Unitary Transforms, Wavelets and Their Applications

EE4830 Lecture 5
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With thanks to G&W website, Mani Thomas, Min Wu, W. Trappe, etc.
Announcements

- PS#3 due Friday March 2\textsuperscript{nd} (no extensions)
  - 4.19 Expand the expression for Laplacian filter, try to visualize/explain what you get

- EXP#2 due Monday March 26\textsuperscript{th}
  - Image compression and eigen face applications
  - New dataset for eigen face: Rice ELEC301
- PS#4 to be posted this week

- Midterm on March 5
  - YES: text book(s), class notes, calculator
  - NO: computer/cellphone/matlab/internet
  - 5 analytical problems, sample midterm + solutions available
  - Coverage: lecture 1-5, summary content in lecture 6
  - Additional instructor office hours: 4:30-6:30pm Monday March 5\textsuperscript{th}
    Mudd 1312, enter from the backdoor, x4-3131
- Midterm class evaluations available next week
Lecture Outline

- Unitary transforms
  - Review of definition, properties
  - Examples: DFT, DCT, KLT, Haar ...
  - Applications

- Wavelet transform and applications

- Readings for *today and last week*: G&W Chap 4, 7, Jain 5.1-5.11
Digital Transform as Basis Expansion

Forward transform
\[ y(u) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n)a(u,n) \]

Inverse transform
\[ x(n) = \frac{1}{\sqrt{N}} \sum_{u=0}^{N-1} y(u)b(u,n) \]

1D-DFT
\[ a(u,n) = e^{-j2\pi \frac{un}{N}} = \cos(2\pi \frac{un}{N}) - j\sin(2\pi \frac{un}{N}) \]

Matrix notation
\[ y = Ax \]
\[ x = A^{-1}y \]
DFT vs. DCT

1D-DCT

\[ a(0, n) = \sqrt{\frac{1}{N}} \quad u = 0 \]

\[ a(u, n) = \sqrt{\frac{2}{N}} \cos\left(\frac{\pi(2n + 1)u}{2N}\right) \quad u = 1, 2, \ldots, N - 1 \]

1D-DFT

\[ a(u, n) = e^{-j2\pi\frac{un}{N}} = \cos(2\pi\frac{un}{N}) + jsin(2\pi\frac{un}{N}) \]

\[ \text{real}(a) \quad \text{imag}(a) \]
**DFT and DCT in Matrix Notations**

Matrix notation for 1D transform

\[ y = Ax \]
\[ x = A^{-1} y \]

**1D-DCT**

\[
a(0, n) = \sqrt{\frac{1}{N}} \quad u = 0
\]

\[
a(u, n) = \sqrt{\frac{2}{N}} \cos \frac{\pi (2n + 1) u}{2N} \quad u = 1, 2, \ldots, N - 1
\]

**1D-DFT**

\[
a(u, n) = e^{-j2\pi \frac{un}{N}} = \cos \left( 2\pi \frac{un}{N} \right) - jsin \left( 2\pi \frac{un}{N} \right)
\]

\(N=32\)
Unitary Transforms

Matrix notation for 1D transform

\[
\begin{align*}
y &= Ax \\
x &= A^{-1}y
\end{align*}
\]

This transform is called “unitary” when \( A \) is a unitary matrix

\[
A^{-1} = A^*T, \quad AA^*T = A^*A^T = I
\]

\[
A = [a_1^T, \ldots, a_{N-1}^T]^T = [a_{un}]_{N \times N}
\]

Unitary Transform implies the following properties

Orthonormality (Eq 5.5 in Jain)
: no two basis represent the same information in the image

\[
\sum_m \sum_n a_{um}a_{vn} = \delta(u - v)
\]

Completeness (Eq 5.6)
: all information in the image are represented in the set of basis functions

\[
\sum_u \sum_v a_{um}a_{vn} = \delta(m - n)
\]
Unitary Transforms in 2D

- Image transform as basis expansion:

  \[ g(u, v) \propto \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) a_{uv}(m, n) \]

  \[ f(m, n) \propto \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} g(u, v) a_{uv}^*(m, n) \]

Separable: \( a_{uv}(m, n) = a_u(m)b_v(n) \)

DFT2 \( a_{uv}(m, n) = e^{-j2\pi(\frac{um}{M} + \frac{vn}{N})} \)
From 1D-DCT to 2D-DCT

- Rows of A form a set of orthonormal basis
- A is not symmetric!
- DCT is not the real part of unitary DFT!
DFT and DCT on Lena

Shift low-freq to the center

DFT2

Assume periodic and zero-padded ...

