Columbia University in the City of New York

Unitary Transforms, Wavelets and Their Applications

EE4830 Lecture 5

Feb 26th, 2007

Lexing Xie

With thanks to G&W website, Mani Thomas, Min Wu, W. Trappe, etc.

Announcements

- PS#3 due Friday March 2nd (no extensions)
  - 4.19 Expand the expression for Laplacian filter, try to visualize/explain what you get

- EXP#2 due Monday March 26th
  - Image compression and eigen face applications
  - New dataset for eigen face: Rice ELEC301
- PS#4 to be posted this week

- Midterm on March 5
  - YES: text book(s), class notes, calculator
    NO: computer/cellphone/matlab/internet
  - 5 analytical problems, sample midterm + solutions available
  - Coverage: lecture 1-5, summary content in lecture 6
  - Additional instructor office hours: 4:30-6:30pm Monday March 5th
    Mudd 1312, enter from the backdoor, x4-3131
- Midterm class evaluations available next week

Lecture Outline

- Unitary transforms
  - Review of definition, properties
  - Examples: DFT, DCT, KLT, Haar …
  - Applications
- Wavelet transform and applications
- Readings for *today and last week*: G&W Chap 4, 7, Jain 5.1-5.11

Digital Transform as Basis Expansion

Forward transform

$$y(u) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) a(u,n)$$

Inverse transform

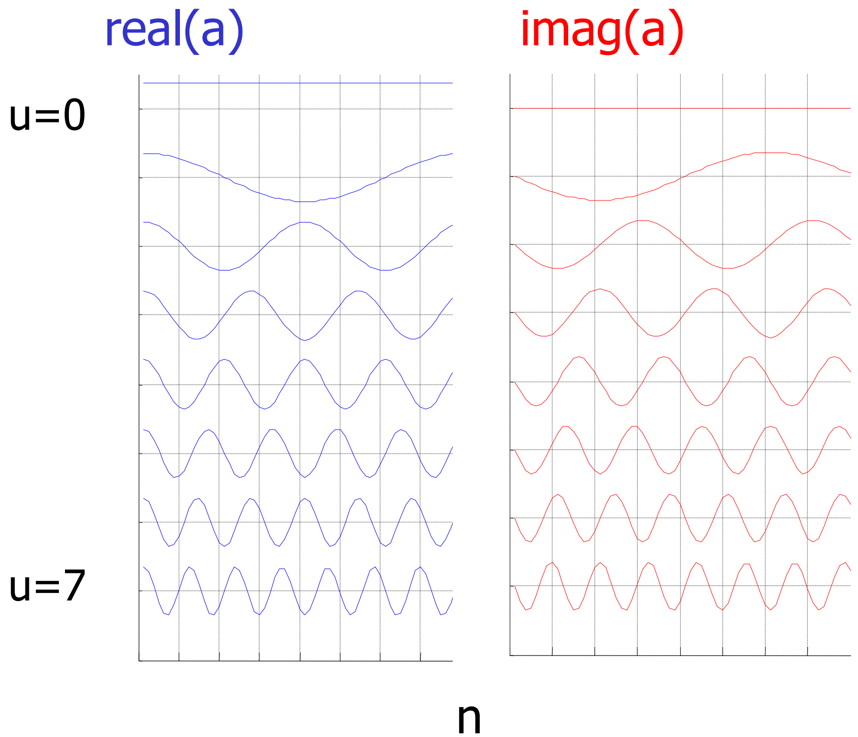
$$x(n) = \frac{1}{\sqrt{N}} \sum_{u=0}^{N-1} y(u) b(u,n)$$

Matrix notation

$$y = Ax$$
$$x = A^{-1} y$$

1D-DFT

$$a(u,n) = e^{-j2\pi \frac{un}{N}} = cos(2\pi \frac{un}{N}) - j sin(2\pi \frac{un}{N})$$



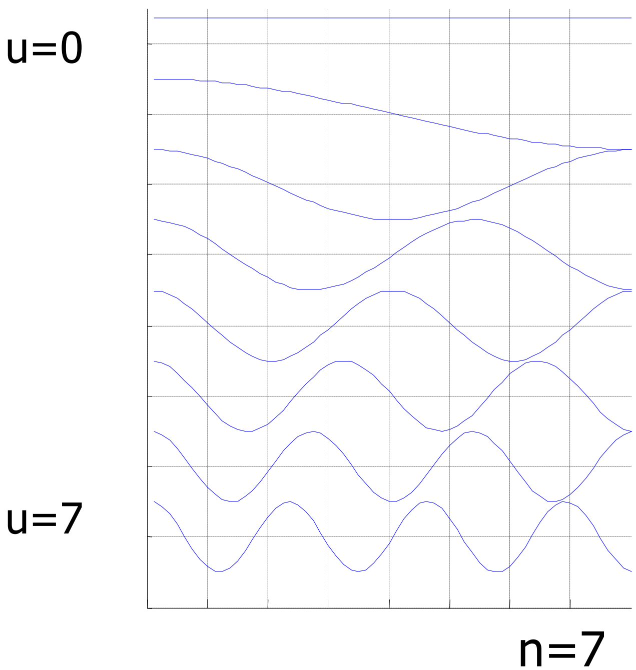
DFT vs. DCT

1D-DCT

$$a(0,n) = \sqrt{\frac{1}{N}} \qquad u = 0$$

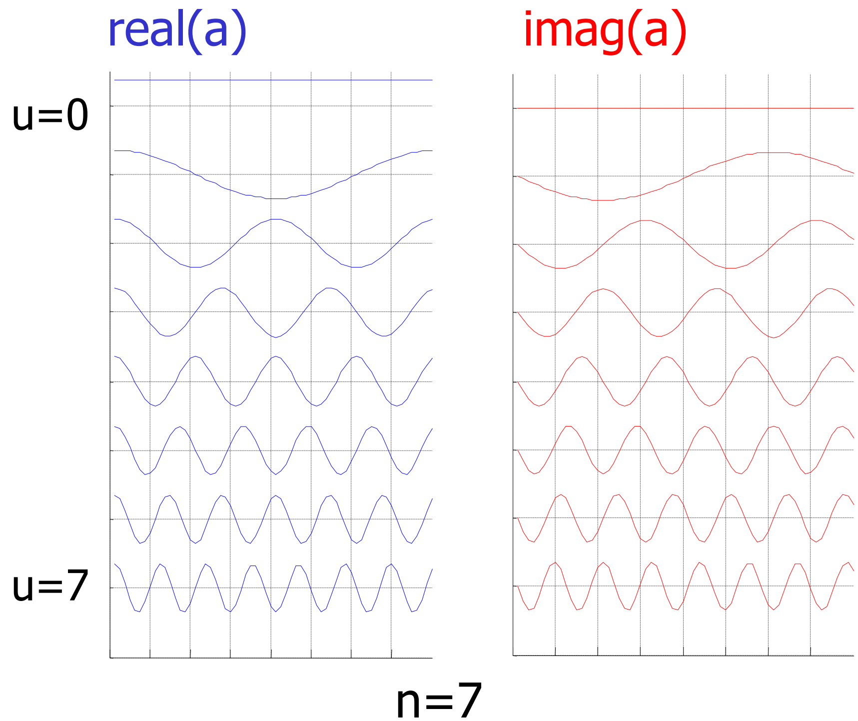
$$a(u,n) = \sqrt{\frac{2}{N}} cos\frac{\pi(2n+1)u}{2N}$$

$$u = 1, 2, \ldots, N-1$$



1D-DFT

$$\begin{aligned} a(u,n) &= e^{-j2\pi\frac{un}{N}} \\ &= cos(2\pi\frac{un}{N}) + jsin(2\pi\frac{un}{N}) \end{aligned}$$



