specify integer indices of arrays and vectors. The symbol roman j will represent $\sqrt{-1}$. The complex conjugate of a complex variable such as z, will be denoted by z^*. Certain symbols will be redefined at appropriate places in the text to keep the notation clear.

Table 2.1 lists several well-known one-dimensional functions that will be often encountered. Their two-dimensional versions are functions of the *separable form*

$$f(x, y) = f_1(x)f_2(y) \tag{2.1}$$

For example, the two-dimensional delta functions are defined as

$$\text{Dirac:} \quad \delta(x, y) = \delta(x)\delta(y) \tag{2.2a}$$

$$\text{Kronecker:} \quad \delta(m, n) = \delta(m)\delta(n) \tag{2.2b}$$

which satisfy the properties

$$\left.\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y')\delta(x - x', y - y')dx'\,dy' = f(x, y) \\ \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon} \delta(x, y)dx\,dy = 1, \end{aligned}\right\} \tag{2.3}$$

$$\left.\begin{aligned} x(m, n) = \sum\sum_{m', n' = -\infty}^{\infty} x(m', n')\delta(m - m', n - n') \\ \sum\sum_{m, n = -\infty}^{\infty} \delta(m, n) = 1 \end{aligned}\right\} \tag{2.4}$$

The definitions and properties of the functions rect(x, y), sinc(x, y), and comb(x, y) can be defined in a similar manner.

TABLE 2.1 Some Special Functions

Function	Definition	Function	Definition
Dirac delta	$\delta(x) = 0, x \neq 0$	*Rectangle*	$\text{rect}(x) = \begin{cases} 1, & \lvert x\rvert \le \frac{1}{2} \\ 0, & \lvert x\rvert > \frac{1}{2} \end{cases}$
	$\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} \delta(x)\,dx = 1$	*Signum*	$\text{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$
Sifting property	$\int_{-\infty}^{\infty} f(x')\delta(x - x')\,dx' = f(x)$	*Sinc*	$\text{sinc}(x) = \frac{\sin \pi x}{\pi x}$
Scaling property	$\delta(ax) = \frac{\delta(x)}{\lvert a\rvert}$	*Comb*	$\text{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)$
Kronecker delta	$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$	*Triangle*	$\text{tri}(x) = \begin{cases} 1 - \lvert x\rvert, & \lvert x\rvert \le 1 \\ 0, & \lvert x\rvert > 1 \end{cases}$
Sifting property	$\sum_{m=-\infty}^{\infty} f(m)\delta(n - m) = f(n)$		

2.3 LINEAR SYSTEMS AND SHIFT INVARIANCE

A large number of imaging systems can be modeled as two-dimensional linear systems. Let $x(m, n)$ and $y(m, n)$ represent the input and output sequences, respectively, of a two-dimensional system (Fig. 2.1), written as

$$y(m, n) = \mathcal{H}[x(m, n)] \tag{2.5}$$

This system is called *linear* if and only if any linear combination of two inputs $x_1(m, n)$ and $x_2(m, n)$ produces the same combination of their respective outputs $y_1(m, n)$ and $y_2(m, n)$, i.e., for arbitrary constants a_1 and a_2

$$\begin{aligned}\mathcal{H}[a_1 x_1(m, n) + a_2 x_2(m, n)] &= a_1 \mathcal{H}[x_1(m, n)] + a_2 \mathcal{H}[x_2(m, n)] \\ &= a_1 y_1(m, n) + a_2 y_2(m, n)\end{aligned} \tag{2.6}$$

This is called *linear superposition*. When the input is the two-dimensional Kronecker delta function at location (m', n'), the output at location (m, n) is defined as

$$h(m, n; m', n') \triangleq \mathcal{H}[\delta(m - m', n - n')] \tag{2.7}$$

and is called the *impulse response* of the system. For an imaging system, it is the image in the output plane due to an ideal point source at location (m', n') in the input plane. In our notation, the semicolon (;) is employed to distinguish the input and output pairs of coordinates.

