## Homework Set no. 7

1. Maximum likelihood decoding. (Gallager 2.5) A source produces independent, equally probable symbols from an alphabet $\left(a_{1}, a_{2}\right)$ at a rate of one symbol every 3 seconds. These symbols are transmitted over a binary symmetric channel which is used once each second by encoding the source symbol $a_{1}$ as 000 and the source symbol $a_{2}$ as 111 . If in the corresponding 3 second interval of the channel output, any of the sequences $000,001,010,100$ is received, $a_{1}$ is decoded; otherwise, $a_{2}$ is decoded. Let $\epsilon<\frac{1}{2}$ be the channel crossover probability.
(a) For each possible received 3-bit sequence in the interval corresponding to a given source letter, find the probability that $a_{1}$ came out of the source given that received sequence.
(b) Using part (a), show that the above decoding rule minimizes the probability of an incorrect decision.
(c) Find the probability of an incorrect decision (using part (a) is not the easy way here).
(d) If the source is slowed down to produce one letter every $2 n+1$ seconds, $a_{1}$ being encoded by $2 n+10$ 's and $a_{2}$ being encoded by $2 n+11$ 's. What decision rule minimizes the probability of error at the decoder? Find the probability of error as $n \rightarrow \infty$.
2. Channels with memory have higher capacity. (EIT 8.4) Consider a binary symmetric channel with $Y_{i}=X_{i} \oplus Z_{i}$, where $\oplus$ is $\bmod 2$ addition, and $X_{i}, Y_{i} \in\{0,1\}$.
Suppose that $\left\{Z_{i}\right\}$ has constant marginal probabilities $\operatorname{Pr}\left\{Z_{i}=1\right\}=p=1-\operatorname{Pr}\left\{Z_{i}=0\right\}$, but that $Z_{1}, Z_{2}, \ldots, Z_{n}$ are not necessarily independent. Assume that $Z^{n}$ is independent of the input $X^{n}$. Let $C=1-H(p, 1-p)$. Show that

$$
\max _{p\left(x_{1}, x_{2}, \ldots, x_{n}\right)} I\left(X_{1}, X_{2}, \ldots, X_{n} ; Y_{1}, Y_{2}, \ldots, Y_{n}\right) \geq n C .
$$

3. Using two channels at once. (EIT8.6) Consider two discrete memoryless channels ( $X_{1}, p\left(y_{1} \mid\right.$ $\left.x_{1}\right), y_{1}$ ) and ( $X_{2}, p\left(y_{2} \mid x_{2}\right), y_{2}$ ) with capacities $C_{1}$ and $C_{2}$ respectively. A new channel ( $X_{1} \times$ $\left.X_{2}, p\left(y_{1} \mid x_{1}\right) \times p\left(y_{2} \mid x_{2}\right), y_{1} \times y_{2}\right)$ is formed in which $x_{1} \in X_{1}$ and $x_{2} \in \mathcal{X}_{2}$, are simultaneously sent, resulting in $y_{1}, y_{2}$. Find the capacity of this channel.

## 4. Double Erasure Channel.

(a) Compute the capacity of the double erasure shown in Figure 1. Give a probability mass function on the channel input that achieves capacity.
(b) The "zero-error" capacity is the capacity for which a zero error rate can be guaranteed even for a fnite blocklength. It is, of course, upper bounded by the capacity we generally compute. It is also difficult in general to compute. However, the channel above is relatively simple. Show that the zero-error capacity of this channel is at least 1 bit.


Figure 1: Double Erasure Channel.