DFT and DCT in Matrix Notations

Matrix notation for 1D transform

$$y = Ax$$
$$x = A^{-1}y$$

1D-DCT

$$a(0,n) = \sqrt{\frac{1}{N}} \qquad u = 0$$

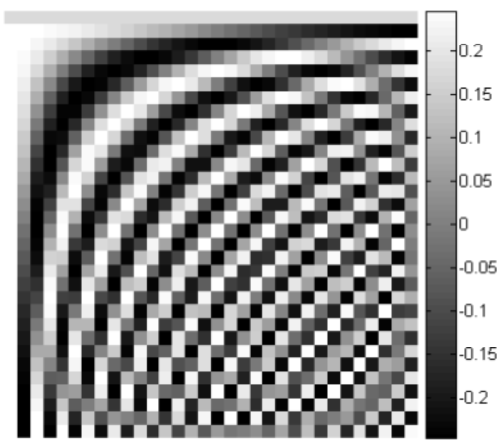
$$a(u,n) = \sqrt{\frac{2}{N}} cos \frac{\pi(2n+1)u}{2N}$$
$$u = 1, 2, \ldots, N-1$$

1D-DFT

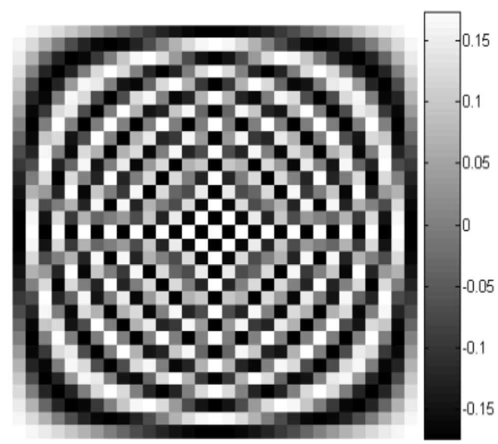
$$a(u,n) = e^{-j2\pi\frac{un}{N}}$$
$$= cos(2\pi\frac{un}{N}) - jsin(2\pi\frac{un}{N})$$

N=32

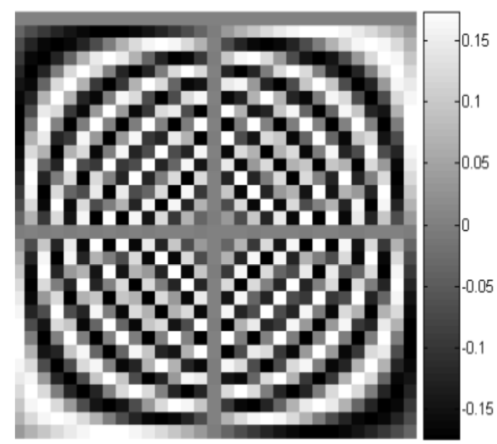
A



real(A)



imag(A)



Unitary Transforms

Matrix notation for 1D transform

$$y = Ax$$
$$x = A^{-1}y$$

This transform is called "unitary" when A is a unitary matrix

$$A^{-1} = A^{*T}, \; AA^{*T} = A^{*}A^{T} = I$$

$$A = [a_1^T, \ldots, a_{N-1}^T]^T \; = [a_{un}]_{N \times N}$$

Unitary Transform implies the following properties

Orthonormality (Eq 5.5 in Jain)

: no two basis represent the same information in the image

$$\sum_m \sum_n a_{um} a_{vn} = \delta(u-v)$$

Completeness (Eq 5.6)

: all information in the image are represented in the set of basis functions

$$\sum_u \sum_v a_{um} a_{vn} = \delta(m-n)$$

Unitary Transforms in 2D

- Image transform as basis expansion:

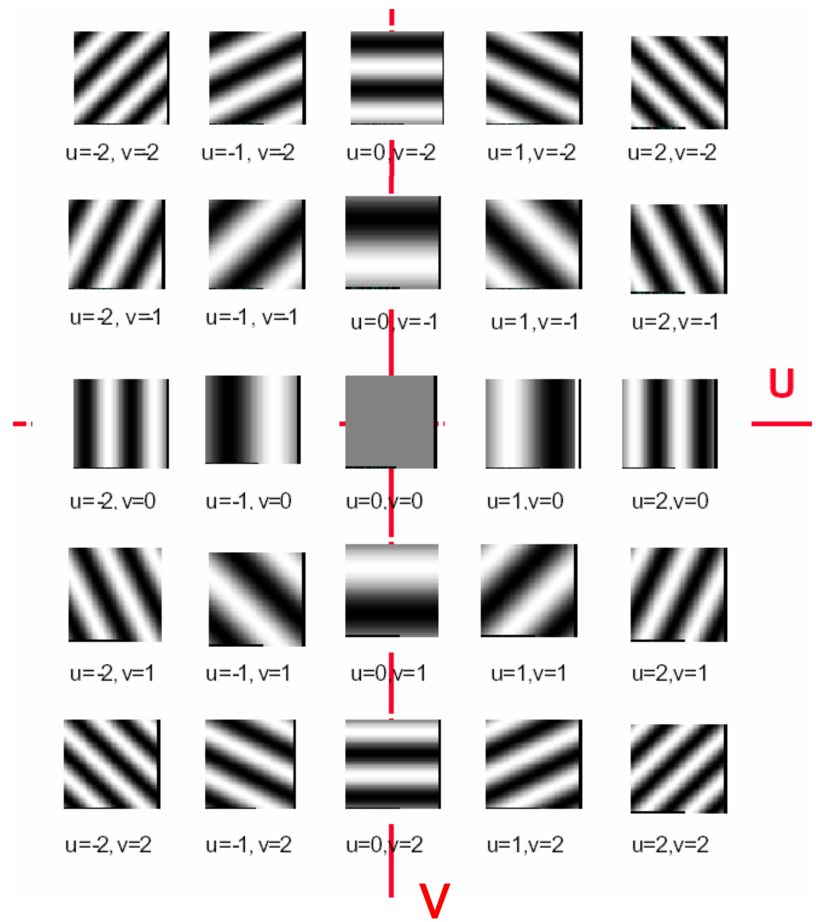
$$g(u,v) \propto \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) a_{uv}(m,n)$$

$$f(m,n) \propto \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} g(u,v) a_{uv}^{*}(m,n)$$

