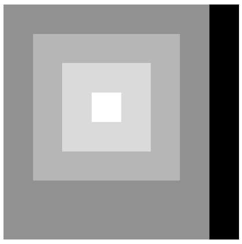
T-61.5100 Digital image processing, Exercise 2/06

Image enhancement in the spatial domain

1. Equalize the histogram of the 8×8 image below. The image has grey levels $0, 1, \ldots, 7$.



4	4	4	4	4	4	4	0
4	5	5	5	5	5	4	0
4	5	6	6	6	5	4	0
4	5	6	7	6	5	4	0
4	5	6	6	6	5	4	0
4	5	5	5	5	5	4	0
4	4	4	4	4	4	4	0
4	4	4	4	4	4	4	0

2. Assume that we have many noisy versions $g_i(x, y)$ of the same image $f(x, y)$, i.e.

$$g_i(x, y) = f(x, y) + \eta_i(x, y)$$

where the noise η_i is zero-mean and all point-pairs (x, y) are uncorrelated. Then we can reduce noise by taking the mean of all the noisy images

$$\bar{g}(x, y) = \frac{1}{M} \sum_{i=1}^{M} g_i(x, y).$$

Prove that

$$E\{\bar{g}(x, y)\} = f(x, y)$$

and

$$\sigma^2_{\bar{g}(x,y)} = \frac{1}{M} \sigma^2_{\eta(x,y)}$$

where $\sigma^2_{\eta(x,y)}$ is the variance of η and $\sigma^2_{\bar{g}(x,y)}$ the variance of $\bar{g}(x, y)$.

3. A two-variable continuous function's Laplace-operator is

$$\bigtriangledown^2 f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

From this, it follows that

$$\mathcal{F}\{\bigtriangledown^2 f(x, y)\} = -(2\pi)^2 (u^2 + v^2) F(u, v),$$

where $F(u, v) = \mathcal{F}\{f(x, y)\}$. Determine the corresponding operator and the Fourier transform in the discrete case. Compare the result obtained to the continuous case.

4. Show that subtracting the Laplacian from an image is proportional to unsharp masking. Use the definition for the Laplacian in the discrete case.

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1.

Histogram equalization enhances contrast in an image (more gray levels are used). It is usually used to improve the visibility of details in images. The discrete equalization is given by

$$s_k = T(r_k) = \sum_{j=0}^{k} \frac{n_j}{n},$$

where r_k is the normalized gray level, n_k is the number of pixels having gray level k, and n is the total number of pixels.

First we calculate the gray level histogram:

k	0	1	2	3	4	5	6	7
r_k	0	0.143	0.286	0.429	0.571	0.714	0.857	1.0
n_k	8	0	0	0	31	16	8	1
$p_r(r_k) = n_k/n$	0.125	0	0	0	0.484	0.250	0.125	0.016

Then we form the cumulative distribution function:

s_k	0.125	0.125	0.125	0.125	0.609	0.859	0.984	1.000

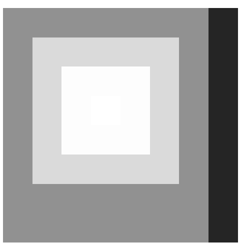
Finally we round the obtained values for the actual gray levels r_k:

s'_k	0.143	0.143	0.143	0.143	0.571	0.857	1.000	1.000

So, the histogram equalization is done by mapping the gray levels k into the new gray levels k':

k	0	1	2	3	4	5	6	7
k'	1	1	1	1	4	6	7	7

Resulting image:



$$\begin{matrix} 4 & 4 & 4 & 4 & 4 & 4 & 4 & 1 \\ 4 & 6 & 6 & 6 & 6 & 6 & 4 & 1 \\ 4 & 6 & 7 & 7 & 7 & 6 & 4 & 1 \\ 4 & 6 & 7 & 7 & 7 & 6 & 4 & 1 \\ 4 & 6 & 7 & 7 & 7 & 6 & 4 & 1 \\ 4 & 6 & 6 & 6 & 6 & 6 & 4 & 1 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 1 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 1 \end{matrix}$$

2.

The noisy image is now given as

$$\underbrace{g_i(x,y)}_{\text{noisy image}} = \underbrace{f(x,y)}_{\text{original image}} + \underbrace{\eta_i(x,y)}_{\text{noise}},$$

where noise has zero mean and it is uncorrelated.