DCT2

Assume reflection ...
Exercise

- Unitary or not?

\[
A_1 = \begin{bmatrix}
\sqrt{2} & j \\
-j & \sqrt{2}
\end{bmatrix}
\quad
A_2 = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & j \\
j & 1
\end{bmatrix}
\]

- How do we decompose this picture?

\[
\text{DCT2} \quad \Rightarrow \quad \begin{bmatrix}
? & ? \\
? & ?
\end{bmatrix}
\]
Properties of 1-D Unitary Transform

- **Energy Conservation**
  - $|| f ||^2 = || g ||^2$
  - $|| g ||^2 = || Af ||^2 = (Af)^* (Af) = f^* A^* A f = f^* f = || f ||^2$

- **Rotation**
  - A unitary transformation is a rotation of a vector in an N-dimension space, i.e., a rotation of basis coordinates
  - The angles between vectors are preserved

- Review: correlation in vectors and images
- De-correlation (example 5.2 in Jain)
  - Highly correlated input elements $\rightarrow$ quite uncorrelated output coefficients
- Energy compaction
  - Many common unitary transforms tend to pack a large fraction of signal energy into just a few transform coefficients
Karhunen-Loeve Transform (KLT)

- a.k.a the Hotelling transform or the Principle Component Analysis (PCA)
- Eigen decomposition of $R_x$: $R_x u_k = \lambda_k u_k$
  - Recall the properties of $R_x$
    - Hermitian (conjugate symmetric $R^H = R$);
    - Nonnegative definite (real non-negative eigen values)
- Karhunen-Loeve Transform (KLT)
  - $y = U^H x \iff x = U y$ with $U = [ u_1, \ldots u_N ]$
  - KLT is a unitary transform with basis vectors in $U$ being the orthonormalized eigenvectors of $R_x$
  - $U^H R_x U = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ i.e. KLT performs decorrelation
  - Often order $\{u_i\}$ so that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$
Properties of K-L Transform

- **Decorrelation**
  - \( E[ y y^H ] = E[ (U^H x) (U^H x)^H ] = U^H E[ x x^H ] U = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \)
  - By construction
  - Note: Other matrices (unitary or nonunitary) may also decorrelate the transformed sequence [Jain’s example 5.5 and 5.7]

- **Minimizing MSE under basis restriction**
  - If only allow to keep \( m \) coefficients for any \( 1 \leq m \leq N \), what’s the best way to minimize reconstruction error?
    - *Keep the coefficients w.r.t. the eigenvectors of the first \( m \) largest eigenvalues*
KLT Basis Restriction

- Basis restriction
  - Keep only a subset of \( m \) transform coefficients and then perform inverse transform \((1 \leq m \leq N)\)
  - Basis restriction error: MSE between original & new sequences

\[
J_m \triangleq E\left[ \sum_{n=1}^{N} |X[n]-\hat{X}[n]|^2 \right] = \text{Tr}\left[ E\left( (X-\bar{X})(X-\bar{X})^H \right) \right]
\]

- Goal: to find the forward and backward transform matrices to minimize the restriction error for each and every \( m \)

\[
\begin{align*}
X & \xrightarrow{A_{N \times N}} Y = AX \\
& \xrightarrow{[1 \ldots 0 \ldots 0]} W = ImY \\
& \xrightarrow{B_{N \times N}} \hat{X} = BW
\end{align*}
\]

\[
\min J_m \Rightarrow A = U^H, \quad B = A^T = U
\]

\( U \) = eig vector of \( Rx \) arranged with decreasing \( \lambda_k \)
Unitary Transforms in Other Flavors

Walsh-Hardamard

\[ H_3 = \frac{1}{2^{3/2}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} \]

Slant

Nassiri et. al, "Texture Feature Extraction using Slant-Hadamard Transform"
Energy Compaction Transforms

- DCT has excellent energy compaction for highly correlated data
- DCT is a good replacement for K-L
  - Close to optimal for highly correlated data
  - Not depend on specific data like K-L does
  - Fast algorithm available

[ref and statistics: Jain’s pp153, 168-175]