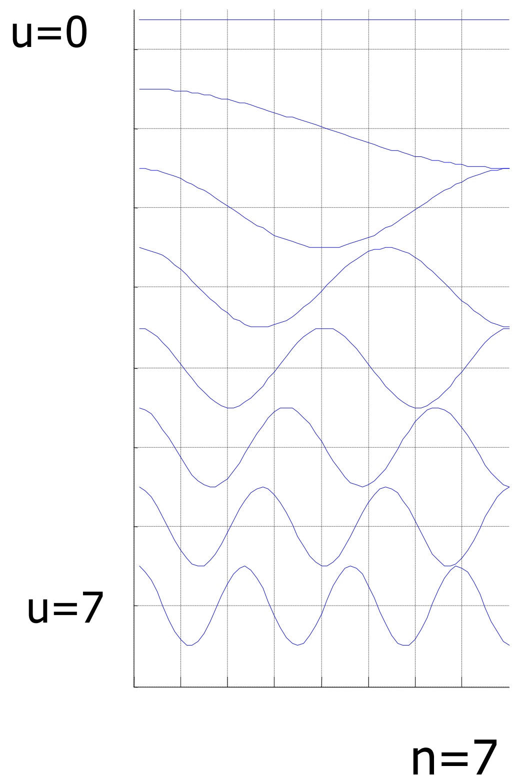
Separable: $a_{uv}(m,n) = a_u(m) b_v(n)$

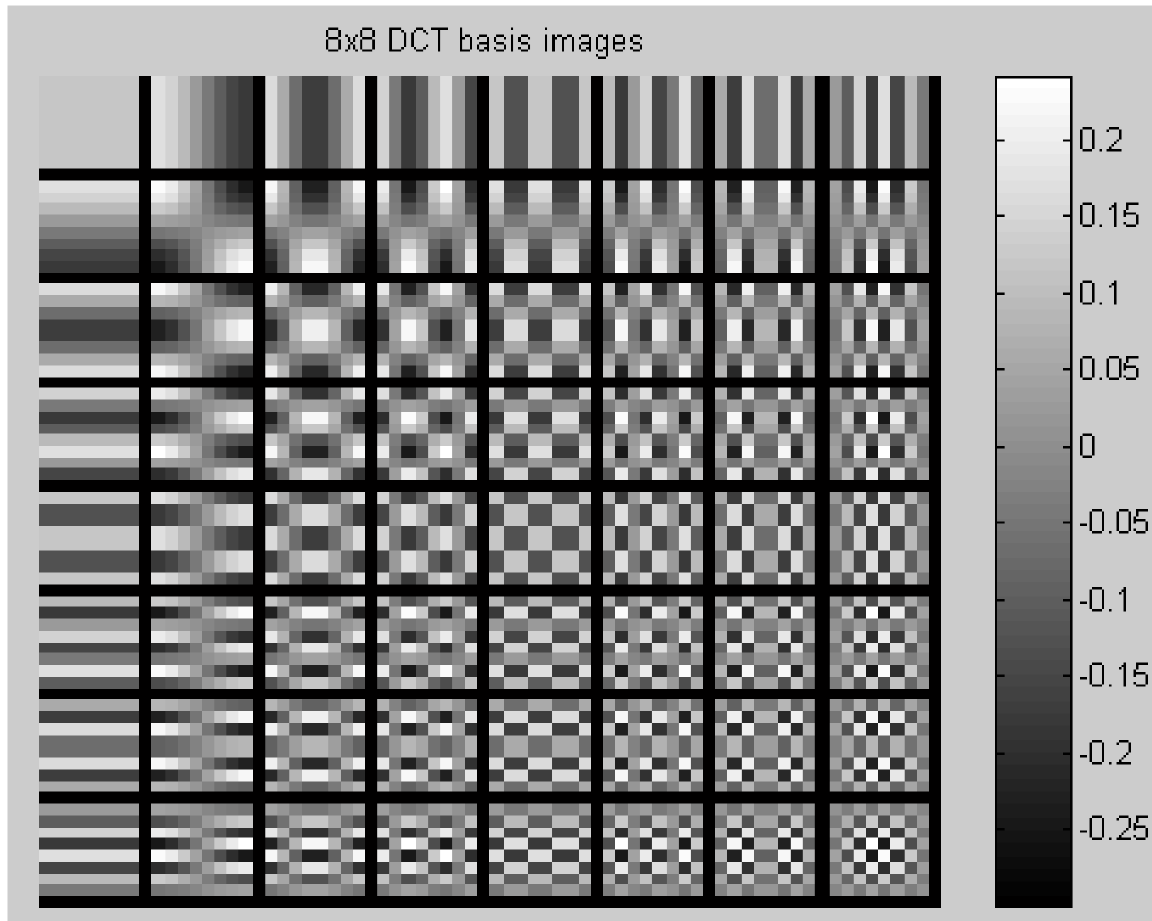
DFT2 $a_{uv}(m,n) = e^{-j2\pi(\frac{um}{M}+\frac{vn}{N})}$

real(DFT2)



From 1D-DCT to 2D-DCT



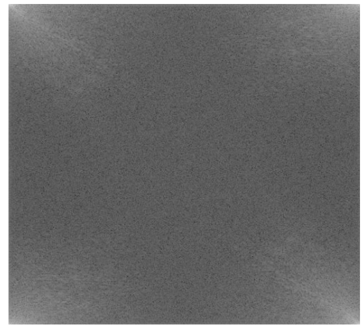


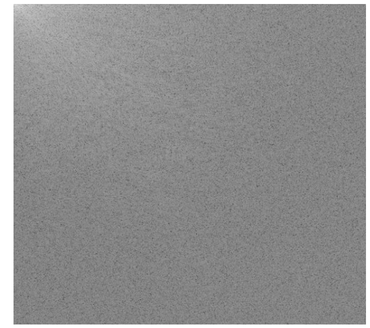
- Rows of A form a set of orthonormal basis
- A is not symmetric!
- DCT is <u>not</u> the real part of unitary DFT!

DFT and DCT on Lena

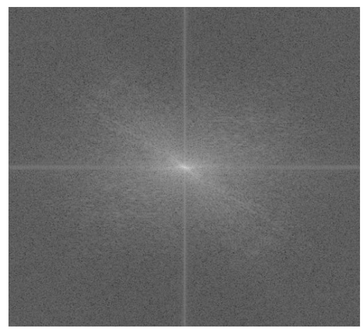
DFT2

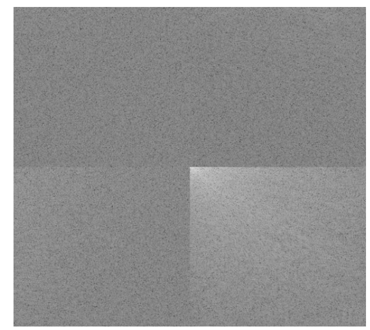
DCT2





Shift low-freq
to the center





Assume periodic and zero-padded …

Assume reflection …

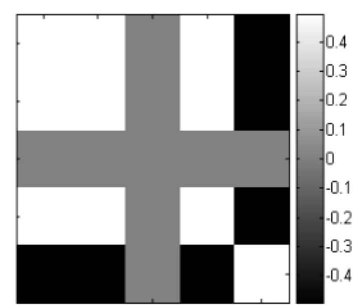
Exercise

- Unitary or not?