The impulse response is called the *point spread function* (PSF) when the inputs and outputs represent a positive quantity such as the intensity of light in imaging systems. The term *impulse response* is more general and is allowed to take negative as well as complex values. The *region of support* of an impulse response is the smallest closed region in the m, n plane outside which the impulse response is zero. A system is said to be a *finite impulse response* (FIR) or an *infinite impulse response* (IIR) system if its impulse response has finite or infinite regions of support, respectively.

The output of any linear system can be obtained from its impulse response and the input by applying the superposition rule of (2.6) to the representation of (2.4) as follows:

$$\begin{aligned} y(m, n) &= \mathcal{H}[x(m, n)] \\ &= \mathcal{H}\left[\sum_{m'}\sum_{n'} x(m', n')\delta(m - m', n - n')\right] \\ &= \sum_{m'}\sum_{n'} x(m', n')\mathcal{H}[\delta(m - m', n - n')] \\ \Rightarrow \qquad y(m, n) &= \sum_{m'}\sum_{n'} x(m', n')h(m, n; m', n') \end{aligned} \tag{2.8}$$

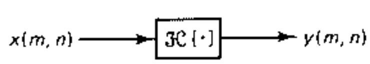


Figure 2.1 A system.

A system is called *spatially invariant* or *shift invariant* if a translation of the input causes a translation of the output. Following the definition of (2.7), if the impulse occurs at the origin we will have

$$\mathcal{H}[\delta(m, n)] = h(m, n; 0, 0)$$

Hence, it must be true for shift invariant systems that

$$\begin{aligned} h(m, n; m', n') &\triangleq \mathcal{H}[\delta(m - m', n - n')] \\ &= h(m - m', n - n'; 0, 0) \\ \Rightarrow \qquad h(m, n; m', n') &= h(m - m', n - n') \end{aligned} \tag{2.9}$$

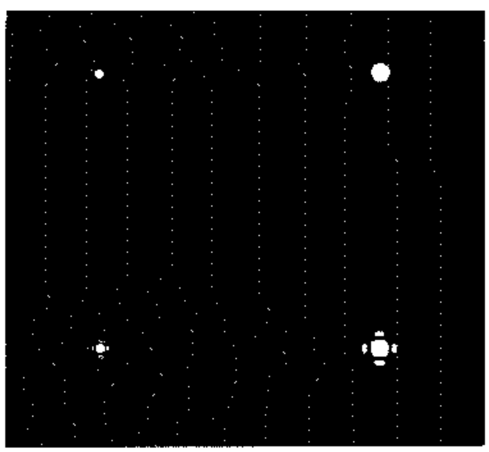
i.e., the impulse response is a function of the two displacement variables only. This means the shape of the impulse response does not change as the impulse moves about the m, n plane. A system is called *spatially varying* when (2.9) does not hold. Figure 2.2 shows examples of PSFs of imaging systems with separable or circularly symmetric impulse responses.

For shift invariant systems, the output becomes

$$y(m, n) = \sum_{m', n' = -\infty}^{\infty}\sum h(m - m', n - n')x(m', n') \tag{2.10}$$

which is called the *convolution* of the input with the *impulse response*. Figure 2.3 shows a graphical interpretation of this operation. The impulse response array is rotated about the origin by 180° and then shifted by (m, n) and overlayed on the array $x(m', n')$. The sum of the product of the arrays $\{x(\cdot,\cdot)\}$ and $\{h(\cdot,\cdot)\}$ in the overlapping regions gives the result at (m, n). We will use the symbol $\circledast$ to denote the convolution operation in both discrete and continuous cases, i.e.,

$$\begin{aligned} g(x, y) = h(x, y) \circledast f(x, y) &\triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x - x', y - y')f(x', y')\,dx'\,dy' \\ y(m, n) = h(m, n) \circledast x(m, n) &\triangleq \sum_{m', n' = -\infty}^{\infty}\sum h(m - m', n - n')x(m', n') \end{aligned} \tag{2.11}$$



a	b
c	d

Figure 2.2 Examples of PSFs. (a) Circularly symmetric PSF of average atmospheric turbulence causing small blur; (b) atmospheric turbulence PSF causing large blur; (c) separable PSF of a diffraction limited system with square aperature; (d) same as (c) but with smaller aperture.

