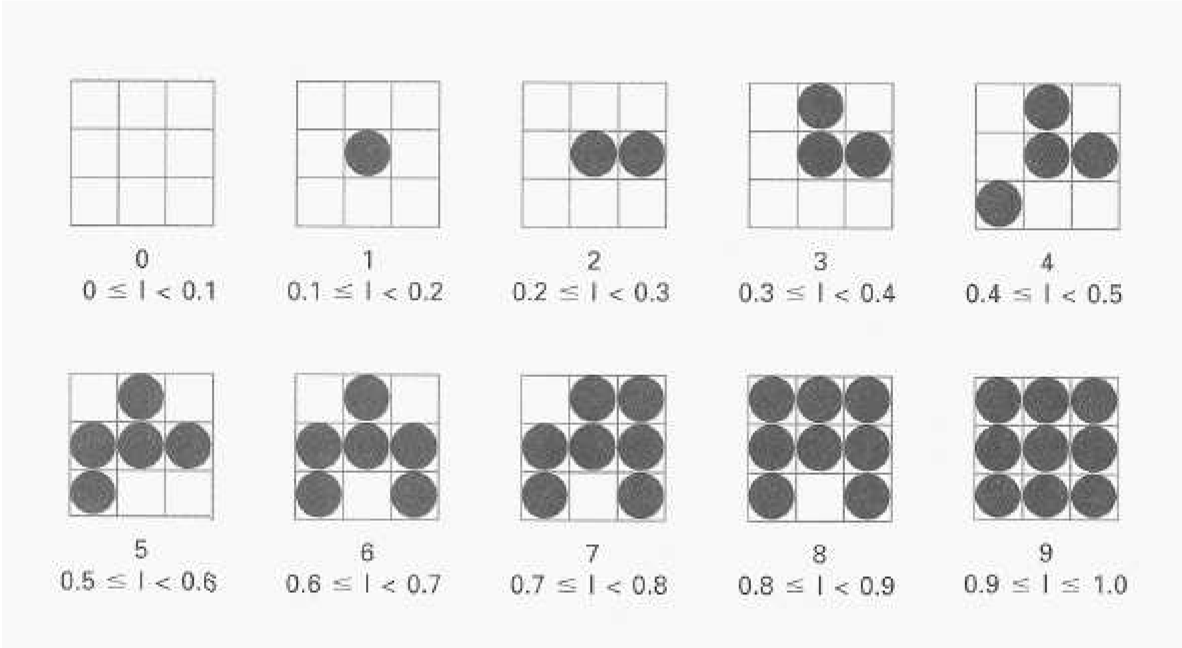


Image from Thomas Funkhouser's lecture in Princeton University.