$$A_1 = \begin{bmatrix} \sqrt{2} & j \\ -j & \sqrt{2} \end{bmatrix} \qquad A_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & j \\ j & 1 \end{bmatrix}$$

- How do we decompose this picture?

$$\overset{\text{DCT2}}{\Rightarrow} \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

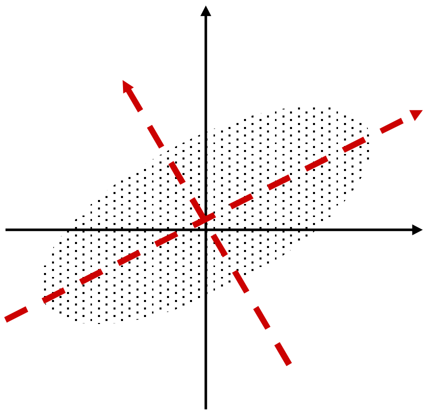


Properties of 1-D Unitary Transform

- Energy Conservation
  - $|| \underline{f}||^2 = || \underline{g}||^2$
    - $|| \underline{g}||^2 = ||A\underline{f}||^2 = (A\underline{f})^{*T}(A\underline{f}) = \underline{f}^{*T} A^{*T} A \underline{f} = \underline{f}^{*T}\underline{f} = ||\underline{f}||^2$
- Rotation
  - A unitary transformation is a rotation of a vector in an N-dimension space, i.e., a rotation of basis coordinates
  - The angles between vectors are preserved

- Review: correlation in vectors and images
- De-correlation (example 5.2 in Jain)
  - Highly correlated input elements → quite uncorrelated output coefficients
- Energy compaction
  - Many common unitary transforms tend to pack a large fraction of signal energy into just a few transform coefficients

Karhunen-Loeve Transform (KLT)



- a.k.a the Hotelling transform or the Principle Component Analysis (PCA)
- Eigen decomposition of R_x: $\quad R_x \underline{u}_k = \lambda_k \underline{u}_k$
  - Recall the properties of R_x
    - Hermitian (conjugate symmetric $R^H = R$);
    - Nonnegative definite (real non-negative eigen values)

- Karhunen-Loeve Transform (KLT)

  $$y = U^H x \Leftrightarrow x = U y \quad \text{with } U = [\, \underline{u}_1, \ldots \underline{u}_N \,]$$

  - KLT is a unitary transform with basis vectors in U being the orthonormalized eigenvectors of R_x

  - $U^H R_x U = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ i.e. KLT performs decorrelation
  - Often order $\{\underline{u}_i\}$ so that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$

Properties of K-L Transform

- Decorrelation
  - $E[\underline{y}\,\underline{y}^H] = E[(U^H x)(U^H x)^H] = U^H E[x\, x^H]\, U = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$
    - By construction
    - Note: Other matrices (unitary or nonunitary) may also decorrelate the transformed sequence [Jain's example5.5 and 5.7]

- Minimizing MSE under basis restriction
  - If only allow to keep *m* coefficients for any $1 \leq m \leq N$, what's the best way to minimize reconstruction error?

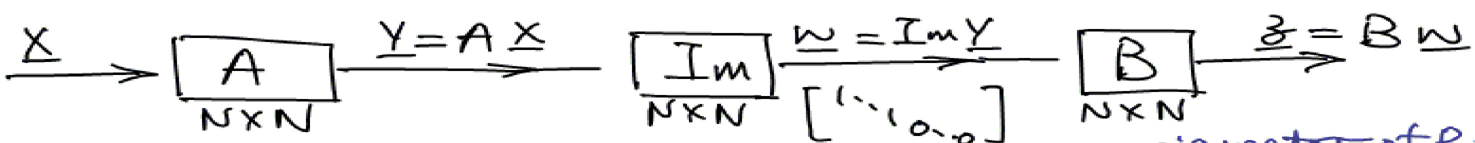
  → *Keep the coefficients w.r.t. the eigenvectors of the first m largest eigenvalues*

KLT Basis Restriction

- Basis restriction
  - Keep only a subset of m transform coefficients and then perform inverse transform $(1 \le m \le N)$
  - Basis restriction error: MSE between original & new sequences

$$J_m \triangleq E\left[\sum_{n=1}^{N} |x[n]-z[n]|^2\right] = Tr\left[E\left((\underline{x}-\underline{z})(\underline{x}-\underline{z})^H\right)\right]$$

- Goal: to find the forward and backward transform matrices to minimize the restriction error for each and every m



$$\min J_m \Rightarrow A = U^H, \; B = A^{-1} = U$$

U: eigvector of R_x arranged with decreasing $\{\lambda_k\}$

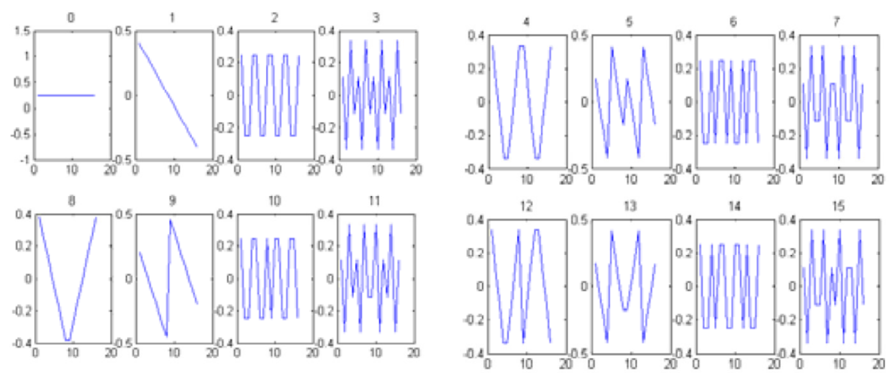
Unitary Transforms in Other Flavors

Walsh-Hardamard

$$H_3 = \frac{1}{2^{3/2}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

Slant

$$\begin{bmatrix} 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 \\ 0.5401 & 0.3858 & 0.2315 & 0.0772 & -0.0772 & -0.2315 & -0.3858 & -0.5401 \\ 0.3536 & -0.3536 & -0.3536 & 0.3536 & 0.3536 & -0.3536 & -0.3536 & 0.3536 \\ 0.1581 & -0.4743 & 0.4743 & -0.1581 & 0.1581 & -0.4743 & 0.4743 & -0.1581 \\ 0.4743 & 0.1581 & -0.1581 & -0.4743 & -0.4743 & -0.1581 & 0.1581 & 0.4743 \\ 0.2415 & -0.0345 & -0.3105 & -0.5866 & 0.5866 & 0.3105 & 0.0345 & -0.2415 \\ 0.3536 & -0.3536 & -0.3536 & 0.3536 & -0.3536 & 0.3536 & 0.3536 & -0.3536 \\ 0.1581 & -0.4743 & 0.4743 & -0.1581 & -0.1581 & 0.4743 & -0.4743 & 0.1581 \end{bmatrix}$$



wikipedia

Nassiri et. al, "Texture Feature Extraction using Slant-Hadamard Transform"