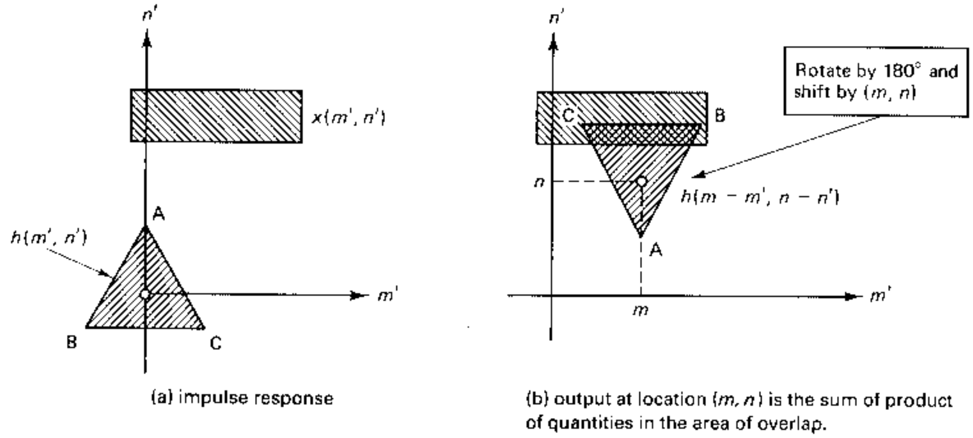
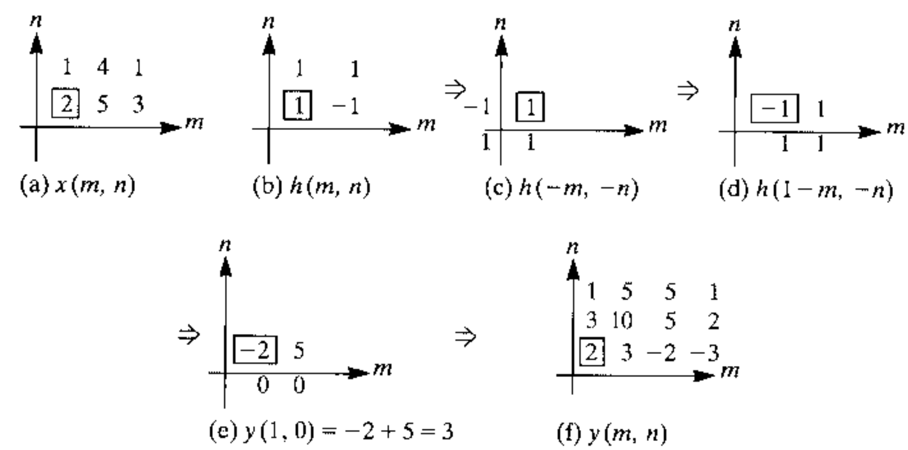


Figure 2.3 Discrete convolution in two dimensions

The convolution operation has several interesting properties, which are explored in Problems 2.2 and 2.3.

Example 2.1 (Discrete convolution)

Consider the 2×2 and 3×2 arrays $h(m, n)$ and $x(m, n)$ shown next, where the boxed element is at the origin. Also shown are the various steps for obtaining the convolution of these two arrays. The result $y(m, n)$ is a 4×3 array. In general, the convolution of two arrays of sizes $(M_1 \times N_1)$ and $(M_2 \times N_2)$ yields an array of size $[(M_1 + M_2 - 1) \times (N_1 + N_2 - 1)]$ (Problem 2.5).



2.4 THE FOURIER TRANSFORM

Two-dimensional transforms such as the Fourier transform and the Z-transform are of fundamental importance in digital image processing as will become evident in the subsequent chapters. In one dimension, the Fourier transform of a complex