## V. Castelli

## Homework no. 1

## Problems 1 to 3 are graded, no group solutions allowed. Due date: January 25.

1. Entropy of functions of a random variable (EIT 2.5). Let X be a discrete random variable. Show that the entropy of a function of X is less than or equal to the entropy of X by justifying the following steps:

$$H(X,g(X)) \stackrel{(a)}{=} H(X) + H(g(X) \mid X) \tag{1}$$

$$\stackrel{(b)}{=} H(X); \tag{2}$$

$$H(X, g(X)) \stackrel{(c)}{=} H(g(X)) + H(X \mid g(X))$$
<sup>(d)</sup>

$$\stackrel{(a)}{\geq} H(g(X)). \tag{4}$$

Thus  $H(g(X)) \leq H(X)$ .

2. Venn diagrams (EIT 2.12). ONE CANNOT USE THE VENN DIAGRAMS to define new quantities OR to prove theorems.

Using Venn diagrams, we could define the mutual information common to three random variables X, Y and Z as

$$I(X;Y;Z) = I(X;Y) - I(X;Y|Z).$$

This quantity is symmetric in X, Y and Z, despite the preceding asymmetric definition. Unfortunately, I(X;Y;Z) is not necessarily nonnegative, so it is not a Mutual Information at all!

- Find X, Y and Z such that I(X;Y;Z) < 0,
- Prove the following two identities:

$$I(X;Y;Z) = H(X,Y,Z) - H(X) - H(Y) - H(Z) + I(X;Y) + I(Y;Z) + I(Z;X)$$
$$I(X;Y;Z) = H(X,Y,Z) - H(X,Y) - H(Y,Z) - H(Z,X) + H(X) + H(Y) + H(Z)$$

The first identity can be understood using the Venn diagram analogy for entropy and mutual information. The second identity follows easily from the first.

3. Infinite entropy. This problem shows that the entropy of a discrete random variable can be infinite. Let  $A = \sum_{n=2}^{\infty} (n \log^2 n)^{-1}$ . (It is easy to show that A is finite by bounding the infinite sum by the integral of  $(x \log^2 x)^{-1}$ .) Show that the integer-valued random variable X defined by  $\Pr(X = n) = (An \log^2 n)^{-1}$  for  $n = 2, 3, \ldots$  has  $H(X) = +\infty$ .

(Turn over)

For your pleasure only (the following two problems will not be graded. You do not have to turn in a solution. You are encouraged to work them out with your colleagues).

These 2 problems are much harder than any other in the course. They are meant for you to talk to your class mates, getting to know each other, and develop, if necessary, the taste for solving puzzles and intriguing problems.

1. 100 people are boarding a plane with 100 seats, including you. The first person loses the boarding pass while stepping into the plane. The flight attendant tells the person to just sit anywhere to speed up the boarding procedures. The flight attendant also tells every other passenger to sit in the assigned seat written on the boarding pass if not taken, and to select any other available place should the assigned seat be taken by someone else.

You are the last person on the plane. Assuming that all possible assignments of seats to people are equiprobable, and that passengers selecting a place when their place is taken do so according to the uniform distribution, what is the probability that you sit in your assigned seat?

2. There are 36 prisoners in a camp. One day, the warden assembles them in a room and proposes the following "game": he will give the prisoners 1 hour to organize themselves. Then they will be separated from each other. The warden will select a prisoner at random (by drawing a name from a urn) and take the prisoner to a room. The room contains two switches, which are not connected to anything. Each switch can be in a up or a down position. The selected prisoner must select one switch and change its position (he cannot switch both or not switch any). By using this mechanism to communicate with each other the prisoners must determine when all of them have been taken (at least once) to the room. When one of the prisoners declares that all prisoners have been taken to the room with the switches, the game is over. If the prisoner making the declaration is correct, all 36 go free, otherwise they all die.

What is the winning strategy (for the prisoners, of course!)?