Energy Compaction Transforms

- DCT has excellent energy compaction for highly correlated data
- DCT is a good replacement for K-L
  - Close to optimal for highly correlated data
  - Not depend on specific data like K-L does
  - Fast algorithm available

[ref and statistics: Jain's pp153, 168-175]

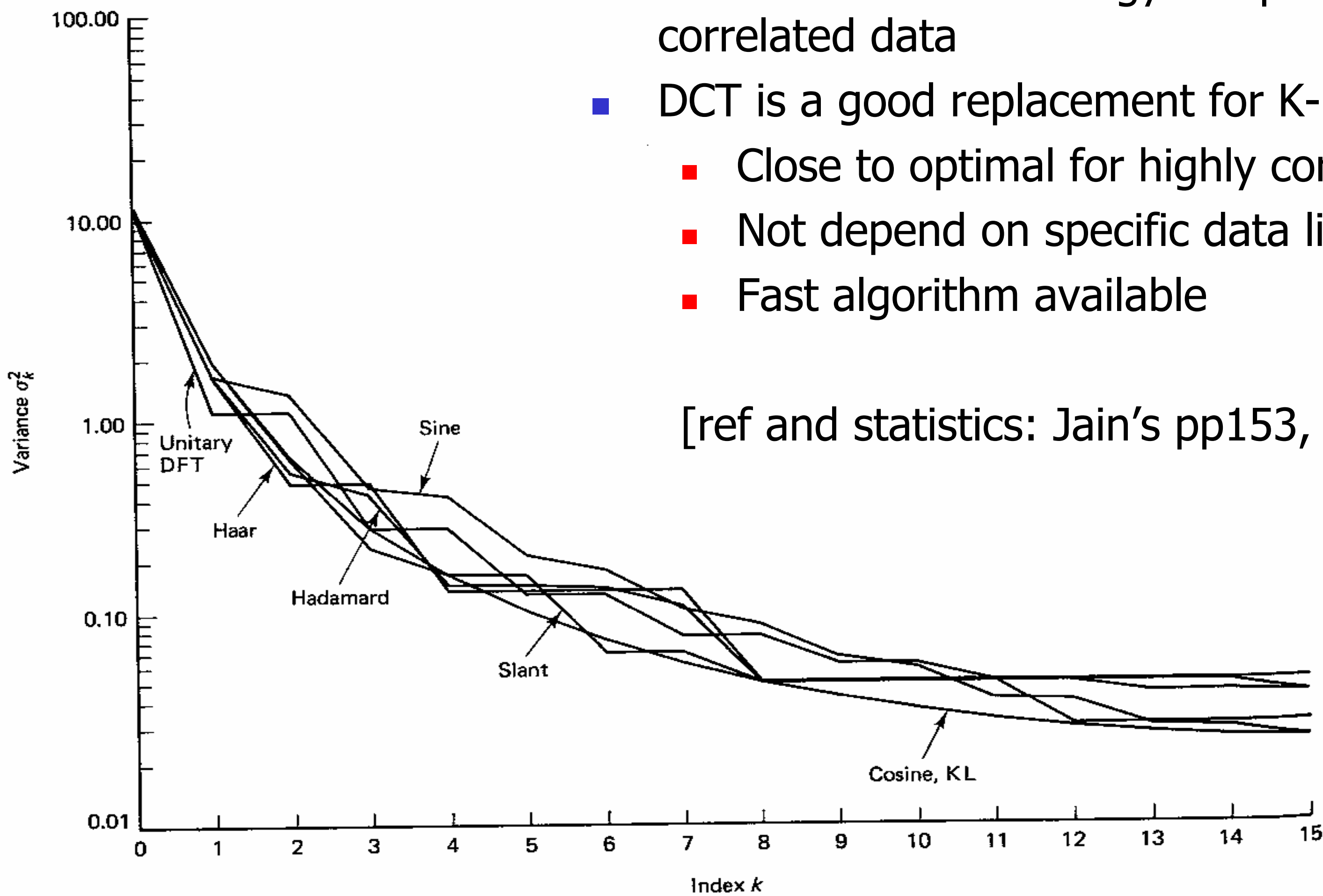
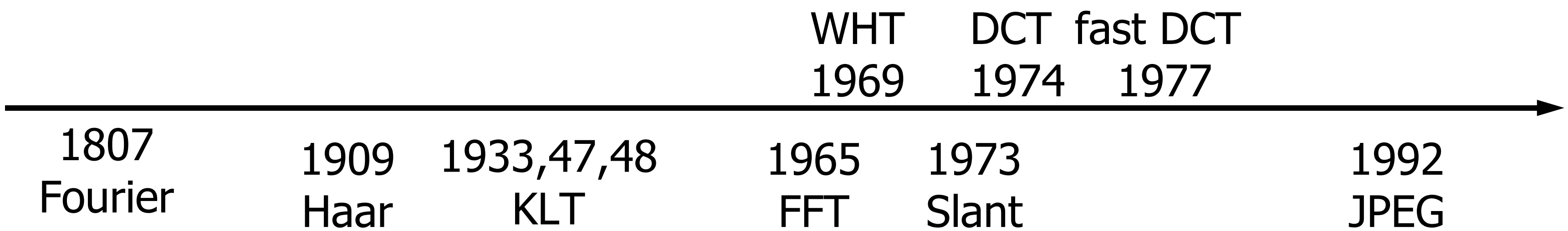


Figure 5.18 Distribution of variances of the transform coefficients (in decreasing order) of a stationary Markov sequence with $N = 16$, $\rho = 0.95$ (see Example 5.9).

The Desirables for Image Transforms

	DFT	DCT	KLT
Theory			
Inverse transform available	✓	✓	✓
Energy conservation (Parsevell)	✓	✓	✓
Good for compacting energy	?	?	✓
Orthonormal, complete basis	✓	✓	✓
(sort of) shift- and rotation invariant	✓	✓	?
Transform basis signal-independent			
Implementation			
Real-valued	x	✓	✓
Separable	✓	✓	x
Fast to compute w. butterfly-like structure	✓	✓	x
Same implementation for forward and inverse transform	✓	✓	x