Figure 5.18  Distribution of variances of the transform coefficients (in decreasing order) of a stationary Markov sequence with $N = 16$, $\rho = 0.95$ (see Example 5.9).
The Desirables for Image Transforms

- **Theory**
  - Inverse transform available
    - DFT: ✓
    - DCT: ✓
    - KLT: ✓
  - Energy conservation (Parsevell)
    - DFT: ✓
    - DCT: ✓
    - KLT: ✓
  - Good for compacting energy
    - DFT: ?
    - DCT: ?
    - KLT: ✓
  - Orthonormal, complete basis
    - DFT: ✓
    - DCT: ✓
    - KLT: ✓
  - (sort of) shift- and rotation invariant
    - DFT: ✓
    - DCT: ✓
    - KLT: ?
  - Transform basis signal-independent

- **Implementation**
  - Real-valued
    - DFT: x
    - DCT: ✓
    - KLT: ✓
  - Separable
    - DFT: ✓
    - DCT: ✓
    - KLT: x
  - Fast to compute w. butterfly-like structure
    - DFT: ✓
    - DCT: ✓
    - KLT: x
  - Same implementation for forward and inverse transform
    - DFT: ✓
    - DCT: ✓
    - KLT: x
A Brief History of Transforms

1807 Fourier Theory
1909 Haar filters “wavelets”
1933 Hotelling transform
1947 1948 Karhunen-Loeve
1965 FFT, Cooley-Tukey
1969 WHT, Shanks “computing fast Walsh-Hadamard transform”
1973 Slant Transform and applications to image coding
1974 DCT, Rao,
1977 Fast DCT

... 1992 JPEG Standard

from wikipedia, “a brief history of wavelets”, and other online sources
Applications of Image Transforms

- Compression
- Feature extraction
- Pattern recognition: e.g., eigen faces
  - analyze the principal ("dominating") components
Image Compression

\[ \text{SNR(dB)} = 10 \log_{10} \left( \frac{P_{\text{signal}}}{P_{\text{noise}}} \right) = 20 \log_{10} \left( \frac{A_{\text{signal}}}{A_{\text{noise}}} \right) \]

where \( P \) is average power and \( A \) is RMS amplitude.
Gabor filters

- Gaussian windowed Fourier Transform
  - Make convolution kernels from product of Fourier basis images and Gaussians

\[ \text{Odd (sin)} \times \text{Even (cos)} = \text{Frequency} \]
Example: Filter Responses

Filter bank

Input image

from Forsyth & Ponce
Texture Representation: Filter Responses

- Choose a group of filters
  - Edge/Bar filters: Something like Gabor filters at different orientations, scales
  - Spot filters: Center-surround filters like a Gaussian/difference of Gaussians at multiple scales
- Run filters over image to get a set of *response images*
  - Each contains specific texture information
- Collect statistics of responses over an image or subimage
  - Mean of squared response
  - Mean and variance of squared response
- Euclidean distance between vectors of response statistics for two images is measure of texture similarity
Eigen Faces

Courtesy of Rice U. eigen face project and Yale face database
Lecture Outline

- Unitary transforms
  - Properties
  - Examples: DFT, DCT, KLT, Haar
  - Applications

- Brief overview of wavelet transform and applications
Image in Multiple Scales

FIGURE 7.3 Two image pyramids and their statistics: (a) a Gaussian (approximation) pyramid and (b) a Laplacian (prediction residual) pyramid.
A Three-scale Wavelet Decomposition
A Brief History of Wavelets

1909 Haar

...  
1946 Gabor, Time-Frequency analysis

...  
1982 Morlet, geophysics
1984 Marseile team, Grossmann/Paul, “mathematical microscope”
1985 Meyer, operator theory
1986 Mallat, signal analysis, filter design
1988 Lemarie, Daubechies, w. exponential decay and compact support
1991 ... general construction, ...
1992 ... biorthogonal wavelets, continuous wavelet transform ...
1995 ... wavelet on domains
2000+ image compression standard (JPEG 2000)

- For in-depth looks ...
  - ELEN E6860y Advanced Digital Signal Processing
  - Wavelet and Subband Coding, Vetterli and Kovacevic
Summary

- **Unitary Transforms**
  - Theory revisited
  - Example transforms: DFT, DCT, KLT, Hadamard, Slant
- **Applications**
  - Compression
  - Feature extraction
  - Image matching (eigen faces)
- **Wavelet decomposition of images**