Nash equilibria of a generic networking game with applications to circuit-switched networks

Outlines

• The Generic Networking Game Model
• User Access Mechanisms
• A Circuit-switched Network Example
• Conclusions
The Generic Networking Game Model

- Settings
  1. $N$ is the population of the users.
  2. $\lambda_n$ is the access control parameter for user $n$.
  3. $\vec{\lambda}$ is the vector of access control parameters for all users.
  4. $\theta_n(\vec{\lambda})$ is the amount of resource received by user $n$.
  5. $U_n(\theta_n)$ defines the utility user $n$ gains by receiving $\theta_n$ amount of resource.
• Assumptions

1. $\lambda_n \in [0, \lambda_n^{max}]$ for some fixed $\lambda_n^{max} > 0$.
2. $\theta_n(\tilde{\lambda})$ is a nondecreasing function of $\lambda_n$.
3. $\theta_n(.)$ is continuous in $\prod_{n=1}^{N}[0, \lambda_n^{max}]$ and differentiable with respect to $\lambda_n$.
4. $\theta_n(\tilde{\lambda}) = 0$ for all $\tilde{\lambda}$ such that $\lambda_n = 0$.
5. $U_n$ is nondecreasing and $U_n(0) = 0$.
6. $U'_n$ is nondecreasing, i.e., $U_n$ is concave.
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• Users’ behavior

1. Net utility maximization problem: \( \arg \max_\theta U_n(\theta) - M \theta \)

2. We define \( y_n \equiv (U'_n)^{-1}(M) \) which is the solution of the above problem.

3. After the \( j^{th} \) iteration, for each user \( n \), its input parameter is:

\[
\lambda_{n}^{j+1} = \min \{ G(y_n, \theta_n(\vec{\lambda}_j), \lambda_n^j), \lambda_n^{\max} \}
\]

where the jointly continuous function \( G \) satisfies the following properties:

\[
G(y, \theta, \lambda) \begin{cases} 
= \lambda & \text{if } y = \theta \\
> \lambda & \text{if } y > \theta \\
< \lambda & \text{if } y < \theta
\end{cases}
\]
• Results

1. Definition: A fix or equilibrium point of the iteration is any \( \vec{\lambda}^* \in \prod_{n=1}^{N} [0, \lambda_{n}^{max}] \) such that

\[
\lambda_n^* = \min\{G(y_n, \theta_n(\vec{\lambda}^*), \lambda_n^*), \lambda_{n}^{max}\}.
\]

By Brouwer’s fixed point theorem, there exists at least one such point.

2. Lemma: (a) If \( \lambda_n^* < \lambda_{n}^{max} \), \( \theta_n(\vec{\lambda}^*) = y_n \). (b) \( \theta_n(\vec{\lambda}^*) \leq y_n \).

Theorem: Each fixed point is a Nash equilibrium point (NEP).

Proof: Interior solution follows lemma (a). Boundary solution has users whose \( \lambda_n = \lambda_{n}^{max} \).
3. Definition: A NEP \( \tilde{\lambda}^* \) is Pareto optimal if, for any other NEP \( \tilde{\alpha}^* \),

\[
U_n(\theta_n(\tilde{\lambda}^*)) - M\theta_n(\tilde{\lambda}^*) \geq U_n(\theta_n(\tilde{\alpha}^*)) - M\theta_n(\tilde{\alpha}^*)
\]

for all \( n \).

Theorem: Fixed points in the interior of \( \prod_{n=1}^{N} [0, \lambda_{n}^{max}] \) are Pareto optimal.
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User Access Mechanisms

- Additive increase and multiplicative decrease (AIMD) iterations:
  \[ G(y, \theta, \lambda) = \lambda - \alpha \lambda 1\{y < \theta\} + \delta 1\{y > \theta\} \]

- Multiplicative increase and multiplicative decrease (MIMD) iterations:
  \[ G(y, \theta, \lambda) = \frac{y}{\theta} \lambda \]
User Access Mechanisms