A Brief History of Transforms



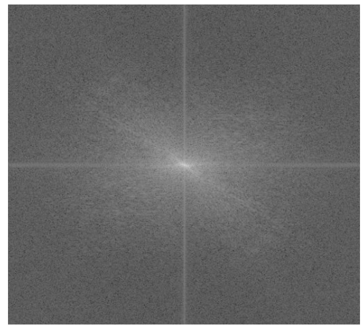
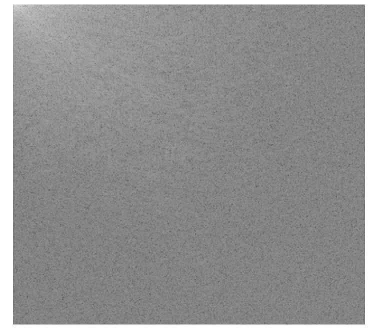
1807 Fourier Theory
1909 Haar filters “wavelets”
1933 Hotelling transform
1947 1948 Karhunen-Loeve
1965 FFT, Cooley-Tukey
1969 WHT, Shanks “computing fast Walsh-Hadamard transform”
1973 Slant Transform and applications to image coding
1974 DCT, Rao,
1977 Fast DCT
…
1992 JPEG Standard

from wikipedia, “a brief history of wavelets”, and other online sources

Applications of Image Transforms

- Compression
- Feature extraction
- Pattern recognition: e.g., eigen faces
  - analyze the principal (“dominating”) components

Image Compression

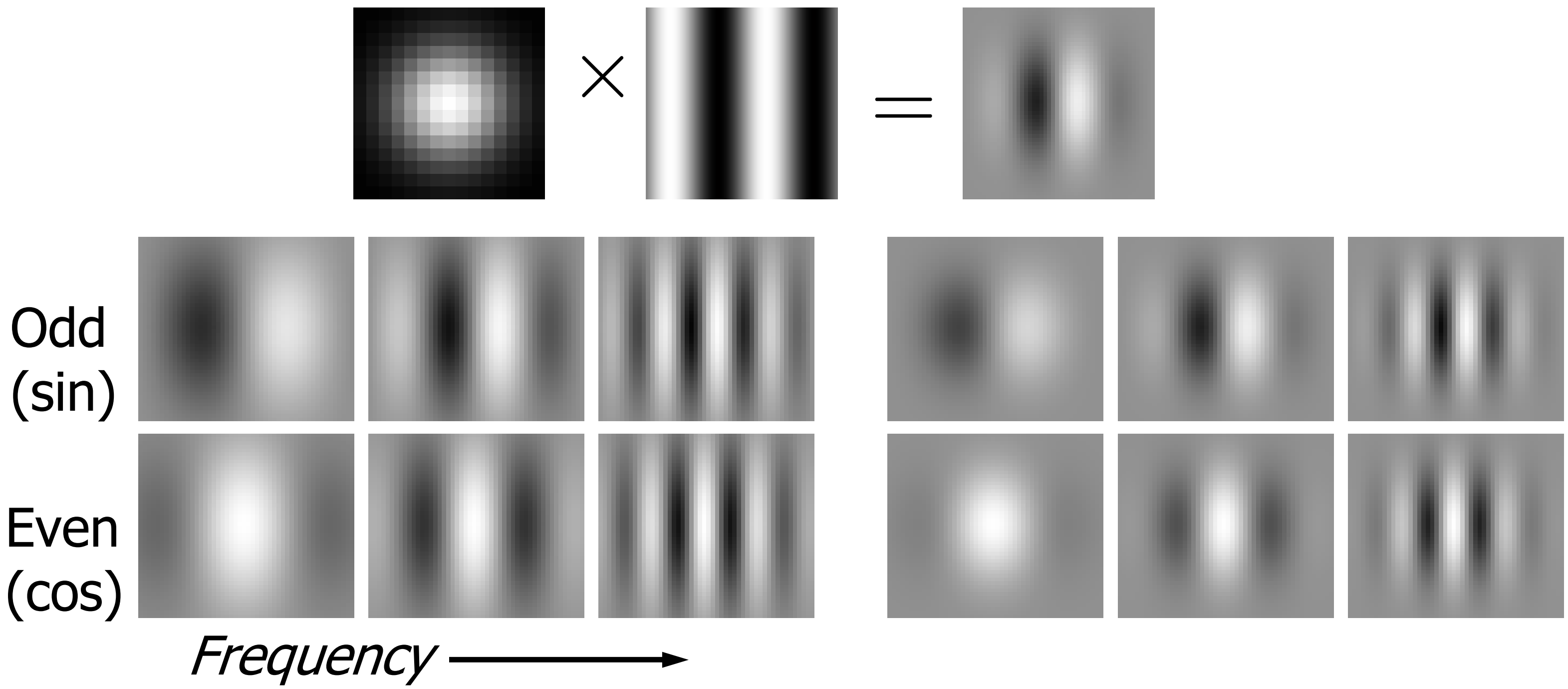
  

$$\mathrm{SNR(dB)} = 10\log_{10}\left(\frac{P_{\mathrm{signal}}}{P_{\mathrm{noise}}}\right) = 20\log_{10}\left(\frac{A_{\mathrm{signal}}}{A_{\mathrm{noise}}}\right)$$

where P is average power and A is RMS amplitude.

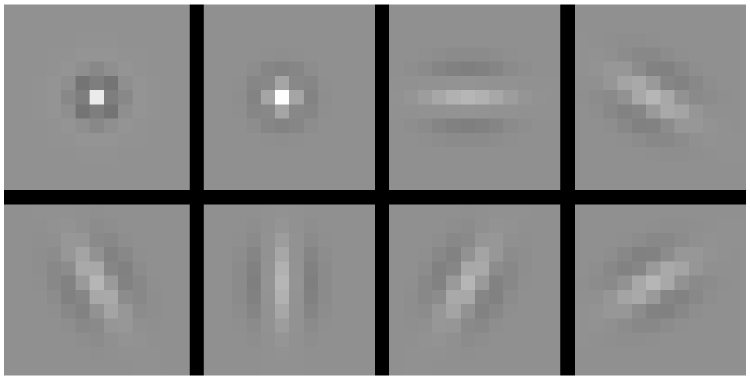
Gabor filters

- Gaussian windowed Fourier Transform
  - Make convolution kernels from product of Fourier basis images and Gaussians