When we average the noisy images

$$\bar{g}(x,y) = \frac{1}{M}\sum_{i=1}^{M} g_i(x,y)$$

we get for the expectation

$$E\left\{\bar{g}(x,y)\right\} = E\left\{\frac{1}{M}\sum_{i=1}^{M} g_i(x,y)\right\} = E\left\{\frac{1}{M}\sum_{i=1}^{M} f(x,y) + \frac{1}{M}\sum_{i=1}^{M}\eta_i(x,y)\right\} =$$

$$\frac{1}{M}\sum_{i=1}^{M} E\left\{f(x,y)\right\} + \frac{1}{M}\sum_{i=1}^{M}\underbrace{E\left\{\eta_i(x,y)\right\}}_{=0 \text{ (zero mean)}} = f(x,y)$$

and for the variance

$$\sigma^2_{\bar{g}(x,y)} = E\left\{(\bar{g} - E\left\{\bar{g}\right\})^2\right\} = E\left\{\left(\frac{1}{M}\sum_{i=1}^{M}(f+\eta_i) - f\right)^2\right\} = E\left\{\left(\frac{1}{M}\sum_{i=1}^{M}\eta_i\right)^2\right\} =$$

$$\frac{1}{M^2}E\left\{\left(\sum_{i=1}^{M}\eta_i\right)^2\right\} = \frac{1}{M^2}E\left\{\sum_{i=1}^{M}\left(\eta_i^2 + \sum_{j=1,\, j\neq i}^{M}\eta_i\eta_j\right)\right\} = \frac{1}{M^2}\left[\sum_{i=1}^{M}\left(\underbrace{E\{\eta_i^2\}}_{=\sigma_\eta^2} + \sum_{j=1,\, j\neq i}^{M}\underbrace{E\{\eta_i\eta_j\}}_{=0}\right)\right]$$

$$= \frac{1}{M}\sigma_\eta^2,$$

since the noise had zero mean and it was uncorrelated.

3.

We approximate the derivative in discrete case:

$$\frac{\partial f(x,y)}{\partial x} \simeq f(x,y) - f(x-1,y)$$

By ‘derivating’ again, we have

$$\frac{\partial^2 f(x,y)}{\partial x^2} \simeq (f(x,y) - f(x-1,y)) - (f(x-1,y) - f(x-2,y)) = f(x,y) - 2f(x-1,y) + f(x-2,y).$$

This is equivalent to filtering with a mask $[1, -2, 1]$. The mask is most practically used symmetrically so that -2 is in the middle, rather than as in the formula above.
Thus

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} \simeq f(x-1,y) + f(x+1,y) + f(x,y-1) + f(x,y+1) - 4f(x,y),$$

which corresponds to mask $h(x,y)$:

	1	
1	-4	1
	1	

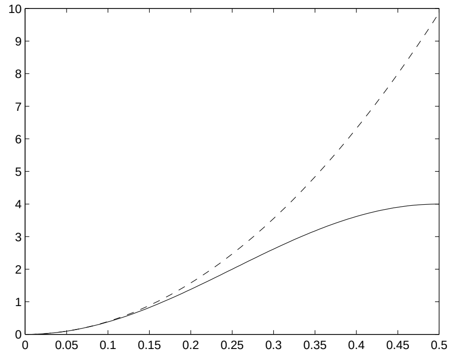
Next we calculate the Fourier transform of the mask:

$$H(u,v) = \frac{1}{N}\sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} h(x,y)e^{-j2\pi(ux+vy)/N} = \frac{1}{N}(e^{j2\pi u/N} + e^{j2\pi v/N} + e^{-j2\pi u/N} + e^{-j2\pi v/N} - 4)$$

$$= \frac{1}{N}\left[2\cos(2\pi u/N) + 2\cos(2\pi v/N) - 4\right]$$

And finally

$$\mathcal{F}\{\bigtriangledown^2 f[x,y]\} = \frac{1}{N}(2\cos(2\pi u/N) + 2\cos(2\pi v/N) - 4) \cdot F(u,v),$$



In the figure there is $|H(u,0)|$ in discrete and continuous case. Solid line: discrete case, i.e. $|H(u,0)| = 2 - 2\cos(2\pi u)$. Dotted line: continuous case, i.e. $|H(u,0)| = (2\pi)^2 u^2$.

4.

Consider the following equation:

$$\begin{aligned} f(x,y) - \bigtriangledown^2 f(x,y) &= f(x,y) - [f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y)] \\ &= 6f(x,y) - [f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) + f(x,y)] \\ &= 5\{1.2f(x,y) - \frac{1}{5}[f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) + f(x,y)]\} \\ &= 5[1.2f(x,y) - \bar{f}(x,y)] \end{aligned}$$

where $\bar{f}(x,y)$ denotes the average of $f(x,y)$ in a predefined neighborhood that is centered at (x,y) and includes the center pixel and its four immediate neighbors. Treating the constants in the last line of the above equation as proportionality factors, we may write

$$f(x,y) - \bigtriangledown^2 f(x,y) \sim f(x,y) - \bar{f}(x,y).$$

The right side of this equation is recognized as the definition of unsharp masking given in Eq. (3.7-7). Thus, it has been demonstrated that subtracting the Laplacian from an image is proportional to unsharp masking.