• Unconstrained Jacobi iteration:
  Without constraints with $\lambda_{n}^{max}$

$$\tilde{\lambda}^{j+1} = \tilde{\lambda}^{j} + \varepsilon(F(\tilde{\lambda}^{j}) - \tilde{\lambda}^{j})$$

for a fixed small $\varepsilon > 0$ where the $n^{th}$ component of the function $F$ is

$$F_{n}(\tilde{\lambda}) = G(y_{n}, \theta_{n}(\tilde{\lambda}), \lambda_{n})$$
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A Circuit-switched Network Example

Consider a group of $K$ circuits connecting an origin node to a destination node in a network. A population of $N$ users competes for these $K$ circuits.

- $M/GI/K/K$ queueing model: each user $n$ has Poisson arrival rate $\lambda_n$ and arbitrary holding time distribution with mean $1/\mu_n$.
- Aggregated arrival rate: $\Lambda \equiv \sum_{n=1}^{N-1} \lambda_n$.
- Mean holding time: $\frac{1}{\mu} = \sum_{n=1}^{N} \frac{1}{\mu_n} \frac{\lambda_n}{\Lambda}$.
- Total traffic intensity: $\rho \equiv \Lambda \frac{1}{\mu} = \sum_{n=1}^{N} \frac{\lambda_n}{\mu_n}$.
A Circuit-switched Network Example

- Aggregated per-user connection blocking probability in steady state is Erlang’s formula:
  \[ E_K(\rho) \equiv \frac{\rho^K / K!}{\sum_{k=0}^{K} \rho^k / k!} \]

- The net arrival rate of the \( n^{th} \) user: \( \lambda_n(1 - E(\rho)) \).

- By Little’s result, the mean number of occupied circuits for the \( n^{th} \) user is: \( \theta_n(\bar{\lambda}) \equiv \frac{1}{\mu_n} \lambda_n(1 - E(\rho)) \).
A Circuit-switched Network Example

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A Circuit-switched Network Example

- MIMD iteration:
  \[ \lambda_{n}^{j+1} = \min\left\{ \frac{y_n}{\theta_n} \lambda_n, \lambda_n^{\text{max}} \right\} = \min\left\{ \frac{y_n \mu_n}{1 - \mathcal{E}_K(\rho(\lambda_j))}, \lambda_n^{\text{max}} \right\} \]

- Nash equilibrium:
  \[ \frac{\lambda_n}{\mu_n} = \frac{y_n}{1 - \mathcal{E}} \]

  After aggregation:
  \[ \rho = \sum_{n=1}^{N} \frac{y_n}{1 - \mathcal{E}} \]
A Circuit-switched Network Example

- Numerical studies ($K=1$ and 2 users):
  1. $(\mu_1, \mu_2, y_1, y_2, \lambda_1^{max}, \lambda_2^{max}) = (1, 2, \frac{1}{3}, \frac{1}{2}, 10, 20)$.
  2. $(\mu_1, \mu_2, y_1, y_2, \lambda_1^{max}, \lambda_2^{max}) = (1, 2, \frac{1}{3}, \frac{5}{7}, 10, 20)$.

- Simulations comparing AIMD ($\alpha = 0.5$ and $\delta = 1$) comparing with MIMD:
  $\vec{y} = (10, 15, 20, 25, 30), \vec{\mu} = (15, 20, 25, 30, 35)$ with initial state $\vec{\lambda} = (10, 30, 10, 30, 10)$.
  1. All $\lambda_n^{max} = 100$.
  2. $\vec{\lambda}^{max} = (1500, 1500, 3000, 3000, 3000)$.
  3. $\vec{\lambda}^{max} = (1500, 1500, 3000, 5000, 8000)$. 
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Conclusions

• A generic networking game model has been proposed.
  1. Resource allocation mechanisms $\theta(\vec{\lambda})$ are not restricted.
  2. Utility functions of users $U$ are not restricted.
  3. Users can specify their own behavior mechanisms $G(y, \theta, \lambda)$.

• Convergence dynamics to a Nash equilibrium were studied.
  1. Use of Jacobi update scheme.
  2. Lyapunov theory of system stability.