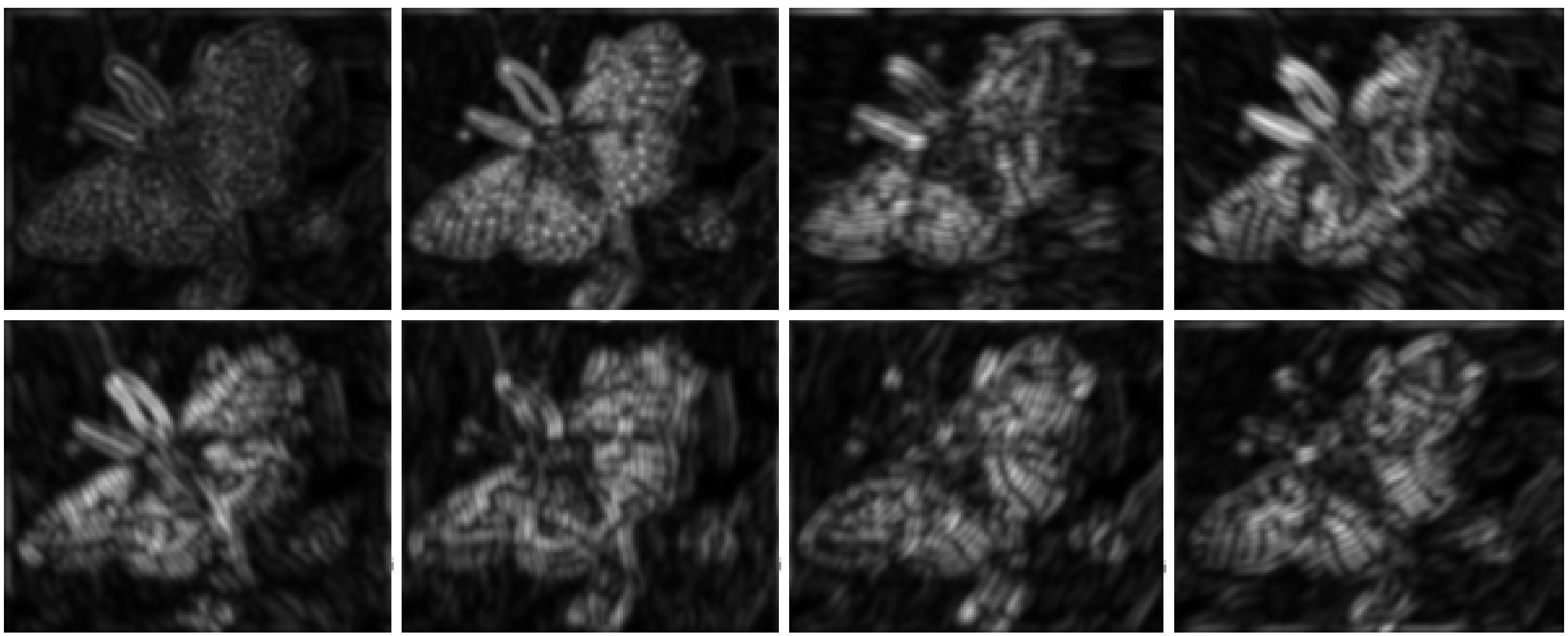


Example: Filter Responses

Filter bank



Input image

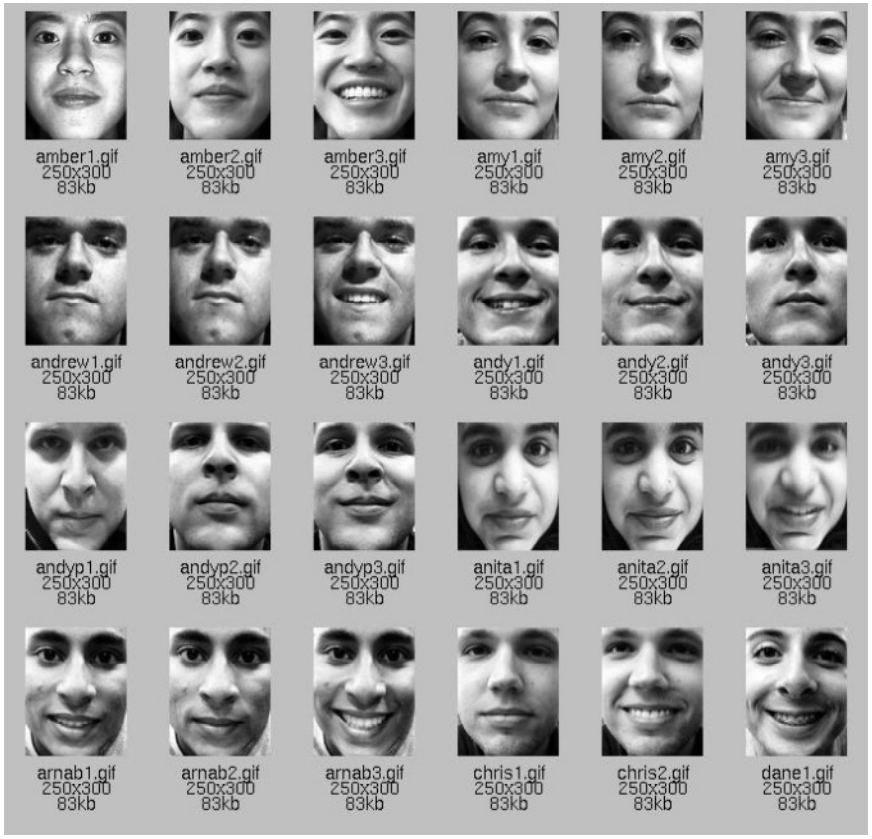


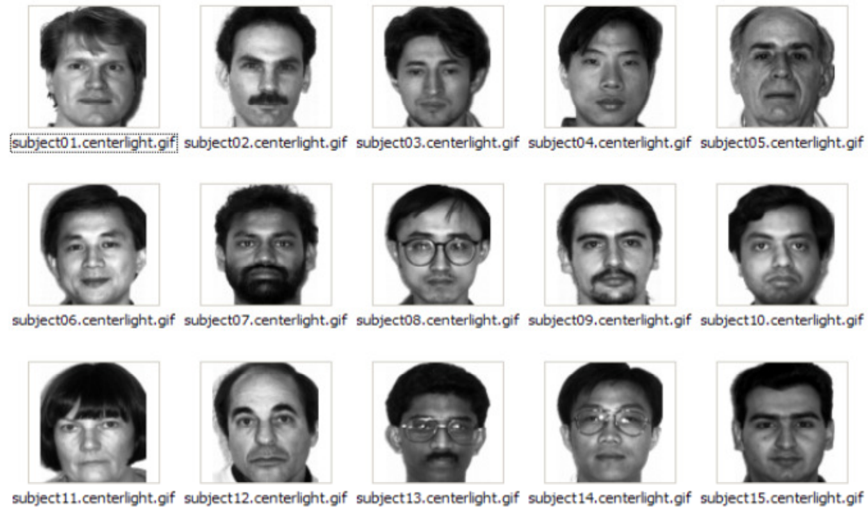
from Forsyth & Ponce

Texture Representation: Filter Responses

- Choose a group of filters
  - Edge/Bar filters: Something like Gabor filters at different orientations, scales
  - Spot filters: Center-surround filters like a Gaussian/difference of Gaussians at multiple scales
- Run filters over image to get a set of *response images*
  - Each contains specific texture information
- Collect statistics of responses over an image or subimage
  - Mean of squared response
  - Mean and variance of squared response
- Euclidean distance between vectors of response statistics for two images is measure of texture similarity

Eigen Faces



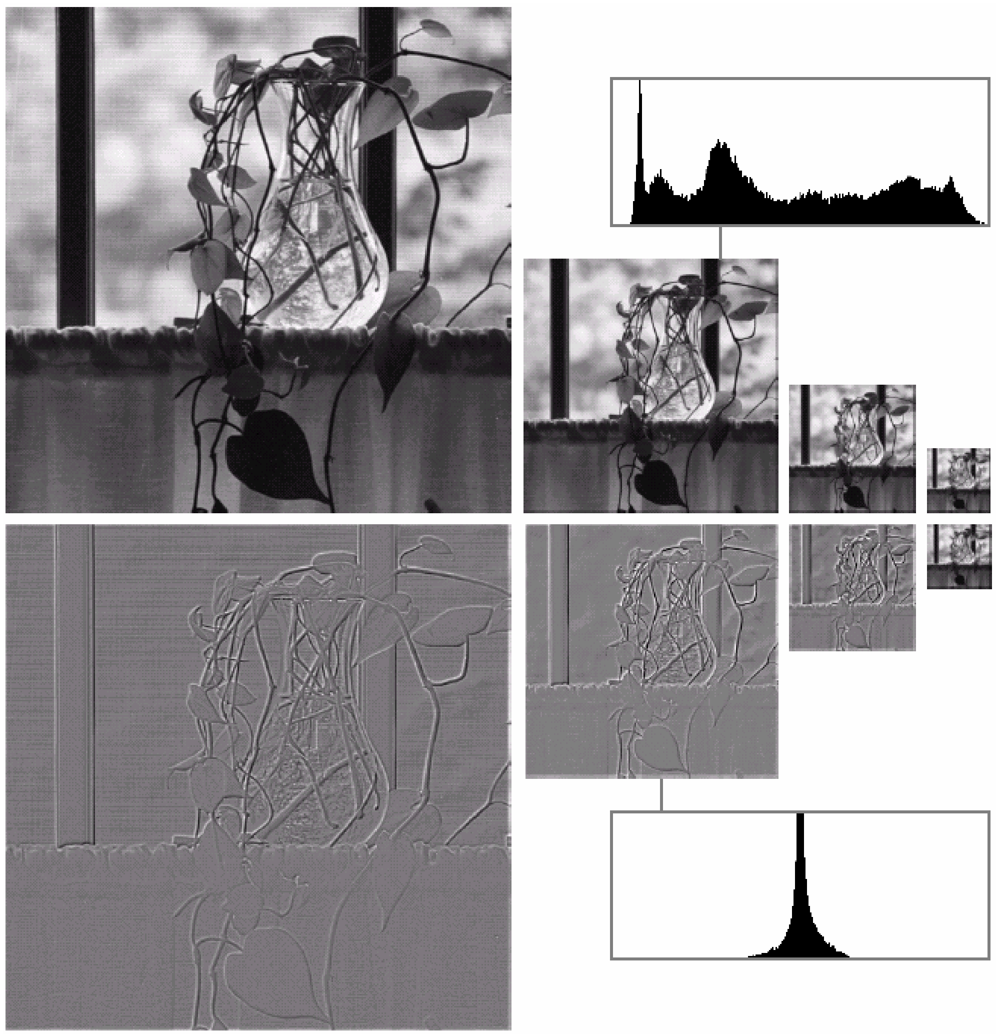


Courtesy of Rice U. eigen face project and Yale face database

Lecture Outline

- Unitary transforms
  - Properties
  - Examples: DFT, DCT, KLT, Haar
  - Applications
- Brief overview of wavelet transform and applications

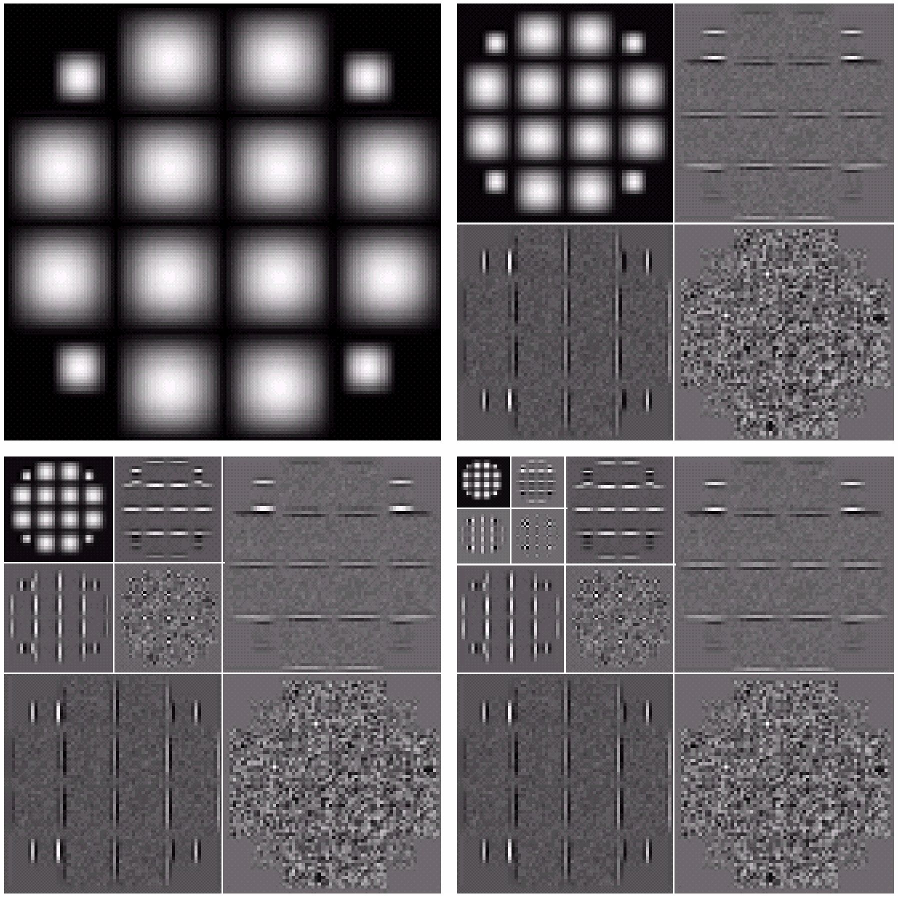
Image in Multiple Scales



a
b

FIGURE 7.3 Two image pyramids and their statistics: (a) a Gaussian (approximation) pyramid and (b) a Laplacian (prediction residual) pyramid.

A Three-scale Wavelet Decompostion



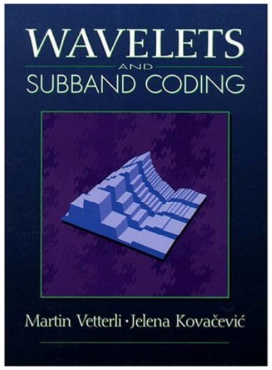
a b
c d

FIGURE 7.23 A three-scale FWT.

A Brief History of Wavelets

1909 Haar
…
1946 Gabor, Time-Frequency analysis
…
1982 Morlet, geophysics
1984 Marseile team, Grossmann/Paul, “mathematical microscope”
1985 Meyer, operator theory
1986 Mallat, signal analysis, filter design
1988 Lemarie, Daubechies, w. exponential decay and compact support
1991 … general construction, …
1992 … biorthongonal wavelets, continuous wavelet transform …
1995 … wavelet on domains
2000+ image compression standard (JPEG 2000)

- For in-depth looks …
  - ELEN E6860y Advanced Digital Signal Processing
  - Wavelet and Subband Coding, Vetterli and Kovacevic



Summary

- Unitary Transforms
  - Theory revisited
  - Example transforms: DFT, DCT, KLT, Hadamard, Slant
  - Applications
    - Compression
    - Feature extraction
    - Image matching (eigen faces)
- Wavelet decomposition of images